CHANGE OF SCALES FOR CRYSTAL DISLOCATION DYNAMICS

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Abstract. We present some recent results obtained in [12, 13, 19, 26, 28, 29, 35, 36, 37, 38] for equations of evolutionary type, run by fractional and possibly anisotropic fractional operators. The models considered arise natural in crystallography, in which the solution of the equation has the physical meaning of the atom dislocation inside the crystal structure. Since different scales come into play in such description, different models have been adopted in order to deal with phenomena at atomic, microscopic, mesoscopic and macroscopic scale. We show that, looking at the asymptotic states of the solutions of equations modeling the dynamics of dislocations at a given scale, one can deduce the model for the motion of dislocations at a larger scale.

1. Introduction

Dislocations are line defects in crystals. Their typical length is of the order of $10^{-6}m$ and their thickness of order of $10^{-9}m$. When the material is submitted to shear stress, these lines can move in the crystallographic planes and their dynamics is one of the main explanation of the plastic behavior of metals. Dislocations can be described at several scales by different models:

- **(a)** atomic scale (Frenkel-Kontorova model),
- **(b)** microscopic scale (Peierls-Nabarro model),
- **(c)** mesoscopic scale (Discrete Dislocation Dynamics),
- **(d)** macroscopic scale (elasto-visco-plasticity with density of dislocations).

We refer the reader to the book of Hirth and Lothe [23] for a detailed introduction to dislocations. In physics and mechanics, it is a great challenge to try to predict macroscopic elasto-visco-plasticity properties of materials (like metals), based on microscopic properties (like dislocations). The classical Frenkel-Kontorova model describes a chain of classical particles evolving in a one dimensional space, coupled with their neighbors and subjected to a periodic potential, see the book of Braun and Kivshar [6] for a detailed presentation of the model. The Peierls-Nabarro model has been originally introduced as a variational (stationary) model (see [32, 23]), in which the microscopic effects are described by a partial differential equation involving a fractional and possibly anisotropic operator of order 1 of elliptic type. The asymptotics of the stationary model have been characterized in a number of mathematical papers within the framework of Γ-convergence, see, e.g., the works by Garroni, Leoni, Müller and collaborators [17, 18, 9, 8, 2] and references therein also for related models. In the face cubic structured (FCC) observed in many metals and alloys, dislocations move at low temperature on the slip plane.The dynamics of dislocations at microscopic scale is then described by the evolutive version of the Peierls-Nabarro model, see for instance [11, 30].

Several changes of scale exist in the literature. In dimension 1, the passage from the Frenkel-Kontorova model to Peierls-Nabarro model (*from (a) to (b)*) has been performed

in [14]. The evolutive Peierls-Nabarro equation in dimension 1 where the fractional operator is the half-Laplacian, has been considered by González and Monneau [19]. The equation models the case of parallel straight edge dislocation lines in the same slip plane. In this setting, after a suitable section of a three-dimensional crystal with a transverse plane, dislocation lines can be identified with points lying in the same line. We will refer to them as *particles* (though no "material" particle is really involved). In [19], looking at the sharp interface limit of the phase transitions of the Peierls-Nabarro model, the authors identify a dynamics of particles that corresponds to the classical discrete dislocation dynamics (*from (b) to (c)*). The results of [19] have been extended to the fractional Laplacian of order $2s \in (0,2)$ by Dipierro, Figalli, Palatucci and Valdinoci [13, 12]. The evolutive, generalized ($s \in (0,1)$) Peierls-Nabarro equation in dimension 1 has been considered again by the author and E. Valdinoci in [35]. Here, differently from the existing literature, dislocation particles are allowed to have different orientations. This produces a new phenomenon: *collision of particles*. Indeed particles with opposite orientations have the tendency to attract each other. The main difficulty here is that when a collision occurs the mesoscopic scale model becomes singular and we loose information about the dynamics of the particles after the collision. We have been able to overcome this difficulty in [37, 38], where we describe the dynamics of dislocations for times bigger than the collision time. We want to mention the papers [5, 1], for related results about the dynamics of dislocations for short times.

For $N \geqslant 2$, the large scale limit of a single phase transition described by Peierls-Nabarro shows that the line tension effect is the much stronger term. The limit model for $s \geqslant \frac{1}{2}$ appears to be the mean curvature motion as proven by Imbert and Souganidis [26]. For $s < \frac{1}{2}$, only partial but significant results have been obtained in [26], suggesting that in this case the front moves by fractional mean curvature. The results of [26] can be seen as the non-local counterpart of those obtained for the Allen-Cahn equation (see [10]). The double limit behavior of the Peierls-Nabarro model in dimensions greater than 1 has been also stressed in [39]. Here the authors show that the Peierls-Nabarro energy in a bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, Γ-converges to the classical minimal surface functional when $s \in \left[\frac{1}{2}, 1\right)$, while it Γ-converges to the non-local minimal surface functional when $s \in \left(0, \frac{1}{2}\right)$.

The passage from the Discrete Dislocation Dynamics model to the Dislocation Density model in dimension 1 has been performed in [15] (*from (c) to (d)*).

In [29] we have investigated the large scale limit of the evolutive Peierls-Nabarro model in any dimension N, in the case of a large number of phase transitions (i.e. of dislocations), recovering at the limit a model with evolution of dislocation densities. In other words, a direct passage from the microscopic scale (Peierls-Nabarro model) to the macroscopic scale (elasto-visco-plasticity with density of dislocations), has been performed (*from (b) to (d)*). From a mathematical point of view, this is an homogenization problem for an evolutive equation run by a fractional and possibly anisotropic operator of order 1 of elliptic type, usually called Lévy operator. The homogenized limit equation can be interpreted as the plastic flow rule in a model for macroscopic crystal plasticity. In [28] we have been able to explicitly characterize the macroscopic equation in the case of parallel straight edge dislocation lines in the same slip plane with the same Burgers' vectors, moving with self-interactions. This result recovers the so called Orowan's law. In the Physics literature this was proposed by Head in [22]. The results of [29, 28] have been extended to equations with anisotropic fractional operators of any order $2s \in (0,2)$ in [36]. The scaling of the system

and the results obtained are different according to the fractional parameter s. Namely, when $s > \frac{1}{2}$ the effective Hamiltonian "localizes" and it only depends on a first order differential operator. Conversely, when $s < \frac{1}{2}$, the non-local features are predominant and the effective Hamiltonian involves the fractional operator of order s.

The aim of the present paper is to describe and explain some recent results for problems involving non-linear and non-local partial differential equations with application to the theory of crystal dislocations. Due to their mathematical interest and in view of the concrete applications in physical models, these problems have been extensively studied in the recent literature. The results presented here, in particular, are contained in [12, 13, 19, 26, 28, 29, 35, 36, 37, 38]. Proofs are quite technical, so we have decided not to include them in the paper. We prefer instead to give heuristic explanations of the theorems we are going to state. We want to mention the paper [21] for a presentation of results prior to [12, 13, 28, 29, 35, 36, 37, 38], related to passages of scales in dimension 1, from the atomic one to the macroscopic one.

Organization of the paper. The paper is organized as follows. We first recall the classical Peierls-Nabarro model, in Section 2. In Section 3 we describe the passage from the Peierls-Nabarro model to the Discrete Dislocation Dynamics model. The results here presented are contained in [12, 13, 19, 35, 37, 38]. In Section 4 we review the homogenization of the Peierls-Nabarro model which represents the passage from the microscopic to the macroscopic scale, and it is studied in [28, 29, 36]. In Section 5, we give a heuristic explanation of some of the results proven in [26], which can be seen as the analogous of the results of Section 3 in dimension greater than 1. Finally, some open problems are presented in Section 6.

