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A NEUMANN EIGENVALUE PROBLEM FOR FULLY NONLINEAR OPERATORS

Isabeau Birindelli and Stefania Patrizi

Dipartimento di matematica "G. Castelnuovo"
Sapienza Università di Roma, Piazzale A. Moro 5
00185 Roma, Italia

Dedicated to Prof. Louis Nirenberg for his 85th birthday

Abstract. We prove the existence of the principal eigenvalues for the Pucci operators in bounded domains with boundary condition $\frac{\partial u}{\partial \vec{n}} = \alpha u$ corresponding respectively to positive and negative eigenfunctions and study their asymptotic behavior when α goes to $+\infty$.

1. **Introduction.** In this introduction and in the rest of the paper we quote some works of Louis Nirenberg that are used explicitly in order to give the right definitions and to prove the results; but the influence of his research, here and in all the papers both the authors have written, goes well beyond the citations. His mathematical ideas have been very important for us, specially for the first named author, but his teaching of how to approach mathematical problems has been as important. We are happy to have this opportunity to thank him for his generosity.

In this paper, for Ω a C^2 bounded domain of $\mathbb{R}^n$ and for any $\alpha > 0$, we consider the eigenvalue problem:

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) + \lambda u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \vec{n}} = \alpha u & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{M}^+_{a,A}$ is the Pucci operator, i.e. $\mathcal{M}^+_{a,A}(M) = \sup_{0<aI\leq\sigma\leq AI} \mathrm{tr}(\sigma \mathrm{M})$.

It is useless to emphasize the importance of the concept of eigenvalue for the understanding of the structural properties of the solutions both for linear and non linear equations. The pioneering work of Berestycki, Nirenberg and Varadhan [4] has open the way to enlarge this fundamental concept to non linear operators. Indeed, even if they treat linear equations, their theory is very well adapted to fully nonlinear operators and viscosity solutions being based primarily on the use of the maximum principle. This has been done by many and in many different contests, let us mention the works of Armstrong, Busca, Demengel, Juutinen, Ishii, Quaas, Sirakov, Yoshimura and the authors of this note ([1, 5, 6, 11, 12, 16, 17]). It should be mentioned that P.-L. Lions in [13], with a completely different approach, first introduces what he called demi-eigenvalues. Indeed when the operator is not odd

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with respect to the Hessian (as is the case of the Pucci operators), eigenvalues corresponding to positive eigenfunctions or to negative eigenfunctions may not coincide and one could interpret these two eigenvalues as a "splitting" of the eigenvalue.

The eigenvalue problem for Robin boundary conditions associated with a fully-nonlinear operator was already treated in [16]. The novelty here is that we consider $\alpha > 0$ which is the "wrong sign" in the sense that the boundary conditions are not "proper", see e.g. [8]. The boundary source and the reaction-diffusion equation are somehow in competition.

In analogy to [4] we define the eigenvalues in the following way:

$$\lambda_\alpha^+ := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ on } \overline{\Omega} \text{ bounded viscosity supersolution of}$$
$$\mathcal{M}^+_{a,A}(D^2 v) + \lambda v = 0 \text{ in } \Omega,\ \frac{\partial v}{\partial \overrightarrow{n}} = \alpha v \text{ on } \partial\Omega\},$$

$$\lambda_\alpha^- := \sup\{\lambda \in \mathbb{R} \mid \exists\, v < 0 \text{ on } \overline{\Omega} \text{ bounded viscosity subsolution of}$$
$$\mathcal{M}^+_{a,A}(D^2 v) + \lambda v = 0 \text{ in } \Omega,\ \frac{\partial v}{\partial \overrightarrow{n}} = \alpha v \text{ on } \partial\Omega\}.$$

The first step is to prove that there exists $u_\alpha^+ > 0$ and $u_\alpha^- < 0$ solutions of (1) when respectively $\lambda = \lambda_\alpha^+$ and $\lambda = \lambda_\alpha^-$ (Proposition 4). We shall also prove that below these eigenvalues there are solutions of the equation with a forcing term $f(x)$ as long as the f has the right sign, i.e. $f \le 0$ below λ_α^+ and $f \ge 0$ below λ_α^-.

We are mainly interested in the asymptotic behavior with respect to α of the eigenvalues. When $\alpha \to 0$, λ_α^+ and λ_α^- tend to 0 which is the principal eigenvalue of the pure Neumann boundary problem

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2 u) + \lambda u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

But our main goal is to study the behavior when $\alpha \to +\infty$, this is done in our main

Theorem 1.1. *The following limits hold:*

$$\lim_{\alpha \to +\infty} \frac{\lambda_\alpha^+}{-\alpha^2} = A, \tag{2}$$

$$\lim_{\alpha \to +\infty} \frac{\lambda_\alpha^-}{-\alpha^2} = a. \tag{3}$$

Interestingly this asymptotic behavior emphasizes the "splitting" of the eigenvalue. In the linear case, i.e. when $a = A = 1$ and the Pucci operator is nothing else but the Laplacian, this problem was treated in [14] by Lou and Zhu with a variational approach. Very recently Daners and Kennedy [9] have proved that this asymptotic behavior is valid for the whole spectrum.

We also prove that for any $K \subset\subset \Omega$, the normalized eigenfunctions u_α^+ and u_α^- satisfy

$$\|u_\alpha^+\|_{L^\infty(K)} \to 0 \quad \text{and} \quad \|u_\alpha^-\|_{L^\infty(K)} \to 0 \quad \text{as } \alpha \to +\infty.$$

So that the eigenfunctions tend to concentrate on the point of the boundary where they reach the sup or the inf.

The idea of the proof of Theorem 1.1 which somehow follows the line adopted in [14], is the following: first we establish that u_α^+ reaches its maximum on the boundary and then we perform a blow up around this point.

Then a key tool will be a Liouville result in the half space (Theorem 5.1). Precisely we prove that for $\gamma > A$ (respectively $\gamma > a$) there are no bounded subsolutions (respectively supersolutions) of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) - \gamma u = 0 & \text{in } \mathbb{R}^n, \\ -\frac{\partial u}{\partial x_n} = u & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

that are positive (respectively negative) somewhere. In [14] the analogous result for the Laplacian is proved using the construction of sub and super solutions in the flavor of what is done in [3]. Let us mention here that it would be interesting to extend the results of Berestycki, Caffarelli, Nirenberg [3] in half spaces, to this class of fully-nonlinear operators and to these boundary conditions.

Lipschitz estimates up to the boundary will be required in the proofs of both the existence results and the asymptotic behavior. These estimates which are interesting in their own right, are established here using an argument inspired by [10] (see also Barles and Da Lio [2] and Milakis and Silvestre [15]).

In the whole paper the fully-nonlinear operator considered is the Pucci operator $\mathcal{M}^+_{a,A}$, but, mutatis mutandis, parallel results can be stated for the Pucci operator $\mathcal{M}^-_{a,A}$ defined by $\mathcal{M}^-_{a,A}(M) = \inf_{0<aI\leq\sigma\leq AI} \mathrm{tr}(\sigma \mathrm{M})$.

2. **Preliminary results.** Let us recall the definition of viscosity sub and supersolution of the Neumann problem associated to a general elliptic operator $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}$. Here $S(n)$ is the space of symmetric matrices on $\mathbb{R}^n$, equipped with the usual ordering. We denote by $USC(\overline{\Omega})$ (resp., $LSC(\overline{\Omega})$) the set of upper (resp., lower) semicontinuous functions on $\overline{\Omega}$. Let $f : \overline{\Omega} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$.

Definition 2.1. A function $u \in USC(\overline{\Omega})$ (resp., $u \in LSC(\overline{\Omega})$) is called *viscosity subsolution* (resp., *supersolution*) of

$$\begin{cases} F(x, u, Du, D^2u) = f(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \vec{n}} = g(x, u) & \text{on} \quad \partial\Omega, \end{cases}$$

if the following conditions hold

(i) For every $x_0 \in \Omega$, for any $\varphi \in C^2(\overline{\Omega})$, such that $u - \varphi$ has a local maximum (resp., minimum) at x_0 then

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq \text{ (resp., } \leq \text{)}\, f(x_0).$$

(ii) For every $x_0 \in \partial\Omega$, for any $\varphi \in C^2(\overline{\Omega})$, such that $u - \varphi$ has a local maximum (resp., minimum) at x_0 then

$$-(F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - f(x_0)) \wedge \left(\frac{\partial \varphi}{\partial \vec{n}}(x_0) - g(x_0, u(x_0)) \right) \leq 0$$

(resp.,

$$-(F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) - f(x_0)) \vee \left(\frac{\partial \varphi}{\partial \vec{n}}(x_0) - g(x_0, u(x_0)) \right) \geq 0).$$

A *viscosity solution* is a continuous function which is both a subsolution and a supersolution.

One of the motivation for these relaxed boundary conditions is the stability under uniform convergence. Actually, if the domain Ω satisfies the exterior sphere condition and F is uniformly elliptic, viscosity subsolutions (resp., supersolutions) satisfy in the viscosity sense $\frac{\partial u}{\partial \vec{n}} \leq$ (resp., $\geq$)$g(x,u)$ for any $x \in \partial\Omega$, see e.g. Proposition 2.1 in [16].