2. The Peierls-Nabarro model

The Peierls-Nabarro model is a phase field model for dislocation dynamics incorporating atomic features into continuum framework. In a phase field approach, the dislocations are represented by transition of a continuous field. We briefly review the model. There are two basic types of dislocations: the edge dislocation and the screw dislocation. In both cases the motion of a dislocation is a result of shear stress, but for a screw dislocation, the defect line movement is perpendicular to direction of the stress and the atom displacement, while for an edge dislocations it is parallel, see [23] for more details. As an example, consider an edge dislocation in a crystal with simple cubic lattice. In a Cartesian system of coordinates $x_1x_2x_3$, we assume that the dislocation is located in the slip plane x_1x_2 (where the dislocation can move) and that the Burgers' vector (i.e. a fixed vector associated to the dislocation) is in the direction of the x_1 axis. We write this Burgers' vector as be_1 for a real b. The disregistry of the upper half crystal $\{x_3 > 0\}$ relative to the lower half $\{x_3 < 0\}$ in the direction of the Burgers' vector is $u(x_1, x_2)$, where u is a phase parameter between 0 and b. Then the dislocation loop can be for instance localized by the level set $u = b/2$. For a closed loop, we expect to have $u \simeq b$ inside the loop and $u \simeq 0$ far outside the loop. In the Peierls-Nabarro model, the total energy is given by

$$\mathcal{E} = \mathcal{E}^{el} + \mathcal{E}^{mis}. \tag{2.1}$$

In (2.1), $\mathcal{E}^{el}$ is the elastic energy induced by the dislocation, and $\mathcal{E}^{mis}$ is the so called *misfit energy* due to the nonlinear atomic interaction across the slip plane,

$$\mathcal{E}^{mis}(u) = \int_{\mathbb{R}^2} W(u(x))\, dx \quad \text{with} \quad x = (x_1, x_2),$$

where $W(u)$ is the interplanar potential. In the classical Peierls-Nabarro model [34, 31], $W(u)$ is approximated by the sinusoidal potential

$$W(u) = \frac{\mu b^2}{4\pi^2 d}\left(1 - \cos\left(\frac{2\pi u}{b}\right)\right),$$

where d is the lattice spacing perpendicular to the slip plane.

The elastic energy $\mathcal{E}^{el}$ induced by the dislocation is (for $X = (x, x_3)$ with $x = (x_1, x_2)$)

$$\mathcal{E}^{el}(u, U) = \frac{1}{2}\int_{\mathbb{R}^3} e : \Lambda : e\, dX \quad \text{with} \quad e = e(U) - u(x)\delta_0(x_3)e^0,$$

and

$$\begin{cases} e(U) = \frac{1}{2}\left(\nabla U + (\nabla U)^T\right) \\ e^0 = \frac{1}{2}\left(e_1 \otimes e_3 + e_3 \otimes e_1\right) \end{cases}$$

where $U : \mathbb{R}^3 \to \mathbb{R}^3$ is the displacement and $\Lambda = \{\Lambda_{ijkl}\}$ are the elastic coefficients. Given the field u, we minimize the energy $\mathcal{E}^{el}(u, U)$ with respect to the displacement U and define

$$\mathcal{E}^{el}(u) = \inf_U \mathcal{E}^{el}(u, U).$$

Following the proof of Proposition 6.1 (iii) in [3], we can see that (at least formally)

$$\mathcal{E}^{el}(u) = -\frac{1}{2}\int_{\mathbb{R}^2} (c_0 \star u)u$$

where c_0 is a certain kernel. In the case of isotropic elasticity, we have

$$\Lambda_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

where λ, μ are the Lamé coefficients. Then the kernel c_0 can be written (see Proposition 6.2 in [3], translated in our framework):

$$c_0(x) = \frac{\mu}{4\pi}\left(\partial_{22}\frac{1}{|x|} + \omega\partial_{11}\frac{1}{|x|}\right) \quad \text{with} \quad \omega = \frac{1}{1-\nu} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda+\mu)}$$

where $\nu \in (-1, 1/2)$ is called the Poisson ratio.

The equilibrium configuration of the edge dislocation is obtained by minimizing the total energy with respect to u, under the constraint that far from the dislocation core, the function u tends to 0 in one half plane and to b in the other half plane. In particular, the phase transition u is then solution of the following equation

$$\mathcal{I}_s[u] = W'(u) \quad \text{on } \mathbb{R}^N, \tag{2.2}$$

where

$$\mathcal{I}_s[u] = \mathrm{PV}\, c_0 \star u = \mathrm{PV}\int_{\mathbb{R}^N} \frac{u(x+y) - u(x)}{|y|^{N+2s}} g\left(\frac{y}{|y|}\right) dy, \tag{2.3}$$

where PV stands for principal value, $s = \frac{1}{2}$, $N = 2$ and $g(z_1, z_2) = \frac{\mu}{4\pi}\left((2\omega - 1)z_1^2 + (2 - \omega)z_2^2\right)$. This operator is known as anisotropic (if $g \not\equiv$ constant) Lévy operator (of order 2s=1). If $\omega = 1$ and $\mu = 2$, then $\mathcal{I}_{\frac{1}{2}} = -(-\Delta)^{\frac{1}{2}}$ (isotropic case). In that special case, we recall that the solution u of (2.2) satisfies $u(x) = \tilde{u}(x, 0)$ where $\tilde{u}(X)$ is the solution of (see [27, 19])

$$\begin{cases} \Delta\tilde{u} = 0 & \text{in} \quad \{x_3 > 0\} \\ \dfrac{\partial\tilde{u}}{\partial x_3} = W'(\tilde{u}) & \text{on} \quad \{x_3 = 0\}. \end{cases}$$

Moreover, we have in particular an explicit solution for $b = 1$, $d = 2$ (with $W'(\tilde{u}) = \frac{1}{2\pi}\sin(2\pi\tilde{u})$)

$$\tilde{u}(X) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x_1}{x_3 + 1}\right). \tag{2.4}$$

In the original Peierls-Nabarro model, the dislocation line is assumed to be straight, say perpendicular to the x_1 axis. In this simplified case, the displacement is a function of only one variable and the phase field function satisfies (2.2) with $N = 1$. It is easy to check that we can recover the explicit solution found in Nabarro [31] by rescaling (2.4):

$$\begin{cases} u(x) = \dfrac{b}{2} + \dfrac{b}{\pi}\arctan\left(\dfrac{2(1-\nu)x_1}{d}\right) & \text{(edge dislocation)} \\ u(x) = \dfrac{b}{2} + \dfrac{b}{\pi}\arctan\left(\dfrac{2x_2}{d}\right) & \text{(screw dislocation).} \end{cases}$$

In a more general model, one can consider a potential W satisfying

$$\begin{aligned} &(i)\ \ W(v + b) = W(v) \quad \text{for all } v \in \mathbb{R}; \\ &(ii)\ \ W(b\mathbb{Z}) = 0 < W(a) \quad \text{for all } a \in \mathbb{R} \setminus b\mathbb{Z}. \end{aligned} \tag{2.5}$$

The periodicity of W reflects the periodicity of the crystal, while the minimum property is consistent with the fact that the perfect crystal is assumed to minimize the energy. We can assume wlog that $b = 1$. Then the 1-D phase transition is solution to:

$$\begin{cases} \mathcal{I}_s[u] = W'(u) & \text{in} \quad \mathbb{R} \\ u' > 0 & \text{in} \quad \mathbb{R} \\ \lim\limits_{x\to-\infty} u(x) = 0, \quad \lim\limits_{x\to+\infty} u(x) = b, \quad u(0) = \dfrac{1}{2}. \end{cases} \tag{2.6}$$

The existence of a unique solution of (2.6), when $\mathcal{I}_s = -(-\Delta)^s$ and under an additional non degeneracy assumption on the second derivative of the potential, has been proven independently by Palatucci, Savin and Valdinoci in [33] and by Cabré and Y. Sire in [7] for any $s \in (0, 1)$. Asymptotic estimates for u and u' are given in [33]. Finer estimates on u are shown in [13] and [12] respectively when $s \in \left[\frac{1}{2}, 1\right)$ and $s \in \left(0, \frac{1}{2}\right)$. We collect these results in the following lemma

Lemma 2.1. *Assume $g \equiv 1$ and*

$$\begin{cases} W \in C^{3,\alpha}(\mathbb{R}) & \text{for some } 0 < \alpha < 1 \\ W(v+1) = W(v) & \text{for any } v \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ W'' > 0 & \text{on } \mathbb{Z}. \end{cases} \tag{2.7}$$

Then there exists a unique solution $u \in C^{2,\alpha}(\mathbb{R})$ of (2.6). Moreover, there exists a constant $C > 0$ and $\kappa > 2s$ (only depending on s) such that

$$\left| u(x) - H(x) + \frac{1}{2sW''(0)} \frac{x}{|x|^{2s+1}} \right| \leqslant \frac{C}{|x|^{\kappa}}, \quad \text{for } |x| \geqslant 1, \tag{2.8}$$

and

$$|u'(x)| \leqslant \frac{C}{|x|^{1+2s}} \quad \text{for } |x| \geqslant 1, \tag{2.9}$$

where H is the Heaviside function.

According to [13], the constant κ in (2.8) can be chosen to be optimal equal to $1 + 2s$. Remark that when $g \equiv 1$, then $-\mathcal{I}_s = C_{N,s}^{-1}(-\Delta)^s$, for a suitable constant $C_{N,s}$ depending on N and s.

In the face cubic structured (FCC) observed in many metals and alloys, dislocations move at low temperature on the slip plane. A collection of dislocations curves all contained in a single slip plane x_1x_2, and moving in a landscape with periodic obstacles (that can be for instance precipitates in the material) are represented by a single phase parameter $u(t, x_1, x_2)$ defined on the plane x_1x_2. The dynamics of dislocations is then described by the evolutive version of the Peierls-Nabarro model (see for instance [30] and [11]):

$$\partial_t u = \mathcal{I}_s[u(t,\cdot)] - W'(u) + \sigma_{13}^{\text{obst}}(t,x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \tag{2.10}$$

with the physical dimension $N = 2$. In the model, the component $\sigma_{13}^{\text{obst}}$ of the stress (evaluated on the slip plane) has been introduced to take into account the shear stress not created by the dislocations themselves. This shear stress is created by the presence of the obstacles and the possible external applied stress on the material.

3. From the Peierls-Nabarro model to the Discrete Dislocation Dynamics model

The evolutive Peierls-Nabarro model (2.10) in dimension $N = 1$, describes at microscopic scale the dynamics of a collection of *parallel* and *straight* edge dislocations all lying in the same slip plane x_1x_2. Suppose that the dislocation lines are perpendicular to the x_1 axis, then, after a section of a three-dimensional crystal with the plane x_1x_3, they can be identified with points lying in the x_1 axis. We will refer to them as *particles*. We want to identify at a larger scale, the mesoscopic one, an evolution model for the Discrete Dislocation Dynamics. In the entire section we will assume that $g \equiv 1$, that is $\mathcal{I}_s$ is, up to a multiplicative constant, the operator $-(-\Delta)^s$. Assume in addition that the exterior stress in (2.10) has the following form $\sigma_{13}^{\text{obst}} = \varepsilon^{2s}\sigma(\varepsilon^{1+2s}t, \varepsilon x)$. We perform the following

rescaling of the solution u of (2.10)

$$v_\varepsilon(t,x) = u\left(\frac{t}{\varepsilon^{1+2s}}, \frac{x}{\varepsilon}\right),$$

where $\varepsilon > 0$ is a small parameter representing the ratio between microscopic scale and the mesoscopic scale. Then, the function $v_\varepsilon(t,x)$ solves

$$\begin{cases} (v_\varepsilon)_t = \dfrac{1}{\varepsilon}\left(\mathcal{I}_s[v_\varepsilon] - \dfrac{1}{\varepsilon^{2s}}W'(v_\varepsilon) + \sigma(t,x)\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ v_\varepsilon(0,\cdot) = v_\varepsilon^0 & \text{on } \mathbb{R}, \end{cases} \tag{3.1}$$

for a suitable initial condition to be chosen. Assume that the potential W satisfies (2.7). We suppose in addition that σ satisfies

$$\begin{cases} \sigma \in BUC([0,+\infty)\times\mathbb{R}) \quad \text{and for some } M>0 \text{ and } \alpha \in (s,1) \\ \|\sigma_x\|_{L^\infty([0,+\infty)\times\mathbb{R})} + \|\sigma_t\|_{L^\infty([0,+\infty)\times\mathbb{R})} \leqslant M \\ |\sigma_x(t,x+h) - \sigma_x(t,x)| \leqslant M|h|^\alpha, \quad \text{for every } x,h\in\mathbb{R} \text{ and } t\in[0,+\infty). \end{cases} \tag{3.2}$$

Given $x_1^0 < x_2^0 < ... < x_N^0$, we say that the function $u\left(\frac{x-x_i^0}{\varepsilon}\right)$, where u is the solution of (2.6), is a transition layer centered at x_i^0 and positively oriented. Similarly, we say that the function $u\left(\frac{x_i^0-x}{\varepsilon}\right)-1$ is a transition layer centered at x_i^0 and negatively oriented. Then the positively oriented transition layer connects the "rest states" 0 and 1, while the negatively oriented one connects 0 with -1. Remark that, since the equation in (2.6) is invariant by translation, the potential W is 1-periodic and $\mathcal{I}_s[u(-\cdot)](x) = \mathcal{I}_s[u(\cdot)](-x)$, we have that $u\left(\zeta\frac{x-x^0}{\varepsilon}\right) - k$ is solution of the same equation for any $x^0\in\mathbb{R}$, $k\in\mathbb{Z}$ and $\zeta\in\{-1,1\}$. The positively (resp., negatively) oriented transition layer identifies a dislocation particle located at the position x_i^0 with Burgers' vector e_1 (resp., $-e_1$). We consider as initial condition in (3.1) the state obtained by superposing N copies of the transition layer, centered at $x_1^0,...,x_N^0$, $N-K$ of them positively oriented and the remaining K negatively oriented, that is

$$v_\varepsilon^0(x) = \frac{\varepsilon^{2s}}{W''(0)}\sigma(0,x) + \sum_{i=1}^N u\left(\zeta_i\frac{x-x_i^0}{\varepsilon}\right) - K, \tag{3.3}$$

where $\zeta_1,...,\zeta_N \in \{-1,1\}$, $\displaystyle\sum_{i=1}^N(\zeta_i)^- = K$, $0\leqslant K\leqslant N$. Here, we denote by $(\zeta)^-$ the function defined by: $(\zeta)^- = 0$ if $\zeta\geqslant 0$ and $(\zeta)^- = \zeta$ if $\zeta<0$. The first term on the right-hand side of (3.3) takes into account the influence of the exterior stress σ. The initial condition (3.3) models an initial configuration in which there are N parallel and straight edge dislocation lines, all lying in the same slip plane, x_1x_2, $N-K$ of them with the same Burgers' vector e_1, the remaining K with Burgers' vector $-e_1$.

Let us introduce the solution $(x_i(t))_{i=1,...,N}$ to the system

$$\begin{cases} \dot{x}_i = \gamma\left(\displaystyle\sum_{j\neq i}\zeta_i\zeta_j\frac{x_i-x_j}{2s|x_i-x_j|^{1+2s}} - \zeta_i\sigma(t,x_i)\right) & \text{in } (0,T_c) \\ x_i(0) = x_i^0, \end{cases} \tag{3.4}$$

where $\gamma := \left(\int_{\mathbb{R}} (u'(x))^2 dx \right)^{-1}$, with u solution of (2.6). The physical properties of the singular potential of this ODE's system depend on the orientation of the dislocation function at the transition points. Namely, if the particles x_i and x_{i+1} have the same orientation, i.e., $\zeta_i \zeta_{i+1} = 1$, then the potential induces a repulsion between them. Conversely, when they have opposite orientation, i.e., $\zeta_i \zeta_{i+1} = -1$, then the potential becomes attractive, and the two particles may collide in a finite time T_c. Therefore, $(0, T_c)$ is the maximal interval where the system (3.4) is well defined. In formulas, in the collision case we have that $x_i(t) \neq x_{i+1}(t)$ for any $t \in [0, T_c)$ and any $i = 1, \dots, N$, with

$$\lim_{t \to T_c^-} x_{i_0}(t) = \lim_{t \to T_c^-} x_{i_0+1}(t), \tag{3.5}$$

for some $i_0 \in \{1, \dots, N\}$. Estimates of the collision time in the case of particles with alternating orientations and in the case in which two consecutive particles with opposite orientation are sufficiently close at the initial time, are given in [35].