We assume throughout the paper that Ω is a bounded domain of $\mathbb{R}^n$ of class C^2.

Theorem 2.2 (Strong Comparison Principle, [16] Theorem 3.1)**.** *Assume that c and f are continuous on $\overline{\Omega}$. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be respectively a sub and a supersolution of*

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) + c(x)u = f(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \vec{n}} = \alpha u & \text{on} \quad \partial\Omega. \end{cases}$$

If $u \leq v$ on $\overline{\Omega}$ then either $u < v$ on $\overline{\Omega}$ or $u \equiv v$ on $\overline{\Omega}$.

Proposition 1 (Maximum Principle for $\lambda < \lambda^+_\alpha$, [16] Theorem 4.5)**.** *Assume $\lambda < \lambda^+_\alpha$. Let $v \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2v) + \lambda v = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \vec{n}} = \alpha v & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

then $v \leq 0$ on $\overline{\Omega}$.

Proposition 2 (Minimum Principle for $\lambda < \lambda^-_\alpha$, [16] Remark 4.6)**.** *Assume $\lambda < \lambda^-_\alpha$. Let $v \in LSC(\overline{\Omega})$ be a viscosity supersolution of*

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2v) + \lambda v = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \vec{n}} = \alpha v & \text{on} \quad \partial\Omega, \end{cases}$$

then $v \geq 0$ on $\overline{\Omega}$.

3. **Lipschitz estimates.** In this section we shall prove a local Lipschitz regularity result for solutions of the Neumann problem associated to general uniformly elliptic operators, that we will use in the next sections. Let us consider the Neumann problem

$$\begin{cases} F(x,u,Du,D^2u) = f(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \vec{n}} = g(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where the operator F is supposed to be continuous on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S(n)$ and satisfying the following assumptions:

(F1) There exist $b, c > 0$ such that for $x \in \overline{\Omega}$, $r, s \in \mathbb{R}$, $p, q \in \mathbb{R}^n$, $X, Y \in S(n)$

$$\begin{aligned} \mathcal{M}^-_{a,A}(Y - X) - b|p-q| - c|r-s| &\leq F(x,r,p,Y) - F(x,s,q,X) \\ &\leq \mathcal{M}^+_{a,A}(Y-X) + b|p-q| + c|r-s|. \end{aligned}$$

(F2) There exists $C_1 > 0$ such that for all $x, y \in \overline{\Omega}$ and $X \in S(n)$

$$|F(x,0,0,X) - F(y,0,0,X)| \leq C_1|x-y|^{\frac{1}{2}}\|X\|.$$

Proposition 3. *Assume that (F1) and (F2) hold. Let $f : \overline{\Omega} \to \mathbb{R}$ be bounded, $g : \partial\Omega \to \mathbb{R}$ be Lipschitz continuous. Let $u \in C(\overline{\Omega})$ be a viscosity solution of (5), then, for any $x_0 \in \overline{\Omega}$ and for any $\rho > 0$, there exists $K > 0$ such that*

$$|u(x) - u(y)| \leq (MK + |g|_{L^\infty(\partial\Omega)})|x-y| \quad \forall x, y \in B_\rho(x_0) \cap \overline{\Omega} \tag{6}$$

and

$$K^2 - bK \le C\left[c|u|_{L^\infty(\overline{B}_{3\rho}(x_0)\cap\overline{\Omega})} + |f|_{L^\infty(\overline{\Omega})} + (1+b)\,|g|_{C^{0,1}(\partial\Omega)} + \frac{b}{\rho} + \frac{1}{\rho^2} + 1\right], \tag{7}$$

where $M \le C(|u|_{L^\infty(\overline{B}_{3\rho}(x_0)\cap\overline{\Omega})} + |g|_{L^\infty(\partial\Omega)} + 1)$ *and* C *depends on* a, A, C_1, n *and* Ω.

The C^2-regularity of Ω implies the existence of a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ on which the distance from the boundary

$$d(x) := \inf\{|x-y|, y \in \partial\Omega\}, \quad x \in \overline{\Omega}$$

is of class C^2. We still denote by d a C^2 extension of the distance function to the whole $\overline{\Omega}$. Without loss of generality we can assume that $|Dd(x)| \le 1$ on $\overline{\Omega}$.

Corollary 1. *Assume* $\lambda \in \mathbb{R}$ *and* $\alpha \ge 0$. *Let* $u \in C(\overline{\Omega})$ *be a viscosity solution of*

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) + \lambda u = f(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \overrightarrow{n}} = \alpha u & \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

Then, for any $\rho > 0$, *there exists* $K > 0$ *such that for any* $x, y \in \Omega_\rho := \{x \in \overline{\Omega} \,|\, d(x) \le \rho\}$

$$|u(x) - u(y)| \le (\alpha|e^{\alpha d(x)}u|_{L^\infty(\Omega_\rho)} + MK)|x-y|$$

and

$$K^2 - C\alpha K \le C\left[\left(\alpha + \alpha^2 + |\lambda|\right)|e^{\alpha d(x)}u|_{L^\infty(\Omega_{3\rho})} + |e^{\alpha d(x)}f|_{L^\infty(\overline{\Omega})} + \frac{\alpha}{\rho} + \frac{1}{\rho^2} + 1\right], \tag{9}$$

where $M \le C(|e^{\alpha d(x)}u|_{L^\infty(\Omega_{3\rho})} + 1)$ *and* C *depends on* a, A, n *and* Ω.

Proof of Proposition 3. We follow the proof of Proposition III.1 of [10], that we modify taking test functions which depend on the distance function and that are suitable for the Neumann boundary conditions. Moreover, as in [2], we are going to use a regularization of g. In order to do so, it is convenient to introduce the following classical lemma.

Lemma 3.1. *Assume* $\rho \in C^\infty(\mathbb{R}^n)$, $\rho > 0$, *supp*$(\rho) \subset B_1(0)$ *and* $\int_{\mathbb{R}^n} \rho(y)dy = 1$. *If* $g \in C^{0,1}(\mathbb{R}^n)$ *and* g *is bounded, then the function* $\widetilde{g} : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ *defined by*

$$\widetilde{g}(x,\varepsilon) := \int_{\mathbb{R}^n} g(z)\rho\left(\frac{x-z}{\varepsilon}\right)\frac{1}{\varepsilon^n}dz, \; \varepsilon > 0,$$

$$\widetilde{g}(x,0) = g(x) \quad \text{for } x \in \mathbb{R}^n,$$

is in $C^{0,1}(\mathbb{R}^n \times [0,+\infty))$. *Moreover, the function* $\widetilde{g}$ *is in* $C^2(\mathbb{R}^n \times (0,+\infty))$ *with*

$$|D_x\widetilde{g}(x,\varepsilon)|, \; |D_\varepsilon\widetilde{g}(x,\varepsilon)| \le C_0,$$

$$|D^2_{xx}\widetilde{g}(x,\varepsilon)|, \; |D^2_{x\varepsilon}\widetilde{g}(x,\varepsilon)|, \; |D^2_{\varepsilon\varepsilon}\widetilde{g}(x,\varepsilon)| \le \frac{C_0}{\varepsilon} \quad \text{in } \mathbb{R}^n \times (0,+\infty)$$

for some positive constant $C_0 \le C|g|_{C^{0,1}(\mathbb{R}^n)}$, *with* C *depending only on* ρ *and* n.

We first extend g to a $C^{0,1}$ function of $\mathbb{R}^n$ and we still denote by g this extension. Then, we consider the function $\widetilde{g}$ associated to g as in Lemma 3.1.

Since Ω is a domain of class C^2, it satisfies the uniform exterior sphere condition, i.e., there exists $r>0$ such that $B(x+r\overrightarrow{n}(x),r)\cap\Omega=\emptyset$ for any $x\in\partial\Omega$. From this property it follows that

$$\langle\overrightarrow{n}(x),y-x\rangle\leq\frac{1}{2r}|y-x|^2\quad\text{for } x\in\partial\Omega \text{ and } y\in\overline{\Omega}. \tag{10}$$

Let $x_0\in\overline{\Omega}$ and $\rho>0$. Let us denote $\overline{B}_{\overline{\Omega}}(x_0,\rho):=\overline{B}_\rho(x_0)\cap\overline{\Omega}$ and $B_{\overline{\Omega}}(x_0,\rho):=B_\rho(x_0)\cap\overline{\Omega}$. We are going to show that u is Lipschitz continuous on $\overline{B}_{\overline{\Omega}}(x_0,\rho)$. For this aim, let us introduce the functions

$$\Phi(x)=MK|x|-M(K|x|)^2,$$

$$\Psi_1(x,y)=e^{-L(d(x)+d(y))}\Phi(x-y),$$

$$\Psi_2(x,y)=\widetilde{g}\left(\frac{x+y}{2},(\delta^2+|x-y|^2)^{\frac{1}{2}}\right)(d(x)-d(y)),$$

and

$$\varphi(x,y)=\Psi_1(x,y)-\Psi_2(x,y),$$

where L is a fixed number greater than $\frac{1}{r}$ with r the radius in (10), K and M are positive constants to be chosen later and δ is a small parameter. We also use the notation

$$\widetilde{g}(Z,T)=\widetilde{g}\left(\frac{x+y}{2},(\delta^2+|x-y|^2)^{\frac{1}{2}}\right).$$