We are now ready, to describe the asymptotic behavior of the dislocation function v_ε. For small ε, the solution v_ε of (3.1)-(3.3) approaches a piecewise constant function. The plateaus of this asymptotic limit correspond to the periodic sites induced by the crystalline structure, but its jump points evolve in time. Roughly speaking, one can imagine that these points behave like a particle system driven by the system of ordinary differential equations (3.4). System (3.4) *can be interpreted as a mesoscopic model for the Discrete Dislocation Dynamics.*

Let us state the result precisely. First, we recall that the (upper and lower) semicontinuous envelopes of a function v are defined as

$$v^*(t,x) := \limsup_{(t',x') \to (t,x)} v(t',x')$$

and

$$v_*(t,x) := \liminf_{(t',x') \to (t,x)} v(t',x').$$

Theorem 3.1 (Theorem 1.1, [35])**.** *Assume that* (2.7), (3.2) *and* (3.3) *hold, and let*

$$v_0(t,x) = \sum_{i=1}^{N} H(\zeta_i(x - x_i(t))) - K, \tag{3.6}$$

where H is the Heaviside function and $(x_i(t))_{i=1,\dots,N}$ is the solution to (3.4)*. Then, for every $\varepsilon > 0$ there exists a unique solution v_ε to* (3.1)*. Furthermore, as $\varepsilon \to 0^+$, the solution v_ε exhibits the following asymptotic behavior:*

$$\limsup_{\substack{(t',x') \to (t,x) \\ \varepsilon \to 0^+}} v_\varepsilon(t',x') \leqslant (v_0)^*(t,x) \tag{3.7}$$

and

$$\liminf_{\substack{(t',x') \to (t,x) \\ \varepsilon \to 0^+}} v_\varepsilon(t',x') \geqslant (v_0)_*(t,x), \tag{3.8}$$

for any $(t,x) \in [0, T_c) \times \mathbb{R}$.

Theorem 3.1 was already proven in [19, 13, 12] respectively in the case $s = \frac{1}{2}$, $s \in \left(\frac{1}{2}, 1\right)$ and $s \in \left(0, \frac{1}{2}\right)$ for a collection of dislocation lines all with the same orientation, i.e, $K = 0$. In this particular case the particles points have the tendency of repel each other, so collisions do not occur, i.e, $T_c = +\infty$.

Let us now give an heuristic explanation of the result of Theorem 3.1.

3.1. **Heuristics of the dynamics.** This subsection is contained in [35]. We think that it could be useful to understand the heuristic derivation of (3.4) in the simpler setting of two particles with different orientations (i.e. $N = 2$ and $K = 1$).

For this, let u be the solution of (2.6). Let us introduce the notation

$$u_{\varepsilon,1}(t,x) := u\left(\frac{x - x_1(t)}{\varepsilon}\right), \quad u_{\varepsilon,2}(t,x) := u\left(\frac{x_2(t) - x}{\varepsilon}\right) - 1,$$

and with a slight abuse of notation

$$u'_{\varepsilon,1}(t,x) := u'\left(\frac{x - x_1(t)}{\varepsilon}\right), \quad u'_{\varepsilon,2}(t,x) := u'\left(\frac{x_2(t) - x}{\varepsilon}\right).$$

Let us consider the following ansatz for v_ε

$$v_\varepsilon(t,x) \simeq u_{\varepsilon,1}(t,x) + u_{\varepsilon,2}(t,x) = u\left(\frac{x - x_1(t)}{\varepsilon}\right) + u\left(\frac{x_2(t) - x}{\varepsilon}\right) - 1.$$

Then, we compute

$$\begin{aligned}(v_\varepsilon)_t &= -u'\left(\frac{x - x_1(t)}{\varepsilon}\right)\frac{\dot{x}_1(t)}{\varepsilon} + u'\left(\frac{x_2(t) - x}{\varepsilon}\right)\frac{\dot{x}_2(t)}{\varepsilon} \\ &= -u'_{\varepsilon,1}(t,x)\frac{\dot{x}_1(t)}{\varepsilon} + u'_{\varepsilon,2}(t,x)\frac{\dot{x}_2(t)}{\varepsilon},\end{aligned}$$

and using the equation (2.6) and the periodicity of W

$$\begin{aligned}\mathcal{I}_s v_\varepsilon(t,x) &= \frac{1}{\varepsilon^{2s}}\mathcal{I}_s u\left(\frac{x - x_1(t)}{\varepsilon}\right) + \frac{1}{\varepsilon^{2s}}\mathcal{I}_s u\left(\frac{x_2(t) - x}{\varepsilon}\right) \\ &= \frac{1}{\varepsilon^{2s}}W'\left(u\left(\frac{x - x_1(t)}{\varepsilon}\right)\right) + \frac{1}{\varepsilon^{2s}}W'\left(u\left(\frac{x_2(t) - x}{\varepsilon}\right)\right) \\ &= \frac{1}{\varepsilon^{2s}}W'(u_{\varepsilon,1}(t,x)) + \frac{1}{\varepsilon^{2s}}W'(u_{\varepsilon,2}(t,x)).\end{aligned}$$

By inserting into (3.1), we obtain

$$-u'_{\varepsilon,1}\frac{\dot{x}_1}{\varepsilon} + u'_{\varepsilon,2}\frac{\dot{x}_2}{\varepsilon} = \frac{1}{\varepsilon^{2s+1}}\Big(W'(u_{\varepsilon,1}) + W'(u_{\varepsilon,2}) - W'(u_{\varepsilon,1} + u_{\varepsilon,2})\Big) + \frac{\sigma}{\varepsilon}. \tag{3.9}$$

Now we make some observations on the asymptotics of the potential W. First of all, we notice that the periodicity of W and the asymptotic behavior of u imply

$$\int_{\mathbb{R}} W'(u(x))u'(x)dx = \int_{\mathbb{R}} \frac{d}{dx}W(u(x))dx = W(1) - W(0) = 0, \tag{3.10}$$

and similarly

$$\int_{\mathbb{R}} W''(u(x))u'(x)dx = 0. \tag{3.11}$$

Next, we use estimate (2.8) and make a Taylor expansion of W' at 0 to compute for $x \neq x_2$

$$\begin{aligned} W'\left(u\left(\frac{x_2-x}{\varepsilon}\right)\right) &\simeq W'\left(H\left(\frac{x_2-x}{\varepsilon}\right) + \frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}}\right) \\ &= W'\left(\frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}}\right) \\ &\simeq W''(0)\frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}} \\ &= \frac{\varepsilon^{2s}(x-x_2)}{2s|x-x_2|^{1+2s}}. \end{aligned}$$

So, we use the substitution $y = (x-x_1)/\varepsilon$ to see that

$$\begin{aligned} \frac{1}{\varepsilon}\int_{\mathbb{R}} W'(u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx &\simeq \frac{1}{\varepsilon}\int_{\mathbb{R}} \frac{\varepsilon^{2s}(x-x_2)}{2s|x-x_2|^{1+2s}}u'\left(\frac{x-x_1}{\varepsilon}\right)dx \\ &= \int_{\mathbb{R}} \frac{\varepsilon^{2s}(\varepsilon y + x_1 - x_2)}{2s|\varepsilon y + x_1 - x_2|^{1+2s}}u'(y)dy \\ &\simeq \frac{\varepsilon^{2s}(x_1-x_2)}{2s|x_1-x_2|^{1+2s}}\int_{\mathbb{R}} u'(y)dy \\ &= \frac{\varepsilon^{2s}(x_1-x_2)}{2s|x_1-x_2|^{1+2s}}, \end{aligned}$$

if $x_1 \neq x_2$. Hence

$$\frac{1}{\varepsilon^{2s+1}}\int_{\mathbb{R}} W'(u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx \simeq \frac{x_1-x_2}{2s|x_1-x_2|^{1+2s}}, \tag{3.12}$$

if $x_1 \neq x_2$. We use again the substitution $y = (x - x_1)/\varepsilon$, (3.10) and (3.11) to get