If $K|x|\leq\frac{1}{4}$, then

$$\Phi(x)\geq\frac{3}{4}MK|x|. \tag{11}$$

We define

$$\Delta_K:=\left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\,|x-y|\leq\frac{1}{4K}\right\}.$$

We fix $M>1$ and $j>0$ such that

$$\max_{\overline{B}_{\overline{\Omega}}(x_0,\rho)^{\,2}}|u(x)-u(y)|+2d_0(|g|_\infty+C_0\delta)\leq e^{-2Ld_0}\frac{M}{8}, \tag{12}$$

$$j=\frac{M}{\rho^2},$$

where $d_0=\max_{x\in\overline{\Omega}}d(x)$, and we claim that taking K large enough, for any small δ one has

$$u(x)-u(y)-\varphi(x,y)-je^{-Ld(x)}|x-x_0|^2\leq 0\quad\text{for } (x,y)\in\Delta_K\cap\overline{B}_{\overline{\Omega}}(x_0,\rho)^2. \tag{13}$$

To show (13) we suppose by contradiction that the maximum of $u(x)-u(y)-\varphi(x,y)-je^{-Ld(x)}|x-x_0|^2$ on $\Delta_K\cap\overline{B}_{\overline{\Omega}}(x_0,\rho)^2$ is positive. Then, for δ small enough, there is $(\overline{x},\overline{y})\in\Delta_K\cap\overline{B}_{\overline{\Omega}}(x_0,\rho)^2$ such that $\overline{x}\neq\overline{y}$ and

$$u(\overline{x})-u(\overline{y})-\widetilde{\varphi}(\overline{x},\overline{y})=\max_{\Delta_K\cap\overline{B}_{\overline{\Omega}}(x_0,\rho)^{\,2}}(u(x)-u(y)-\widetilde{\varphi}(x,y))>0, \tag{14}$$

where

$$\widetilde{\varphi}(x,y)=\varphi(x,y)+je^{-Ld(x)}|x-x_0|^2-C_0\delta(d(x)+d(y)),$$

with C_0 the constant defined as in Lemma 3.1.

The point $(\overline{x},\overline{y})$ belongs to $\operatorname{int}(\Delta_K)\cap B_{\overline{\Omega}}(x_0,\rho)^2$. Indeed, if $|x-y|=\frac{1}{4K}$, by (12) and (11), we have

$$\begin{aligned} u(x)-u(y) \leq& e^{-2Ld_0}\frac{M}{8}-2d_0(|g|_\infty+C_0\delta)\\ \leq& e^{-L(d(x)+d(y))}\frac{1}{2}MK|x-y|-\Psi_2(x,y)\\ &-C_0\delta(d(x)+d(y))\leq\widetilde{\varphi}(x,y). \end{aligned}$$

On the other hand, if $|x-x_0|=\rho$, then

$$\begin{aligned} u(x)-u(y) \leq& e^{-Ld_0}M-2d_0(|g|_\infty+C_0\delta)\\ \leq& e^{-Ld(x)}\frac{M}{\rho^2}|x-x_0|^2+\Psi_1(x,y)-\Psi_2(x,y)\\ &-C_0\delta(d(x)+d(y))=\widetilde{\varphi}(x,y). \end{aligned}$$

Similarly, if $|y-x_0|=\rho$ and $K>K_0/\rho$, for some constant $K_0>0$, then $u(x)-u(y)\leq\widetilde{\varphi}(x,y)$. Hence, $(\overline{x},\overline{y})\in\operatorname{int}(\Delta_K)\cap B_{\overline{\Omega}}(x_0,\rho)^2$.

Since $\overline{x}\neq\overline{y}$ we can compute the derivatives of $\widetilde{\varphi}$ at $(\overline{x},\overline{y})$ obtaining

$$\begin{aligned} D_x\widetilde{\varphi}(\overline{x},\overline{y})=&-Le^{-L(d(\overline{x})+d(\overline{y}))}MK|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)Dd(\overline{x})\\ &+e^{-L(d(\overline{x})+d(\overline{y}))}MK(1-2K|\overline{x}-\overline{y}|)\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}-C_0\delta Dd(\overline{x})\\ &-jLe^{-Ld(\overline{x})}|\overline{x}-x_0|^2Dd(\overline{x})+2je^{-Ld(\overline{x})}(\overline{x}-x_0)-D_x\Psi_2(\overline{x},\overline{y}),\\ D_y\widetilde{\varphi}(\overline{x},\overline{y})=&-Le^{-L(d(\overline{x})+d(\overline{y}))}MK|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)Dd(\overline{y})\\ &-e^{-L(d(\overline{x})+d(\overline{y}))}MK(1-2K|\overline{x}-\overline{y}|)\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}-C_0\delta Dd(\overline{y})\\ &-D_y\Psi_2(\overline{x},\overline{y}), \end{aligned}$$

where

$$\begin{aligned} D_x\Psi_2(\overline{x},\overline{y})=&\frac{d(\overline{x})-d(\overline{y})}{2}D_Z\widetilde{g}\left(\frac{\overline{x}+\overline{y}}{2},(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}\right)\\ &+(d(\overline{x})-d(\overline{y}))\frac{(\overline{x}-\overline{y})}{(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}}D_T\widetilde{g}\left(\frac{\overline{x}+\overline{y}}{2},(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}\right)\\ &+\widetilde{g}\left(\frac{\overline{x}+\overline{y}}{2},(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}\right)Dd(\overline{x}) \end{aligned}$$

and

$$\begin{aligned} D_y\Psi_2(\overline{x},\overline{y})=&\frac{d(\overline{x})-d(\overline{y})}{2}D_Z\widetilde{g}\left(\frac{\overline{x}+\overline{y}}{2},(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}\right)\\ &-(d(\overline{x})-d(\overline{y}))\frac{(\overline{x}-\overline{y})}{(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}}D_T\widetilde{g}\left(\frac{\overline{x}+\overline{y}}{2},(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}\right)\\ &-\widetilde{g}\left(\frac{\overline{x}+\overline{y}}{2},(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}\right)Dd(\overline{y}). \end{aligned}$$

Observe that

$$|D_x\widetilde{\varphi}(\overline{x},\overline{y})|,|D_y\widetilde{\varphi}(\overline{x},\overline{y})|\leq C(MK+C_0+j\rho). \tag{15}$$

Here and henceforth C denotes various positive constants independent of K,b,c,f,g and u.

By Lemma 3.1

$$\left|\widetilde{g}\left(\frac{\overline{x}+\overline{y}}{2},(\delta^2+|\overline{x}-\overline{y}|^2)^{\frac{1}{2}}\right)-g(\overline{x})\right|\le C_0(2|\overline{x}-\overline{y}|+\delta),$$

then, if $\overline{x}\in\partial\Omega$ we have

$$-\langle\overrightarrow{n}(\overline{x}),D_x\Psi_2(\overline{x},\overline{y})\rangle-g(\overline{x})\ge -C_0(4|\overline{x}-\overline{y}|+\delta).$$

Hence, using (10), if $\overline{x}\in\partial\Omega$ we get

$$\begin{aligned}\langle\overrightarrow{n}(\overline{x}),D_x\widetilde{\varphi}(\overline{x},\overline{y})\rangle-g(\overline{x})=&Le^{-Ld(\overline{y})}MK|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)\\&+e^{-Ld(\overline{y})}MK(1-2K|\overline{x}-\overline{y}|)\langle\overrightarrow{n}(\overline{x}),\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}\rangle\\&+jL|\overline{x}-x_0|^2+2j\langle\overrightarrow{n}(\overline{x}),\overline{x}-x_0\rangle\\&-\langle\overrightarrow{n}(\overline{x}),D_x\Psi_2(\overline{x},\overline{y})\rangle-g(\overline{x})+C_0\delta\\\ge&\frac{1}{2}e^{-Ld(\overline{y})}MK|\overline{x}-\overline{y}|\left(\frac{3}{2}L-\frac{1}{r}\right)\\&+j|\overline{x}-x_0|^2\left(L-\frac{1}{r}\right)-4C_0|\overline{x}-\overline{y}|>0,\end{aligned}\tag{16}$$

for $MK>\frac{16rC_0e^{Ld_0}}{(3rL-2)}$, since $\overline{x}\neq\overline{y}$ and $L>\frac{1}{r}$. Similarly, if $\overline{y}\in\partial\Omega$ then

$$\begin{aligned}\langle\overrightarrow{n}(\overline{y}),-D_y\widetilde{\varphi}(\overline{x},\overline{y})\rangle-g(\overline{y})\le&\frac{1}{2}e^{-Ld(\overline{x})}MK|\overline{x}-\overline{y}|\left(-\frac{3}{2}L+\frac{1}{r}\right)\\&+4C_0|\overline{x}-\overline{y}|<0.\end{aligned}$$