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathbb{R}} W'(u_{\varepsilon,1}(t,x) + u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx \\
&\simeq \frac{1}{\varepsilon}\int_{\mathbb{R}} W'\left(u\left(\frac{x-x_1}{\varepsilon}\right) + H(x) + \frac{\varepsilon^{2s}(x-x_2)}{2sW''(0)|x-x_2|^{1+2s}}\right)u'\left(\frac{x-x_1}{\varepsilon}\right)dx \\
&= \int_{\mathbb{R}} W'\left(u(y) + \frac{\varepsilon^{2s}(\varepsilon y + x_1 - x_2)}{2sW''(0)|\varepsilon y + x_1 - x_2|^{1+2s}}\right)u'(y)dy \\
&\simeq \int_{\mathbb{R}} W'(u(y))u'(y)dy + \int_{\mathbb{R}} W''(u(y))\frac{\varepsilon^{2s}(\varepsilon y + x_1 - x_2)}{2sW''(0)|\varepsilon y + x_1 - x_2|^{1+2s}}u'(y)dy \\
&\simeq \frac{\varepsilon^{2s}(x_1 - x_2)}{2sW''(0)|x_1 - x_2|^{1+2s}}\int_{\mathbb{R}} W''(u(y))u'(y)dy \\
&= 0.
\end{aligned}$$

We deduce

$$\frac{1}{\varepsilon^{1+2s}}\int_{\mathbb{R}} W'(u_{\varepsilon,1}(t,x) + u_{\varepsilon,2}(t,x))u'_{\varepsilon,1}(t,x)dx \simeq 0. \tag{3.13}$$

Moreover, we have

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\mathbb{R}} \sigma(t,x)u'_{\varepsilon,1}(t,x)dx &= \int_{\mathbb{R}} \sigma(t,\varepsilon y + x_1)u'(y)dy \\
&\simeq \sigma(t,x_1)\int_{\mathbb{R}} u'(y)dy \\
&= \sigma(t,x_1).
\end{aligned} \tag{3.14}$$

Finally

$$\frac{1}{\varepsilon}\int_{\mathbb{R}} (u'_{\varepsilon,1}(t,x))^2 dx = \int_{\mathbb{R}} (u'(y))^2 dy = \gamma^{-1}, \tag{3.15}$$

and using (2.9)

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\mathbb{R}} u'_{\varepsilon,1}(t,x)u'_{\varepsilon,2}(t,x)dx &\simeq \frac{1}{\varepsilon}\int_{\mathbb{R}} u'\left(\frac{x-x_1}{\varepsilon}\right)\frac{\varepsilon^{1+2s}}{|x-x_2|^{1+2s}}dx \\
&= \int_{\mathbb{R}} u'(y)\frac{\varepsilon^{1+2s}}{|\varepsilon y + x_1 - x_2|^{1+2s}}dy \\
&\simeq \frac{\varepsilon^{1+2s}}{|x_1 - x_2|^{1+2s}}\int_{\mathbb{R}} u'(y)dy \\
&\simeq 0,
\end{aligned} \tag{3.16}$$

if $x_1 \neq x_2$. Now we multiply (3.9) by $u'_{\varepsilon,1}(t,x)$, we integrate on $\mathbb{R}$ and we use (3.10), (3.12), (3.13), (3.14), (3.15) and (3.16), to get

$$-\gamma^{-1}\dot{x}_1 = \frac{x_1 - x_2}{2s|x_1 - x_2|^{1+2s}} + \sigma(t, x_1).$$

A similar equation is obtained if we multiply (3.9) by $u'_{\varepsilon,2}(t, x)$ and integrate on $\mathbb{R}$. Therefore we get the system

$$\begin{cases} \dot{x}_1 = -\gamma \dfrac{x_1 - x_2}{2s|x_1 - x_2|^{1+2s}} - \gamma\sigma(t, x_1) \\ \dot{x}_2 = -\gamma \dfrac{x_2 - x_1}{2s|x_2 - x_1|^{1+2s}} + \gamma\sigma(t, x_2), \end{cases} \tag{3.17}$$

which is (3.4) with $N = 2$ and $K = 1$. This is a heuristic justification of the link between the partial differential equation in (3.1) and the system of ordinary differential equations in (3.4).

3.2. **Dislocation dynamics after the collision time.** System (3.4), describing at mesoscopic scale the dynamics of dislocation lines, becomes singular at the collision time $t = T_c$, and so no information about the dynamics of dislocations for times bigger than T_c can be inferred from it. To overcome this difficulty the idea consists in looking instead to the solution of the PDE (3.1) for small but fixed ε. Indeed such a function v_ε is well defined for any positive time. For simplicity here we present only the cases with two and three particles. We refer to [38] for the general case of N particles. Roughly speaking, in case of two particles, the collision of the two particles "annihilate" all the dynamics, nothing more is left and the system relaxes to the trivial equilibrium.

The case of three particles is, on the other hand, different from the case of two particles, since the steady state associated with the case of three particles is the heteroclinic orbit (and not the trivial function as in the case of two particles). In the case of three particles, one has that two particles "annihilate" each other, but the third particle "survives", and this produces a jump in the dislocation function – indeed, as explained in the previous sections, these "purely mathematical" particles correspond to an excursion of the dislocation, from two equilibria, which is modeled by the standard transition layer in (2.6). The precise results, which are proven in [37], are stated in the next two subsections.

3.3. **The case of two transition layers.** Given $x_1^0 < x_2^0$ let us consider as initial condition in (3.1)

$$v_\varepsilon^0(x) = \frac{\varepsilon^{2s}}{W''(0)}\sigma(0, x) + u\left(\frac{x - x_1^0}{\varepsilon}\right) + u\left(\frac{x_2^0 - x}{\varepsilon}\right) - 1, \tag{3.18}$$

where u is the solution of (2.6).

In general, it may happen that $T_c = +\infty$, i.e. no collision occurs. On the other hand, it can be shown that when either the external stress is small or the particles are initially close to collision, then $T_c < +\infty$. More precisely, in [36] we proved that if the following condition is satisfied

$$\text{either } \sigma \leqslant 0 \quad \text{or} \quad x_2^0 - x_1^0 < \left(\frac{1}{2s\|\sigma\|_\infty}\right)^{\frac{1}{2s}},$$

then the collision time T_c is finite.

In the setting of finite collision time, the dislocation function v_ε, after a time T_ε, which is only slightly larger than the collision time T_c, becomes small with ε. Indeed, the following two theorems are proven in [37].

Theorem 3.2 (Theorem 1.1, [37]). *Assume that* (2.7), (3.2) *hold and* $T_c < +\infty$. *Let* v_ε *be the solution of* (3.1)- (3.18). *Then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *there exist* T_ε, $\varrho_\varepsilon > 0$ *such that*

$$T_\varepsilon = T_c + o(1), \quad \varrho_\varepsilon = o(1) \quad \text{as } \varepsilon \to 0$$

and

$$v_\varepsilon(T_\varepsilon, x) \leqslant \varrho_\varepsilon \quad \text{for any } x \in \mathbb{R}. \tag{3.19}$$

The result above can be made precise by saying that, if the system is not subject to any external stress, then the dislocation function v_ε decays in time exponentially fast. More precisely, we have:

Theorem 3.3 (Theorem 1.2, [37]). *Assume that* (2.7), (3.2) *hold and that* $\sigma \equiv 0$. *Let* v_ε *be the solution of* (3.1)- (3.18). *Then there exist* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *we have*

$$|v_\varepsilon(t, x)| \leqslant \varrho_\varepsilon e^{c\frac{T_\varepsilon - t}{\varepsilon^{2s+1}}}, \quad \text{for any } x \in \mathbb{R} \text{ and } t \geqslant T_\varepsilon, \tag{3.20}$$

where T_ε *and* ϱ_ε *are given in Theorem 3.2.*

The evolution of the two particle system and of the associated dislocation function, as obtained in Theorems 3.2 and 3.3, is described in Figure 1.