Then, by definition of sub and supersolution

$$F(\overline{x},u(\overline{x}),D_x\widetilde{\varphi}(\overline{x},\overline{y}),X)\ge f(\overline{x}),\quad \text{if } (D_x\widetilde{\varphi}(\overline{x},\overline{y}),X)\in\overline{J}^{2,+}u(\overline{x}),$$

$$F(\overline{y},u(\overline{y}),-D_y\widetilde{\varphi}(\overline{x},\overline{y}),Y)\le f(\overline{y})\quad \text{if } (-D_y\widetilde{\varphi}(\overline{x},\overline{y}),Y)\in\overline{J}^{2,-}u(\overline{y}).$$

Since $(\overline{x},\overline{y})\in\mathrm{int}\Delta_K\cap B_{\overline{\Omega}}(x_0,\rho)^2$, it is a local maximum point of $u(x)-u(y)-\widetilde{\varphi}(x,y)$ in $\overline{\Omega}^2$. Then applying Theorem 3.2 in [8], for every $\epsilon>0$ there exist $X,Y\in S(n)$ such that

$$(D_x\widetilde{\varphi}(\overline{x},\overline{y}),X-C_0\delta D^2d(\overline{x})+D^2(je^{-Ld(x)}|x-x_0|^2))\in\overline{J}^{2,+}u(\overline{x}),$$
$$(-D_y\widetilde{\varphi}(\overline{x},\overline{y}),Y+C_0\delta D^2d(\overline{y}))\in\overline{J}^{2,-}u(\overline{y})$$

and

$$\begin{aligned}\begin{pmatrix}X&0\\0&-Y\end{pmatrix}\le&D^2\varphi(\overline{x},\overline{y})+\epsilon(D^2\varphi(\overline{x},\overline{y}))^2\\\le&D^2\Psi_1(\overline{x},\overline{y})-D^2\Psi_2(\overline{x},\overline{y})\\&+2\varepsilon(D^2\Psi_1(\overline{x},\overline{y}))^2+2\varepsilon(D^2\Psi_2(\overline{x},\overline{y}))^2.\end{aligned}\tag{17}$$

Now we want to estimate the matrix on the right-hand side of the last inequality.

Using Lemma 3.1, it is easy to check that

$$-CC_0\begin{pmatrix}I&0\\0&I\end{pmatrix}\le D^2\Psi_2(\overline{x},\overline{y})\le CC_0\begin{pmatrix}I&0\\0&I\end{pmatrix}.\tag{18}$$

Next, let us estimate $D^2\Psi_1(\overline{x},\overline{y})$.

$$\begin{aligned}D^2\Psi_1(\overline{x},\overline{y})=&\Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))})+D(e^{-L(d(\overline{x})+d(\overline{y}))})\otimes D(\Phi(\overline{x}-\overline{y}))\\&+D(\Phi(\overline{x}-\overline{y}))\otimes D(e^{-L(d(\overline{x})+d(\overline{y}))})+e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})).\end{aligned}$$

We set

$$A_1 := \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_2 := D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_3 := e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})).$$

Observe that

$$A_1 \le CMK|\overline{x}-\overline{y}| \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{19}$$

For A_2 we have the following estimate

$$A_2 \le CMK \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + CMK \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \le CMK \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{20}$$

Indeed for ξ, $\eta \in \mathbb{R}^n$ we compute

$$\begin{aligned} \langle A_2(\xi,\eta),(\xi,\eta)\rangle &= 2Le^{-L(d(\overline{x})+d(\overline{y}))}\{\langle Dd(\overline{x}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\xi\rangle \\ &+ \langle Dd(\overline{y}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\eta\rangle\} \\ &\le CMK(|\xi|+|\eta|)|\eta-\xi| \\ &\le CMK(|\xi|^2+|\eta|^2) + CMK|\eta-\xi|^2. \end{aligned}$$

Now we consider A_3. The matrix $D^2(\Phi(\overline{x}-\overline{y}))$ has the form

$$D^2(\Phi(\overline{x}-\overline{y})) = \begin{pmatrix} D^2\Phi(\overline{x}-\overline{y}) & -D^2\Phi(\overline{x}-\overline{y}) \\ -D^2\Phi(\overline{x}-\overline{y}) & D^2\Phi(\overline{x}-\overline{y}) \end{pmatrix},$$

and the Hessian matrix of $\Phi(x)$ is

$$D^2\Phi(x) = \frac{MK}{|x|}\left(I - \frac{x \otimes x}{|x|^2}\right) - 2MK^2 I. \tag{21}$$

If we choose

$$\epsilon = \frac{|\overline{x}-\overline{y}|}{4MKe^{-L(d(\overline{x})+d(\overline{y}))}}, \tag{22}$$

then we have the following estimates

$$\begin{gathered} \epsilon A_1^2 \le CMK|\overline{x}-\overline{y}|^3 I_{2n}, \quad \epsilon A_2^2 \le CMK|\overline{x}-\overline{y}| I_{2n}, \\ \epsilon(A_1A_2+A_2A_1) \le CMK|\overline{x}-\overline{y}|^2 I_{2n}, \\ \epsilon(A_1A_3+A_3A_1) \le CMK|\overline{x}-\overline{y}| I_{2n}, \quad \epsilon(A_2A_3+A_3A_2) \le CMK I_{2n}, \end{gathered} \tag{23}$$

where $I_{2n} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then using (18), (19), (20), (23) and observing that

$$(D^2(\Phi(\overline{x}-\overline{y})))^2 = \begin{pmatrix} 2(D^2\Phi(\overline{x}-\overline{y}))^2 & -2(D^2\Phi(\overline{x}-\overline{y}))^2 \\ -2(D^2\Phi(\overline{x}-\overline{y}))^2 & 2(D^2\Phi(\overline{x}-\overline{y}))^2 \end{pmatrix},$$

from (17) we can conclude that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le (MO(K) + CC_0) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

where

$$B = e^{-L(d(\overline{x})+d(\overline{y}))}\left[D^2\Phi(\overline{x}-\overline{y}) + \frac{|\overline{x}-\overline{y}|}{MK}(D^2\Phi(\overline{x}-\overline{y}))^2\right]. \tag{24}$$

The last inequality can be rewritten as follows

$$\begin{pmatrix} \widetilde{X} & 0 \\ 0 & -\widetilde{Y} \end{pmatrix} \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with $\widetilde{X} = X - (MO(K) + CC_0)I$ and $\widetilde{Y} = Y + (MO(K) + CC_0)I$.

Now we want to get a good estimate for $\mathrm{tr}(\widetilde{X} - \widetilde{Y})$, as in [10]. For that aim let

$$0 \le P := \frac{(\overline{x} - \overline{y}) \otimes (\overline{x} - \overline{y})}{|\overline{x} - \overline{y}|^2} \le I.$$

Since $\widetilde{X} - \widetilde{Y} \le 0$ and $\widetilde{X} - \widetilde{Y} \le 4B$, we have

$$\mathrm{tr}(\widetilde{X} - \widetilde{Y}) \le \mathrm{tr}(P(\widetilde{X} - \widetilde{Y})) \le 4\mathrm{tr}(PB).$$

We have to compute $\mathrm{tr}(PB)$. From (21), observing that the matrix $(1/|x|^2)x \otimes x$ is idempotent, i.e., $[(1/|x|^2)x \otimes x]^2 = (1/|x|^2)x \otimes x$, we compute

$$(D^2\Phi(x))^2 = \frac{M^2K^2}{|x|^2}(1 - 4K|x|)\left(I - \frac{x \otimes x}{|x|^2}\right) + 4M^2K^4I.$$

Then, since $\mathrm{tr}P = 1$ and $4K|\overline{x} - \overline{y}| \le 1$, we have

$$\mathrm{tr}(PB) = e^{-L(d(\overline{x})+d(\overline{y}))}MK^2(-2 + 4K|\overline{x} - \overline{y}|) \le -e^{-L(d(\overline{x})+d(\overline{y}))}MK^2 < 0.$$

This gives

$$|\mathrm{tr}(\widetilde{X} - \widetilde{Y})| = -\mathrm{tr}(\widetilde{X} - \widetilde{Y}) \ge 4e^{-L(d(\overline{x})+d(\overline{y}))}MK^2 \ge CMK^2. \tag{25}$$

Since $\|B\| \le \frac{CMK}{|\overline{x} - \overline{y}|}$, we have

$$\|B\|^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X} - \widetilde{Y})|^{\frac{1}{2}} \le \left(\frac{CMK}{|\overline{x} - \overline{y}|}\right)^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X} - \widetilde{Y})|^{\frac{1}{2}} \le \frac{C}{K^{\frac{1}{2}}|\overline{x} - \overline{y}|^{\frac{1}{2}}}|\mathrm{tr}(\widetilde{X} - \widetilde{Y})|.$$

The Lemma III.I in [10] ensures the existence of a universal constant C depending only on n such that

$$\|\widetilde{X}\|, \|\widetilde{Y}\| \le C\{|\mathrm{tr}(\widetilde{X} - \widetilde{Y})| + \|B\|^{\frac{1}{2}}|\mathrm{tr}(\widetilde{X} - \widetilde{Y})|^{\frac{1}{2}}\}.$$

Thanks to the above estimates we can conclude that

$$\|\widetilde{X}\|,\ \|\widetilde{Y}\| \le C|\mathrm{tr}(\widetilde{X} - \widetilde{Y})|\left(1 + \frac{1}{K^{\frac{1}{2}}|\overline{x} - \overline{y}|^{\frac{1}{2}}}\right). \tag{26}$$

Now, using assumptions (F1) and (F2) concerning F, the definition of $\widetilde{X}$ and $\widetilde{Y}$ and the fact that u is sub and supersolution we compute

$$\begin{aligned}
f(\overline{x}) \le& F(\overline{x}, u(\overline{x}), D_x\widetilde{\varphi}(\overline{x}, \overline{y}), X - C_0\delta D^2 d(\overline{x}) + D^2(je^{-Ld(x)}|x - x_0|^2))\\
\le& F(\overline{x}, u(\overline{x}), D_x\widetilde{\varphi}(\overline{x}, \overline{y}), \widetilde{X}) + MO(K) + CC_0 + Cj\\
\le& F(\overline{x}, u(\overline{y}), -D_y\widetilde{\varphi}(\overline{x}, \overline{y}), \widetilde{Y}) + c|u(\overline{x}) - u(\overline{y})| + b|D_x\widetilde{\varphi}(\overline{x}, \overline{y}) + D_y\widetilde{\varphi}(\overline{x}, \overline{y})|\\
&+ a\mathrm{tr}(\widetilde{X} - \widetilde{Y}) + MO(K) + CC_0 + Cj\\
\le& F(\overline{y}, u(\overline{y}), -D_y\widetilde{\varphi}(\overline{x}, \overline{y}), \widetilde{Y}) + C_1|\overline{x} - \overline{y}|^{\frac{1}{2}}\|\widetilde{Y}\| + 2c|u(\overline{y})| + 2b|D_y\widetilde{\varphi}(\overline{x}, \overline{y})|\\
&+ c|u(\overline{x}) - u(\overline{y})| + b|D_x\widetilde{\varphi}(\overline{x}, \overline{y}) + D_y\widetilde{\varphi}(\overline{x}, \overline{y})|\\
&+ a\mathrm{tr}(\widetilde{X} - \widetilde{Y}) + MO(K) + CC_0 + Cj\\
\le& f(\overline{y}) + C_1|\overline{x} - \overline{y}|^{\frac{1}{2}}\|\widetilde{Y}\| + 2c|u(\overline{y})| + 2b|D_y\widetilde{\varphi}(\overline{x}, \overline{y})| + c|u(\overline{x}) - u(\overline{y})|\\
&+ b|D_x\widetilde{\varphi}(\overline{x}, \overline{y}) + D_y\widetilde{\varphi}(\overline{x}, \overline{y})| + a\mathrm{tr}(\widetilde{X} - \widetilde{Y}) + MO(K) + CC_0 + Cj.
\end{aligned}$$

From these inequalities, using (15), (26) and (25), for $K > \overline{K}$, where $\overline{K}$ is a constant depending only on a, A, C_1, n and Ω, we get

$$\begin{aligned}
&-2|f|_{L^\infty(\overline{\Omega})} - 4c|u|_{L^\infty(\overline{B}_{\overline{\Omega}}(x_0,\rho))} - C|g|_{C^{0,1}(\partial\Omega)} \\
\leq& Cb|D\widetilde{\varphi}|_\infty + MO(K) + Cj + C|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|(|\overline{x}-\overline{y}|^{\frac{1}{2}} + K^{-\frac{1}{2}}) + a\mathrm{tr}(\widetilde{X}-\widetilde{Y}) \\
\leq& CM\left(-K^2 + bK + \frac{b}{\rho} + \frac{1}{\rho^2}\right) + Cb|g|_{C^{0,1}(\partial\Omega)}.
\end{aligned} \tag{27}$$

Then, since we have chosen $M > 1$, for $K > \overline{K}$ we obtain

$$K^2 - bK \leq C\left(|f|_{L^\infty(\overline{\Omega})} + c|u|_{L^\infty(\overline{B}_{\overline{\Omega}}(x_0,\rho))} + (1+b)|g|_{C^{0,1}(\partial\Omega)} + \frac{b}{\rho} + \frac{1}{\rho^2}\right), \tag{28}$$

and this is a contradiction for K large enough. This implies that there exists K satisfying (28), such that (13) holds true. Next, choosing $x = x_0$, (13) gives

$$u(x_0) - u(y) \leq \varphi(x_0, y) \quad \forall y \in \overline{B}_{\overline{\Omega}}(x_0,\rho) \cap \overline{\Omega}.$$

Repeating the proof in $\overline{B}_{\overline{\Omega}}(x, 2\rho)$ for any $x \in \overline{B}_{\overline{\Omega}}(x_0,\rho)$, we finally find the u satisfies (6) and (7). □

Proof of Corollary 1. Let us define

$$v(x) := e^{\alpha d(x)} u(x).$$

Then, v is a solution of

$$\begin{cases} F(x, v, Dv, D^2 v) = e^{\alpha d(x)} f(x) & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \vec{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$F(x, r, p, X) = \mathcal{M}^+_{a,A}\{X - \alpha(Dd \otimes p + p \otimes Dd) + \alpha^2 r(Dd \otimes Dd) - \alpha r D^2 d\} + \lambda r.$$

It is easy to check that F satisfies assumptions (F1) and (F2) with $C_1 = 0$, and

$$c = C(\alpha^2 + \alpha + |\lambda|), \quad b = C\alpha,$$

where C depends on a, A, n and Ω. Then, by Proposition 3, the Lipschitz constant of v on Ω_ρ is bounded from above by $M_v K_v$, where $M_v \leq C(|e^{\alpha d(x)} u|_{L^\infty(\Omega_{3\rho})} + 1)$ and K_v satisfies (9). Hence, for any $x, y \in \Omega_\rho$, we have

$$\begin{aligned}
|u(x) - u(y)| \leq& |e^{-\alpha d(x)} - e^{-\alpha d(y)}||v(x)| + e^{-\alpha d(y)}|v(x) - v(y)| \\
\leq& (\alpha|e^{\alpha d(x)} u|_{L^\infty(\Omega_\rho)} + M_v K_v)|x - y|.
\end{aligned}$$

□

4. Properties of the principal eigenvalues.

Proposition 4 (Existence of principal eigenfunctions). *There exists $u^+_\alpha > 0$ and $u^-_\alpha < 0$ on $\overline{\Omega}$ respectively viscosity solution of*

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2 u^+_\alpha) + \lambda^+_\alpha u^+_\alpha = 0 & \text{in} \quad \Omega, \\ \frac{\partial u^+_\alpha}{\partial \vec{n}} = \alpha u^+_\alpha & \text{on} \quad \partial\Omega, \end{cases} \tag{29}$$

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2 u^-_\alpha) + \lambda^-_\alpha u^-_\alpha = 0 & \text{in} \quad \Omega, \\ \frac{\partial u^-_\alpha}{\partial \vec{n}} = \alpha u^-_\alpha & \text{on} \quad \partial\Omega. \end{cases} \tag{30}$$

Proof. We follow the arguments of [5]. To show the existence of positive eigenfunctions, the first step is to prove that if f is a continuous function such that $f \leq 0$, $f \not\equiv 0$, then for any $\lambda < \lambda_\alpha^+$ there exists a positive solution of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) + \lambda u = f(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \overrightarrow{n}} = \alpha u & \text{on} \quad \partial\Omega. \end{cases} \tag{31}$$

Observe that $v \equiv 1$ is a positive subsolution of (4) for $\lambda \geq 0$. This implies, by Proposition 1, that if $\lambda < \lambda_\alpha^+$ then $\lambda < 0$. Let $(v_n)_n$ be the sequence defined by $v_1 = 0$ and v_{n+1} be the solution of

$$\begin{cases} F(x, v_{n+1}, Dv_{n+1}, D^2v_{n+1}) - (c-\lambda)v_{n+1} = e^{\alpha d(x)}f(x) - cv_n & \text{in} \quad \Omega, \\ \frac{\partial v_{n+1}}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$F(x, r, p, X) = \mathcal{M}^+_{a,A}\{X - \alpha(Dd \otimes p + p \otimes Dd) + \alpha^2 r(Dd \otimes Dd) - \alpha r D^2 d\}$$

and $c = C(\alpha^2 + \alpha)$. By comparison, the sequence is positive and increasing. Let $(u_n)_n$ be the sequence defined by $u_n(x) := e^{-\alpha d(x)}v_n(x)$, then u_{n+1} is solution of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u_{n+1}) - (c-\lambda)u_{n+1} = f(x) - cu_n & \text{in} \quad \Omega, \\ \frac{\partial u_{n+1}}{\partial \overrightarrow{n}} = \alpha u_{n+1} & \text{on} \quad \partial\Omega. \end{cases}$$

We claim that $(u_n)_n$ is bounded. Suppose that it is not, then defining $w_n := \frac{u_n}{|u_n|_\infty}$ one gets that w_{n+1} is a solution of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2w_{n+1}) - (c-\lambda)w_{n+1} = \frac{f(x)}{|u_{n+1}|_\infty} - c\frac{|u_n|_\infty}{|u_{n+1}|_\infty}w_n & \text{in} \quad \Omega, \\ \frac{\partial w_{n+1}}{\partial \overrightarrow{n}} = \alpha w_{n+1} & \text{on} \quad \partial\Omega. \end{cases}$$

By Corollary 1, $(w_n)_n$ converges along a subsequence to a positive function w which satisfies

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2w) + \lambda w = c(1-k)w \geq 0 & \text{in} \quad \Omega, \\ \frac{\partial w}{\partial \overrightarrow{n}} = \alpha w & \text{on} \quad \partial\Omega, \end{cases}$$

where $k := \limsup_{n\to+\infty} \frac{|u_n|_\infty}{|u_{n+1}|_\infty} \leq 1$. This contradicts the Maximum Principle, Proposition 1. Then $(u_n)_n$ is bounded and letting n go to infinity, by the compactness result, the sequence converges uniformly to a function u which is a solution of (31). Moreover, u is positive by the Strong Comparison Principle, Theorem 2.2.