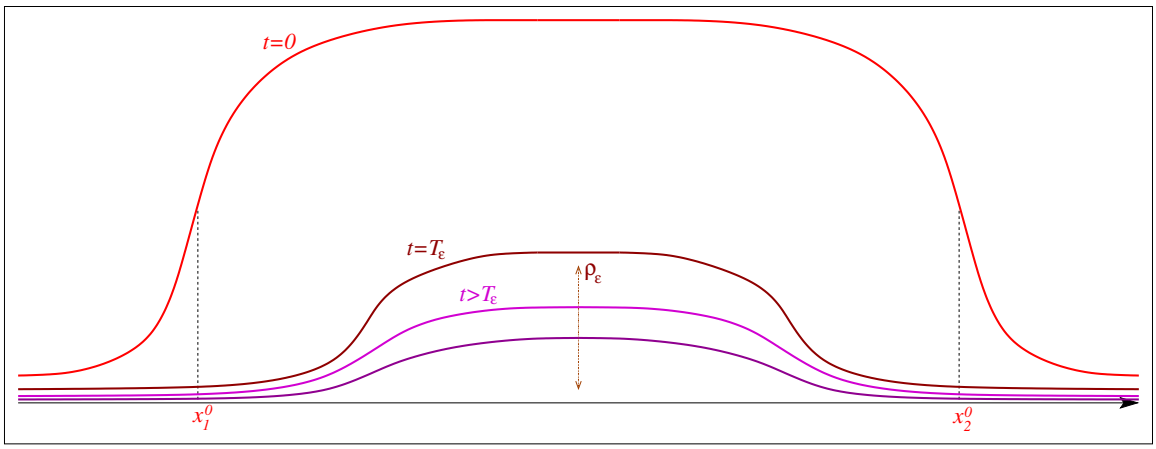

Figure 1: (Figure 1 in [37]) Evolution of the dislocation function in case of two particles.

3.4. **The case of three transition layers.** Next, we consider the case in which the initial condition in (3.1) is a superposition of three transition layers with different orientation. Precisely, let $\zeta_1 = 1$, $\zeta_2 = -1$, $\zeta_3 = 1$. Given $x_1^0 < x_2^0 < x_3^0$, let us consider as initial condition in (3.1)

$$v_\varepsilon^0(x) = \frac{\varepsilon^{2s}}{W''(0)}\sigma(0,x) + \sum_{i=1}^{3} u\left(\zeta_i \frac{x - x_i^0}{\varepsilon}\right) - 1, \tag{3.21}$$

where u is the solution of (2.6).

Theorem 3.4 (Theorem 1.3, [37]). *Assume that* (2.5), (3.2), *hold and* $T_c < +\infty$. *Let* v_ε *be the solution of* (3.1)-(3.21). *Then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *there exist* T_ε^1, T_ε^2, $\varrho_\varepsilon > 0$ *and* y_ε, z_ε *such that*

$$T_\varepsilon^1,\, T_\varepsilon^2 = T_c + o(1), \quad \varrho_\varepsilon = o(1) \quad \text{as } \varepsilon \to 0,$$
$$|z_\varepsilon - y_\varepsilon| = o(1) \quad \text{as } \varepsilon \to 0$$

and for any $x \in \mathbb{R}$

$$v_\varepsilon(T_\varepsilon^1, x) \leqslant u\left(\frac{x - y_\varepsilon}{\varepsilon}\right) + \varrho_\varepsilon \tag{3.22}$$

and

$$v_\varepsilon(T_\varepsilon^2, x) \geqslant u\left(\frac{x - z_\varepsilon}{\varepsilon}\right) - \varrho_\varepsilon, \tag{3.23}$$

where u *is the solution of* (2.6).

Next result is the analogue of Theorem 3.3 in the three particle setting. Roughly speaking, it says that, after a small transition time after the collision, the dislocation function relaxes towards the standard layer solution exponentially fast. The formal statement is the following:

Theorem 3.5 (Theorem 1.4, [37]). *Assume that* (2.5), (3.2), *hold and that* $\sigma \equiv 0$. *Let* v_ε *be the solution of* (3.1)-(3.21). *Then there exist* $\varepsilon_0 > 0$ *and* $\mu > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *there exists* $K_\varepsilon = o(1)$ *as* $\varepsilon \to 0$ *such that*

(3.24)

$$v_\varepsilon(t,x) \leqslant u\left(\frac{x - y_\varepsilon + K_\varepsilon\varrho_\varepsilon\left(1 - e^{-\frac{\mu(t-T_\varepsilon^1)}{\varepsilon^{2s+1}}}\right)}{\varepsilon}\right) + \varrho_\varepsilon e^{-\frac{\mu(t-T_\varepsilon^1)}{\varepsilon^{2s+1}}}, \quad \text{for any } x \in \mathbb{R} \text{ and } t \geqslant T_\varepsilon^1,$$

(3.25)

$$v_\varepsilon(t,x) \geqslant u\left(\frac{x - z_\varepsilon - K_\varepsilon\varrho_\varepsilon\left(1 - e^{-\frac{\mu(t-T_\varepsilon^2)}{\varepsilon^{2s+1}}}\right)}{\varepsilon}\right) - \varrho_\varepsilon e^{-\frac{\mu(t-T_\varepsilon^2)}{\varepsilon^{2s+1}}}, \quad \text{for any } x \in \mathbb{R} \text{ and } t \geqslant T_\varepsilon^2$$

where T_ε^1, T_ε^2, ϱ_ε, y_ε *and* z_ε *are given in Theorem 3.4 and* u *is the solution of* (2.6).

Corollary 3.6 (Corollary 1.5, [37]). *Under the assumptions of Theorem 3.5, there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$, *there exist a sequence* $t_k \to +\infty$ *as* $k \to +\infty$, *and a point* $x_\varepsilon \in \mathbb{R}$ *with*

$$y_\varepsilon - K_\varepsilon\varrho_\varepsilon < x_\varepsilon < z_\varepsilon + K_\varepsilon\varrho_\varepsilon, \tag{3.26}$$

such that

$$v_\varepsilon(t_k, x) \to u\left(\frac{x - x_\varepsilon}{\varepsilon}\right) \quad \text{as } k \to +\infty, \tag{3.27}$$

where y_ε, z_ε, K_ε *and* ϱ_ε *are given in Theorem 3.4 and* u *is the solution of* (2.6).

The results of Theorems 3.4 and 3.5 and Corollary 3.6 are represented in Figure 2, where we sketched the evolution of the dislocation function and of the associated particle system in the case of three particles with alternate orientations.

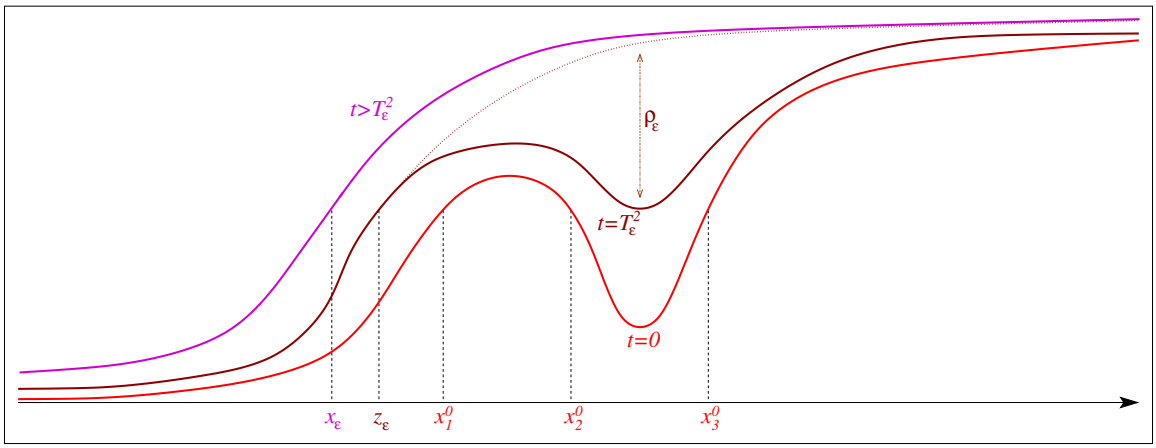

Figure 2: (Figure 2 in [37]) Evolution of the dislocation function in case of three particles.