We are now in position to construct a sequence $(u_n)_n$ of positive solutions of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u_n) + \lambda_n u_n = -1 & \text{in} \quad \Omega, \\ \frac{\partial u_n}{\partial \overrightarrow{n}} = \alpha u_n & \text{on} \quad \partial\Omega, \end{cases}$$

where $(\lambda_n)_n$ is an increasing sequence which converges to λ_α^+. The sequence $(u_n)_n$ is unbounded, otherwise one would contradict the definition of λ_α^+ (see Theorem 8 of [5]). Then, up to subsequence, $|u_n|_\infty \to +\infty$ as $n \to +\infty$ and defining $\phi_n := \frac{u_n}{|u_n|_\infty}$ one gets that ϕ_n satisfies

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2\phi_n) + \lambda_n\phi_n = -\frac{1}{|u_n|_\infty} & \text{in} \quad \Omega, \\ \frac{\partial \phi_n}{\partial \overrightarrow{n}} = \alpha\phi_n & \text{on} \quad \partial\Omega. \end{cases}$$

By Corollary 1, an extracted subsequence converges to a function u_α^+ with $|u_\alpha^+|_\infty = 1$, which is a solution of (29). Moreover, by Theorem 2.2, $u_\alpha^+ > 0$ on $\overline{\Omega}$.

Similar arguments show the existence of negative solutions of (30). □

Proposition 5 (Simplicity of the first eigenvalues, [16] Proposition 7.1)**.** *Let $v \in C(\overline{\Omega})$ be a viscosity subsolution (resp. supersolution) of (29) (resp. (30)), then there exists $t \in \mathbb{R}$ such that $v \equiv t u_\alpha^+$ (resp. $v \equiv t u_\alpha^-$).*

Remark 1. Remark that

$$\lambda_\alpha^+ < -A\alpha^2, \tag{32}$$

$$\lambda_\alpha^- < -a\alpha^2. \tag{33}$$

Indeed, the function $v(x) := e^{\alpha x_1}$, where x_1 is the first coordinate of $x \in \mathbb{R}^n$, is a positive subsolution of

$$\begin{cases} \mathcal{M}_{a,A}^+(D^2 v) - A\alpha^2 v = 0 & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \vec{n}} = \alpha v & \text{on} \quad \partial\Omega. \end{cases} \tag{34}$$

Then the Maximum Principle, Proposition 1, implies that $\lambda_\alpha^+ \leq -A\alpha^2$. If $\lambda_\alpha^+ = -A\alpha^2$, then by Proposition 5, $v(x)$ is a solution of (29) and this implies that $\Omega = \mathbb{R}^n$. Hence (32) holds true. Similarly, inequality (33) is a consequence of the Minimum Principle, Proposition 2, of Proposition 5 and the fact that $-v(x)$ is a negative supersolution of (34) with A replaced by a.

Remark 2. Since $\lambda_\alpha^+, \lambda_\alpha^- < 0$ the operator $\mathcal{M}_{a,A}^+(D^2 u) + \lambda u$, with $\lambda = \lambda_\alpha^+$ or $\lambda = \lambda_\alpha^-$ satisfies the Dirichlet Comparison Principle.

Proposition 6. *The sequences $(\lambda_\alpha^+)_\alpha$ and $(\lambda_\alpha^-)_\alpha$ are decreasing.*

Proof. Let us prove that $(\lambda_\alpha^+)_\alpha$ is decreasing. Consider $0 < \alpha_1 < \alpha_2$ and let $u_{\alpha_1}^+$ be a solution of (29) with $\alpha = \alpha_1$. Then $u_{\alpha_1}^+$ is a positive subsolution of

$$\begin{cases} \mathcal{M}_{a,A}^+(D^2 u) + \lambda_{\alpha_1}^+ u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \vec{n}} = \alpha_2 u & \text{on} \quad \partial\Omega, \end{cases}$$

and the Maximum Principle, Proposition 1, implies $\lambda_{\alpha_1}^+ \geq \lambda_{\alpha_2}^+$. The strict inequality $\lambda_{\alpha_1}^+ > \lambda_{\alpha_2}^+$ follows from Proposition 5. □

Lemma 4.1. *Let u_α^+ and u_α^- be respectively a positive solution of (29) and a negative solution of (30), then*

$$u_\alpha^+(x) < \max_{\partial\Omega} u_\alpha^+ \quad \forall x \in \Omega,$$

$$u_\alpha^-(x) > \min_{\partial\Omega} u_\alpha^- \quad \forall x \in \Omega.$$

Proof. Let us show the result for u_α^+. Suppose by contradiction that the maximum of u_α^+ is attained at some point $x_0 \in \Omega$ and let $v(x) := u_\alpha^+(x) - u_\alpha^+(x_0)$. Since $u_\alpha^+(x_0) > 0$ and $\lambda_\alpha^+ < 0$, v satisfies

$$\mathcal{M}_{a,A}^+(D^2 v) + \lambda_\alpha^+ v \geq 0 \quad \text{in } \Omega$$

and $v \leq 0$ in Ω, $v(x_0) = 0$. Then the Strong Maximum Principle implies $u_\alpha^+ \equiv u_\alpha^+(x_0)$ in Ω and this contradicts the fact that u_α^+ solves (29). □

5. **Liouville type results.** For $\gamma > 0$ let us introduce the system

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) - \gamma u = 0 & \text{in } \mathbb{R}^n_+, \\ -\frac{\partial u}{\partial x_n} = u & \text{on } \partial\mathbb{R}^n. \end{cases} \tag{35}$$

Theorem 5.1. *If $\gamma > A$, any bounded subsolution of (35) is non-positive in $\mathbb{R}^n_+$.*
If $\gamma > a$, any bounded supersolution of (35) is non-negative in $\mathbb{R}^n_+$.
Hence, if $\gamma > A$ there are no, non trivial bounded solutions of (35).

Remark 3. It turns out that Theorem 5.1 is sharp: $u(x) = e^{-x_n}$ (resp., $u(x) = -e^{-x_n}$) is a positive bounded subsolution (resp., negative bounded supersolution) of (35) for every $\gamma \le A$ (resp., $\gamma \le a$).

Theorem 5.1 also fails without the boundedness condition. Indeed, $u(x) = e^{\nu\cdot x}$ (resp., $u(x) = -e^{\nu\cdot x}$), with $\nu = (\nu_1, ..., \nu_{n-1}, -1)$, $|\nu| > 1$, is an unbounded subsolution (resp., supersolution) of (35) for $A < \gamma \le A|\nu|^2$ (resp., $a < \gamma \le |\nu|^2 a$).

We assume that $u(x)$ is a bounded subsolution of (35) with $\gamma > 0$, which is positive somewhere. We normalize u so that

$$\sup_{\mathbb{R}^n_+} u = 1. \tag{36}$$

Then u is a viscosity subsolution of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) - \gamma u = 0 & \text{in } \mathbb{R}^n_+, \\ -\frac{\partial u}{\partial x_n} = 1 & \text{on } \partial\mathbb{R}^n. \end{cases} \tag{37}$$

Proposition 7. *Assume $\gamma > 0$ and $k \in \mathbb{R}$. Let $u \in USC(\mathbb{R}^n_+)$ and $v \in LSC(\mathbb{R}^n_+)$ be respectively bounded viscosity sub and supersolution of*

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) - \gamma u = 0 & \text{in } \mathbb{R}^n_+, \\ -\frac{\partial u}{\partial x_n} = k & \text{on } \partial\mathbb{R}^n. \end{cases} \tag{38}$$

Then $u \le v$ in $\mathbb{R}^n_+$.

Proof. Suppose by contradiction that $\sup_{\mathbb{R}^n_+}(u - v) = M > 0$. Let ψ be a smooth positive function with bounded derivatives and such that $\psi(x) \to +\infty$ as $|x| \to +\infty$. Let $\chi(x) = \chi(x_n)$ be a smooth function such that $\chi(x_n) = x_n$ for $|x_n| \le 1$ and $\chi(x_n) \equiv 0$ for $|x_n| > 2$. Let

$$\varphi(x, y) = \frac{j}{2}|x - y|^2 - k(x_n - y_n) + \beta\psi(x) - \varepsilon(\chi(x) + \chi(y)).$$

Then, for β and ε small enough and $j > 0$, the supremum of the function $u(x) - v(y) - \varphi(x, y)$ is greater than $\frac{M}{2}$ and it is reached at some point $(\overline{x}, \overline{y}) \in \overline{\mathbb{R}^n_+} \times \overline{\mathbb{R}^n_+}$.

If $\overline{x} \in \partial\mathbb{R}^n_+$ then, for $\overline{y} \in \mathbb{R}^n_+$,

$$-\partial_{x_n}\varphi(\overline{x}, \overline{y}) - k = -j(\overline{x}_n - \overline{y}_n) + k - \beta\partial_{x_n}\psi(\overline{x}) + \varepsilon - k = j\overline{y}_n - \beta\partial_{x_n}\psi(\overline{x}) + \varepsilon > 0$$

for $\varepsilon > \beta|D\psi|_\infty$.

If $\overline{y} \in \partial\mathbb{R}^n_+$ then, for $\overline{x} \in \mathbb{R}^n_+$

$$\partial_{y_n}\varphi(\overline{x}, \overline{y}) - k = -j(\overline{x}_n - \overline{y}_n) - \varepsilon = -j\overline{x}_n - \varepsilon < 0.$$

Both inequalities contradict the definition of sub and supersolution, therefore $\overline{x}, \overline{y} \in \mathbb{R}^n_+$.

Applying Theorem 3.2 of [8], there exist $X, Y \in S(n)$ such that $(D_x\varphi(\overline{x},\overline{y}), X + \beta D^2\psi(\overline{x}) - \varepsilon D^2\chi(\overline{x})) \in \overline{J}^{2,+}u(\overline{x})$, $(-D_y\varphi(\overline{x},\overline{y}), Y + \varepsilon D^2\chi(\overline{y})) \in \overline{J}^{2,-}v(\overline{y})$ and

$$-3j\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3j\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Since u and v are respectively sub and supersolution, we have

$$\mathcal{M}^+_{a,A}(X + \beta D^2\psi(\overline{x}) - \varepsilon D^2\chi(\overline{x})) \geq \gamma u(\overline{x}),$$

$$\mathcal{M}^+_{a,A}(Y + \varepsilon D^2\chi(\overline{y})) \leq \gamma v(\overline{y}).$$

Subtracting the two previous inequalities, using the properties of Pucci's operators and that

$$u(\overline{x}) - v(\overline{y}) > \frac{M}{2} + \frac{j}{2}|\overline{x}_n - \overline{y}_n|^2 - k(\overline{x}_n - \overline{y}_n) - \varepsilon(\chi(\overline{x}) + \chi(\overline{y})) \geq \frac{M}{2} - \frac{k^2}{2j} - C\varepsilon,$$

we finally get

$$\begin{aligned}\frac{\gamma}{2}\left(M - \frac{k^2}{j} - C\varepsilon\right) <&\gamma(u(\overline{x}) - v(\overline{y})) \leq \mathcal{M}^+_{a,A}(X + \beta D^2\psi(\overline{x}) - \varepsilon D^2\chi(\overline{x})) \\ &- \mathcal{M}^+_{a,A}(Y + \varepsilon D^2\chi(\overline{y})) \\ \leq& \mathcal{M}^+_{a,A}(X - Y) + \beta\mathcal{M}^+_{a,A}(D^2\psi(\overline{x})) - \varepsilon\mathcal{M}^-_{a,A}(D^2\chi(\overline{x}) + D^2\chi(\overline{y})) \\ \leq& \beta\mathcal{M}^+_{a,A}(D^2\psi(\overline{x})) - \varepsilon\mathcal{M}^-_{a,A}(D^2\chi(\overline{x}) + D^2\chi(\overline{y})).\end{aligned}$$

This is a contradiction for β and ε small enough and j large. Then $u \leq v$ in $\mathbb{R}^n_+$. □

Proof of Theorem 5.1. The function

$$v(x) = \sqrt{\frac{A}{\gamma}}e^{-\sqrt{\frac{\gamma}{A}}x_n}$$

is the bounded viscosity solution of (37). Then by Proposition 7

$$u(x) \leq \sqrt{\frac{A}{\gamma}}e^{-\sqrt{\frac{\gamma}{A}}x_n}, \quad \text{for any } x \in \mathbb{R}^n_+.$$

It follows from (36) that

$$1 = \sup_{\mathbb{R}^n_+} u \leq \sqrt{\frac{A}{\gamma}},$$

i.e. $\gamma \leq A$.

Similarly, if u is a supersolution of (35), normalized so that $\min_{\mathbb{R}^n_+} u = -1$, then u is a supersolution of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2u) - \gamma u = 0 & \text{in } \mathbb{R}^n_+, \\ -\frac{\partial u}{\partial x_n} = -1 & \text{on } \partial\mathbb{R}^n, \end{cases}$$

and by comparison

$$u(x) \geq -\sqrt{\frac{a}{\gamma}}e^{-\sqrt{\frac{\gamma}{a}}x_n}.$$

This implies $\gamma \leq a$ and Theorem 5.1 is proved. □

6. **Asymptotic behavior and Proof of Theorem 1.1.** We start by the following simple result:

Proposition 8. $\lim_{\alpha\to 0}\lambda_\alpha^\pm = 0.$

Proof. By Proposition 6, λ_α^+ increases to some value $\lambda_0 \le 0$. On the other hand, the sequence of normalized solutions $(u_\alpha^+)_\alpha$, by the Lipschitz estimates Corollary 1, converges to u_0 a positive solution of

$$\begin{cases} \mathcal{M}_{a,A}^+(D^2u) + \lambda_0 u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \overrightarrow{n}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

which satisfies $|u_0| = 1$. Recall that 0 is the principal eigenvalue for the Neumann problem. If $\lambda_0 < 0$, the Maximum Principle below the first eigenvalue, i.e. Proposition 1, implies that $u_0 \le 0$ a contradiction. □

We consider now the asymptotic behavior at infinity. By Remark 1, it is enough to show that

$$\limsup_{\alpha\to+\infty} \frac{\lambda_\alpha^+}{-\alpha^2} \le A, \tag{39}$$

and

$$\limsup_{\alpha\to+\infty} \frac{\lambda_\alpha^-}{-\alpha^2} \le a. \tag{40}$$

We are going to show (39). For $\alpha > 0$, let u_α^+ be a positive solution of (29). By Lemma 4.1, we know that u_α^+ attains its maximum at $x_\alpha \in \partial\Omega$. After normalization, we can assume that $\max_{\overline{\Omega}} u_\alpha^+ = 1$ and $x_\alpha \to 0$ as $\alpha \to +\infty$. Furthermore, we can assume that there is a C^2 function ϕ and $r > 0$ such that

$$\begin{aligned} & x_n = \phi(x'), \quad \forall (x', x_n) \in \partial\Omega \cap B_r(0) \\ & x_n > \phi(x'), \quad \forall (x', x_n) \in \Omega \cap B_r(0) \\ & \phi(0) = 0, \\ & \partial_{x_i}\phi(0) = 0, \quad \text{for } i = 1, ..., n-1. \end{aligned}$$

We flatten $\partial\Omega$ near the origin. Let $\Phi(x) : \Omega \cap B_r(0) \to \Omega_\Phi := \Phi(\Omega \cap B_r(0))$, be such that

$$\begin{aligned} & \Phi_i(x) = x_i, \quad i = 1, ..., n-1, \\ & \Phi_n(x) = x_n - \phi(x'). \end{aligned} \tag{41}$$

Denote by $x = \Psi(y)$ the inverse of $y = \Phi(x)$. The function

$$v_\alpha(y) = u_\alpha^+(\Psi(y))$$

is solution of

$$\begin{cases} \mathcal{M}_{a,A}^+ \Big[\Big(\sum_{l,k=1}^n \partial^2_{y_l y_k} v_\alpha \partial_{x_j}\Phi_k(\Psi(y))\partial_{x_i}\Phi_l(\Psi(y)) \Big)_{ij} & \\ \quad + \Big(\sum_{k=1}^n \partial_{y_k} v_\alpha \partial^2_{x_i x_j}\Phi_k(\Psi(y)) \Big)_{ij} \Big] + \lambda_\alpha^+ v_\alpha = 0 & y \in \Omega_\Phi, \\ \sum_{k,j=1}^n \partial_{y_k} v_\alpha \partial_{x_j}\Phi_k(\Psi(y)) \overrightarrow{n}_j(\Psi(y)) = \alpha v_\alpha & y \in \partial\Omega_\Phi. \end{cases} \tag{42}$$

Since the exterior normal $\overrightarrow{n}(x)$ at $x \in \partial\Omega \cap B_r(0)$ is

$$\overrightarrow{n}(x) = \frac{(D\phi(x'), -1)}{\sqrt{|D\phi(x')|^2 + 1}},$$

by (41), the boundary condition in (42) can be rewritten as follows

$$\frac{1}{\sqrt{|D\phi(y')|^2+1}} \sum_{k=1}^{n-1} \partial_{y_k} v_\alpha \partial_{x_k} \phi(y') - \left(\sqrt{|D\phi(y')|^2+1}\right) \partial_{y_n} v_\alpha = \alpha v_\alpha, \quad y \in \partial\Omega_\Phi.$$

Notice that, since $D\phi(x') \to 0$ as $x' \to 0$, $D\Phi(\Psi(y)) \to I$ as $y \to 0$, where I is the identity matrix of *S(n)*.