It is worth to point out that the case of three particles provides structurally richer phenomena than the case of two particles. Indeed, in the case of three particles we have two different types of collision: simple and triple. The simple collision occurs when only two particles collide at time T_c, i.e., either

$$x_1(T_c) = x_2(T_c) \quad \text{and} \quad x_3(T_c) > x_2(T_c),$$

or

$$x_2(T_c) = x_3(T_c) \quad \text{and} \quad x_1(T_c) < x_2(T_c).$$

In the triple collision case, the three particles collide together and simultaneously, i.e.

$$x_1(T_c) = x_2(T_c) = x_3(T_c).$$

In [35], we proved that if $\sigma \equiv 0$, then for any choice of the initial condition (x_1^0, x_2^0, x_3^0) we have a collision in a finite time. Moreover a triple collision is possible if and only if

$$x_2^0 - x_1^0 = x_3^0 - x_2^0.$$

4. From the Peierls-Nabarro model to the Dislocation Density Model

Consider the evolutive Peierls-Nabarro model in any dimension (2.10), where $\mathcal{I}_s$ is the anisotropic Lévy operator of order 2s, defined in (2.3). Let us first consider the case $s = \frac{1}{2}$, which is studied in [29, 28].

We want to identify at *macroscopic scale* an evolution model for the dynamics of a density of dislocations. We consider the following rescaling

$$u^\varepsilon(t,x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

where ε is the ratio between the typical length scale for dislocation (of the order of the micrometer) and the typical macroscopic length scale in mechanics (millimeter or centimeter). Moreover, assuming suitable initial data

$$u(0,x) = \frac{1}{\varepsilon} u_0(\varepsilon x) \quad \text{on } \mathbb{R}^N, \tag{4.1}$$

(where u_0 is a regular bounded function), we see that the function u^ε is solution of

$$\begin{cases} \partial_t u^\varepsilon = \mathcal{I}_1[u^\varepsilon(t,\cdot)] - W'\left(\frac{u^\varepsilon}{\varepsilon}\right) + \sigma\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\varepsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{4.2}$$

This indicates that at the limit $\varepsilon \to 0$, we will recover a model for the dynamics of (renormalized) densities of dislocations. For $N = 2$, (2.10) with initial condition (4.1) models, at microscopic scale, the dynamics of a collections of edge dislocation lines moving in the same slip plane, with same Burgers' vectors, such that the number of dislocations is of the order of $1/\varepsilon$ per unit of macroscopic scale.

Here, we assume that the function g in (2.3) satisfies

(H1) $g \in C(\mathbf{S}^{N-1})$, $g > 0$, g even.

On the functions W, σ and u_0 we assume:

(H2) $W \in C^{1,1}(\mathbb{R})$ and $W(v+1) = W(v)$ for any $v \in \mathbb{R}$;
(H3) $\sigma \in C^{0,1}(\mathbb{R}^+ \times \mathbb{R}^N)$ and $\sigma(t+1,x) = \sigma(t,x)$, $\sigma(t,x+k) = \sigma(t,x)$ for any $k \in \mathbb{Z}^N$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$;
(H4) $u_0 \in W^{2,\infty}(\mathbb{R}^N)$.

Identifying the limit solution of the function u_ε, when $\varepsilon \to 0$, means solving an homogenization problem. In homogenization, both the limit function and the equation satisfied by it are unknown of the problem. In [29], we show that the limit u^0 of u^ε as $\varepsilon \to 0$ exists and is the unique solution of the homogenized problem

$$\begin{cases} \partial_t u = \overline{H}(\nabla_x u, \mathcal{I}_1[u(t,\cdot)]) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.3}$$

for some continuous function $\overline{H}$ usually called *effective Hamiltonian*. The function u^0 will be interpreted later as a macroscopic plastic strain satisfying the macroscopic plastic flow rule (4.3). Moreover $\mathcal{I}_s[u^0]$ will be the stress created by the macroscopic density of dislocations.

As usual in periodic homogenization, the limit equation is determined by a *cell problem*. In our case, such a problem is for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the following:

$$\begin{cases} \lambda + \partial_\tau v = \mathcal{I}_1[v(\tau,\cdot)] + L - W'(v + \lambda\tau + p\cdot y) + \sigma(\tau, y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ v(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.4}$$

where $\lambda = \lambda(p, L)$ is the unique number for which there exists a solution v of (4.4) which is bounded on $\mathbb{R}^+ \times \mathbb{R}^N$. In order to solve (4.4), we show for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$ the existence of a unique solution of

$$\begin{cases} \partial_\tau w = \mathcal{I}_1[w(\tau,\cdot)] + L - W'(w + p\cdot y) + \sigma(\tau, y) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ w(0,y) = 0 & \text{on} \quad \mathbb{R}^N, \end{cases} \tag{4.5}$$

and we look for some $\lambda \in \mathbb{R}$ for which $w - \lambda\tau$ is bounded. Precisely we have:

Theorem 4.1 (Theorem 1.1, [29])**.** *Assume (H1)-(H4). For $L \in \mathbb{R}$ and $p \in \mathbb{R}^N$, there exists a unique viscosity solution $w \in C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ of (4.5) and there exists a unique $\lambda \in \mathbb{R}$ such that w satisfies: $\frac{w(\tau,y)}{\tau}$ converges towards λ as $\tau \to +\infty$, locally uniformly in y. The real number λ is denoted by $\overline{H}(p, L)$. The function $\overline{H}(p, L)$ is continuous on $\mathbb{R}^N \times \mathbb{R}$ and non-decreasing in L.*

In Theorem 4.1, we denoted by $C_b(\mathbb{R}^+ \times \mathbb{R}^N)$ the set of continuous functions on $\mathbb{R}^+ \times \mathbb{R}^N$ which are bounded on $(0, T) \times \mathbb{R}^N$ for any $T > 0$. The non-local equation (4.2) is related to the local equation

$$\begin{cases} \partial_t u^\varepsilon = F\left(\frac{x}{\varepsilon}, \frac{u^\varepsilon}{\varepsilon}, \nabla u^\varepsilon\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\varepsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N, \end{cases}$$

that was studied in [24] under the assumption that $F(x, u, p)$ is periodic in (x, u) and coercive in p. As in the local case, the presence of the term $\frac{u^\varepsilon}{\varepsilon}$ in (4.2) does not allow to use directly the bounded solution of (4.4), usually called corrector. Indeed, a corrector in dimension $N+1$ needs to be introduced. Nevertheless, we have the following convergence result:

Theorem 4.2 (Theorem 1.2, [29])**.** *Assume (H1)-(H4). The solution u^ε of (4.2) converges towards the solution u^0 of (4.3) locally uniformly in (t, x), where $\overline{H}$ is defined in Theorem 4.1.*

4.1. **Viscosity solutions for non-local operators.** The classical notion of viscosity solution can be adapted for a quite general class of equations involving non-local operators, which includes equations (4.2) and (4.3), see for instance [4]. For equation (4.3), the property of the effective Hamiltonian $\overline{H}(p, L)$ to be non-decreasing with respect to L is a sort of ellipticity condition, which allows to define a well-posed notion of viscosity solution. The definition of viscosity solution for equations involving the Lévy operator $\mathcal{I}_s$, comes from this simple observation: for a smooth function ϕ, one has that, for any $r > 0$,

$$\begin{aligned} &\mathrm{PV} \int_{|z|\leqslant r} (\phi(x+z) - \phi(x)) \frac{1}{|z|^{N+2s}} g\left(\frac{z}{|z|}\right) dz \\ =&\mathrm{PV} \int_{|z|\leqslant r} (\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z) \frac{1}{|z|^{N+2s}} g\left(\frac{z}{|z|}\right) dz, \end{aligned}$$

as

$$\mathrm{PV}\int\limits_{|z|\leqslant r}\nabla(\phi(x)\cdot z)\frac{1}{|z|^{N+2s}}g\left(\frac{z}{|z|}\right)dz=\lim_{\delta\to 0^+}\int\limits_{\delta<|z|\leqslant r}\nabla(\phi(x)\cdot z)\frac{1}{|z|^{N+2s}}g\left(\frac{z}{|z|}\right)dz=0,$$

being the integrand an odd function. Now, if ϕ is sufficiently regular, then the following integrand is convergent,

$$\int\limits_{|z|\leqslant r}(\phi(x+z)-\phi(x)-\nabla\phi(x)\cdot z)\frac{1}{|z|^{N+2s}}g\left(\frac{z}{|z|}\right)dz.$$