We now consider two different cases.
Case 1.

$$\limsup_{\alpha\to+\infty} \frac{\lambda_\alpha^+}{-\alpha^2} = \gamma < +\infty.$$

Without loss of generality, we may assume that $\frac{\lambda_\alpha^+}{-\alpha^2} \to \gamma$ as $\alpha \to +\infty$, and $u_\alpha^+(x_\alpha) = \max_{\overline{\Omega}} u_\alpha^+ = 1$, $x_\alpha \to 0$ as $\alpha \to +\infty$. We let

$$z = \alpha(y - y_\alpha),$$

where $y_\alpha = \Phi(x_\alpha) = (x'_\alpha, 0)$. We set

$$w_\alpha(z) = v_\alpha(y) = u_\alpha^+(x),$$

then for any $R > 0$, as α becomes sufficiently large, w_α is solution of

$$\begin{cases} \mathcal{M}_{a,A}^+ \Big[\left(\sum_{l,k=1}^n \partial^2_{z_l z_k} w_\alpha \partial_{x_j} \Phi_k(\Psi(y)) \partial_{x_i} \Phi_l(\Psi(y)) \right)_{ij} \\ \quad + \frac{1}{\alpha} \left(\sum_{k=1}^n \partial_{z_k} w_\alpha \partial^2_{x_i x_j} \Phi_k(\Psi(y)) \right)_{ij} \Big] + \frac{\lambda_\alpha^+}{\alpha^2} w_\alpha = 0 & z \in B_R^+, \\ \frac{1}{\sqrt{|D\phi(y')|^2+1}} \sum_{k=1}^{n-1} \partial_{z_k} w_\alpha \partial_{x_k} \phi(y') \\ \quad - \left(\sqrt{|D\phi(y')|^2 + 1} \right) \partial_{z_n} w_\alpha = w_\alpha & z \in \Gamma_R, \end{cases} \tag{43}$$

where

$$y = y(z) = \frac{z}{\alpha} + y_\alpha$$

and

$$B_R^+ := B_R(0) \cap \mathbb{R}^n_+, \quad \Gamma_R := B_R(0) \cap \partial\mathbb{R}^n_+.$$

Since for $z \in B_R^+$, $z/\alpha + y_\alpha \to 0$ as $\alpha \to +\infty$ and $\partial_{x_i}\phi(0) = 0$ for $i = 1, ..., n-1$, for α sufficiently large, $I/2 \le D\Psi(z/\alpha + y_\alpha) \le 2I$. Hence, if L_α is the Lipschitz constant of u_α^+ in the set $\{x = \Psi(z/\alpha + y_\alpha), |z| \le R\}$, we have

$$|w_\alpha(z_1) - w_\alpha(z_2)| = \left| u_\alpha^+ \left(\Psi\left(\frac{z_1}{\alpha} + y_\alpha \right) \right) - u_\alpha^+ \left(\Psi\left(\frac{z_2}{\alpha} + y_\alpha \right) \right) \right| \le \frac{2L_\alpha}{\alpha} |z_1 - z_2|.$$

Remark that if $|z| \le R$, then $d(x) \le CR/\alpha$ for $x = \Psi(z/\alpha + y_\alpha)$, where C depends on ϕ. Hence, since for $\rho = CR/\alpha$, $|e^{\alpha d(x)} u_\alpha^+|_{L^\infty(\Omega_{3\rho})} \le e^{3CR}$, Corollary 1 gives

$$L_\alpha \le Ce^{3CR}(\alpha + K_\alpha),$$

where K_α satisfies

$$K_\alpha^2 - C\alpha K_\alpha \le C\left[(\alpha + \alpha^2 + |\lambda_\alpha^+|) e^{3CR} + \frac{\alpha^2}{CR^2} + 1 \right].$$

This implies that the sequence $(w_\alpha)_\alpha$ is bounded in the space of Lipschitz continuous functions of $\overline{B}_R^+$ for any fixed $R>0$, and then, up to subsequence, $w_\alpha \to w_0$ uniformly on $\overline{B}_R^+$, with $\sup_{\mathbb{R}^n_+} w = 1$, viscosity solution of (35). Moreover by the Strong Comparison Principle, $w>0$ on $\overline{\mathbb{R}^n_+}$. Then, by Theorem 5.1, $\gamma \le A$ and this proves (39).

Case 2.

$$\limsup_{\alpha\to+\infty} \frac{\lambda_\alpha^+}{-\alpha^2} = +\infty.$$

Let u_α^+ be the sequence of positive solutions of (29) such that

$$\frac{\lambda_\alpha^+}{-\alpha^2} =: l_\alpha \to +\infty \quad \text{as } \alpha \to +\infty,$$

and $u_\alpha^+(x_\alpha) = \max_{\overline{\Omega}} u_\alpha^+ = 1$. Define

$$z = \sqrt{l_\alpha}\,\alpha(y-y_\alpha) \quad \text{and} \quad w_\alpha(z) = u_\alpha(x),$$

where $y=\Phi(x)$ and $y_\alpha = \Phi(x_\alpha)$. Then, for any $R>0$, as α becomes sufficiently large, w_α satisfies

$$\begin{cases} \mathcal{M}^+_{a,A}\Bigg[\left(\sum_{l,k=1}^n \partial^2_{z_l z_k} w_\alpha \partial_{x_j}\Phi_k(\Psi(y))\partial_{x_i}\Phi_l(\Psi(y))\right)_{ij} \\ \quad + \frac{1}{\sqrt{-\lambda_\alpha^+}}\left(\sum_{k=1}^n \partial_{z_k} w_\alpha \partial^2_{x_i x_j}\Phi_k(\Psi(y))\right)_{ij}\Bigg] - w_\alpha = 0 \quad z\in B_R^+, \\ \frac{1}{\sqrt{|D\phi(y')|^2+1}}\sum_{k=1}^{n-1}\partial_{z_k} w_\alpha \partial_{x_k}\phi(y') - \left(\sqrt{|D\phi(y')|^2+1}\right)\partial_{z_n} w_\alpha = \frac{1}{\sqrt{l_\alpha}} w_\alpha z \in \Gamma_R, \end{cases}$$

where $y = y(z) = \frac{z}{\alpha} + y_\alpha$. As in Case 1, we can show that $w_\alpha \to w_0$ which is a bounded positive viscosity solution of

$$\begin{cases} \mathcal{M}^+_{a,A}(D^2 w_0) - w_0 = 0 & \text{in } \mathbb{R}^n_+, \\ -\frac{\partial w_0}{\partial x_n} = 0 & \text{on } \partial\mathbb{R}^n. \end{cases} \tag{44}$$

On the other hand, by Proposition 7, the only bounded viscosity solution of (44) is $u \equiv 0$ and we reach a contradiction. This ends the proof of Theorem 1.1.

Proposition 9. *Let u_α^+ and u_α^- be respectively the normalized solution of* (29) *and* (30)*, i.e. $\|u_\alpha^\pm\|_\infty = 1$, then for any compact set $K \subset \Omega$*

$$\|u_\alpha^\pm\|_{L^\infty(K)} \to 0 \quad \textit{as } \alpha \to +\infty.$$

Proof. Let u_α^+ be the normalized solution of (29) and let K be a compact set contained in Ω. Let $x_\alpha \in K$ be such that $\max_K u_\alpha^+ = u_\alpha^+(x_\alpha)$. Define $z = \alpha(x - x_\alpha)$ and $w_\alpha(z) = u_\alpha^+(x)$ for $|z| < \alpha r$ where $r = $dist$(K, \partial\Omega)$. Then for any $R>0$, as α becomes large, $w_\alpha(z)$ satisfies

$$\mathcal{M}^+_{a,A}(D^2 w_\alpha) + \frac{\lambda_\alpha^+}{\alpha^2} w_\alpha = 0 \quad \text{in } B_{2R},$$

and $\|w_\alpha\|_\infty \le 1$. By standard elliptic estimates, see e.g. [7] and Theorem 1.1, $w_\alpha \to w_0$ non-negative solution of

$$\mathcal{M}^+_{a,A}(D^2 w_0) - A w_0 = 0 \quad \text{in } \mathbb{R}^n.$$

It is well-know that there are no nontrivial bounded solutions of the above equation, see e.g. [8], hence $w_\alpha(0) = \max_K u_\alpha^+ \to 0$ as $\alpha \to +\infty$ and Proposition 9 is proved. □

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E-mail address: isabeau@mat.uniroma1.it
E-mail address: patrizi@mat.uniroma1.it