On the other hand, if ϕ is bounded, the following integral is convergent too

$$\int\limits_{|z|>r}(\phi(x+z)-\phi(x))\frac{1}{|z|^{N+2s}}g\left(\frac{z}{|z|}\right)dz.$$

Taking into account this simple remark, a well-posed definition of viscosity solution for a general non-local equation with associated initial condition can be given. Consider

$$\begin{cases}u_t=F(t,x,u,Du,\mathcal{I}_s[u]) & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^N\\ u(0,x)=u_0(x) & \text{on}\quad \mathbb{R}^N,\end{cases}\tag{4.6}$$

where $F(t,x,u,p,L)$ is continuous and non-decreasing in L. Set

$$\mathcal{I}_s^{1,r}[\phi,x]:=\int\limits_{|z|\leqslant r}(\phi(x+z)-\phi(x)-\nabla\phi(x)\cdot z)\frac{1}{|z|^{N+2s}}g\left(\frac{z}{|z|}\right)dz,$$

$$\mathcal{I}_s^{2,r}[\phi,x]:=\int\limits_{|z|>r}(\phi(x+z)-\phi(x))\frac{1}{|z|^{N+2s}}g\left(\frac{z}{|z|}\right)dz.$$

Denote by $USC_b(\mathbb{R}^+\times\mathbb{R}^N)$ (resp., $LSC_b(\mathbb{R}^+\times\mathbb{R}^N)$) the set of upper (resp., lower) semi-continuous functions on $\mathbb{R}^+\times\mathbb{R}^N$ which are bounded on $(0,T)\times\mathbb{R}^N$ for any $T>0$.

Definition 4.1 (r-viscosity solution)**.** *A function $u\in USC_b(\mathbb{R}^+\times\mathbb{R}^N)$ (resp., $u\in LSC_b(\mathbb{R}^+\times\mathbb{R}^N)$) is a r-viscosity subsolution (resp., supersolution) of* (4.6) *if $u(0,x)\leqslant(u_0)^*(x)$ (resp., $u(0,x)\geqslant(u_0)_*(x)$) and for any $(t_0,x_0)\in\mathbb{R}^+\times\mathbb{R}^N$, any $\tau\in(0,t_0)$ and any test function $\phi\in C^2(\mathbb{R}^+\times\mathbb{R}^N)$ such that $u-\phi$ attains a local maximum (resp., minimum) at the point (t_0,x_0) on $Q_{(\tau,r)}(t_0,x_0)$, then we have*

$$\partial_t\phi(t_0,x_0)-F(t_0,x_0,u(t_0,x_0),\nabla_x\phi(t_0,x_0),\mathcal{I}_s^{1,r}[\phi(t_0,\cdot),x_0]+\mathcal{I}_s^{2,r}[u(t_0,\cdot),x_0])\leqslant 0$$
$$\text{(resp., }\geqslant 0).$$

A function $u\in C_b(\mathbb{R}^+\times\mathbb{R}^N)$ is a r-viscosity solution of (4.6) *if it is a r-viscosity sub and supersolution of* (4.6)*.*

It is classical that the maximum in the above definition can be supposed to be global. We have also the following property, see e.g., [4]:

Proposition 4.3 (Equivalence of the definitions)**.** *Assume $F(t,x,u,p,L)$ continuous and non-decreasing in L. Let $r>0$ and $r'>0$. A function $u\in USC_b(\mathbb{R}^+\times\mathbb{R}^N)$ (resp., $u\in LSC_b(\mathbb{R}^+\times\mathbb{R}^N)$) is a r-viscosity subsolution (resp., supersolution) of* (4.6) *if and only if it is a r′-viscosity subsolution (resp., supersolution) of* (4.6)*.*

The non-decreasing property of $F(t,x,u,p,L)$ with respect to L is crucial to prove comparison principles between viscosity sub and supersolutions. Comparison principles for viscosity sub and supersolutions of non-local equations including (4.2) are proven for instance in [20]. The comparison principle for (4.3) has been proven in [25]. Existence of viscosity solutions for non-local equations for which the comparison principle holds, follows by using the Perron's method, after providing a suitable sub and supersolution.

4.2. **Mechanical interpretation of the homogenization.** Let us briefly explain the meaning of the homogenization result. In the macroscopic model, the function $u^0(t,x)$ can be interpreted as the plastic strain (localized in the slip plane $\{x_3=0\}$). Then the three-dimensional displacement $U(t,X)$ is obtained as a minimizer of the elastic energy

$$U(t,\cdot) = \arg\min_{\tilde{U}} \mathcal{E}^{el}(u^0(t,\cdot),\tilde{U})$$

and the stress is

$$\sigma = \Lambda : e \quad \text{with} \quad e = e(U) - u^0(t,x)\delta_0(x_3)e^0.$$

Then the resolved shear stress is

$$\mathcal{I}_1[u^0] = \sigma_{13}^{\text{obst}}.$$

The homogenized equation (4.3), i.e.,

$$\partial_t u^0 = \overline{H}(\nabla_x u^0, \mathcal{I}_1[u^0(t,\cdot)])$$

which is the evolution equation for u^0, can be interpreted as the plastic flow rule in a model for macroscopic crystal plasticity. This is the law giving the plastic strain velocity $\partial_t u^0$ as a function of the resolved shear stress $\sigma_{13}^{\text{obst}}$ and the dislocation density ∇u^0.

The typical example of such a plastic flow rule is the Orowan's law:

$$\overline{H}(p,L) \simeq |p|L.$$

This is also the law that we recover in dimension $N=1$ in paper [29] in the case where there are no obstacles (i.e., $\sigma_{13}^{\text{obst}} \equiv 0$) and for small stress L and small density $|p|$.

4.3. **The Orowan's law.** The limit equation of an homogenization problem is defined through a cell problem, but its explicit expression is usually unknown. In [28] we are able to explicitly characterize the effettive Hamiltonian $\overline{H}(p,L)$ defined in Theorem 4.1 for small values of p and L, in the case $\mathcal{I}_1 = -(-\Delta)^{\frac{1}{2}}$, $N=1$ and $\sigma \equiv 0$. In this setting, equation (4.2) models the dynamics of parallel straight edge dislocation lines in the same slip plane with the same Burgers' vector, moving with self-interactions. In other words equation (4.2) simply describes the motion of dislocations by relaxation of the total energy (elastic + misfit). In [28] we study the behavior of $\overline{H}(p,L)$ for small p and L, and in this regime we recover Orowan's law, which claims that

$$\overline{H}(p,L) \simeq \gamma|p|L \tag{4.7}$$

for some constant of proportionality $\gamma > 0$. The precise result is stated in the following

Theorem 4.4 (Theorem 1.2, [28]). *Assume $g = \frac{1}{\pi}$, $N=1$, $\sigma \equiv 0$, $W \in C^{4,\alpha}$, for some $0<\alpha<1$, and (2.7). Let p_0, $L_0 \in \mathbb{R}$. Then the function $\overline{H}$ defined in Theorem 4.1 satisfies*

$$\frac{\overline{H}(\delta p_0, \delta L_0)}{\delta^2} \to \gamma|p_0|L_0 \quad \text{as } \delta \to 0^+ \quad \text{with} \quad \gamma = \left(\int_{\mathbb{R}} (u')^2\right)^{-1}, \tag{4.8}$$

where u is the solution of (2.6).

4.4. **Heuristic for the proof of Orowan's law.**
Define $u(\tau, y) := w(\tau, y) + py$, where w is the corrector solution of (4.5). Then, u satisfies

$$\begin{cases} \partial_\tau u = L + \mathcal{I}_1[u(\tau,\cdot)] - W'(u) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R} \\ u(0,y) = py & \text{on} \quad \mathbb{R}. \end{cases} \tag{4.9}$$

Moreover, by Theorem 4.1, we have that

$$u(\tau, y) \sim py + \lambda\tau + \text{bounded},$$

where $\lambda = \overline{H}(p, L)$. The idea underlying the proof of Orowan's law is related to a fine asymptotics of equation (4.9). From Theorem 3.1 we know that if u solves (4.9) with $L = \delta L_0$, i.e.

$$\partial_\tau u = \delta L_0 + \mathcal{I}_1[u(\tau,\cdot)] - W'(u) \tag{4.10}$$

for a choice of initial data with a finite number of indices i:

$$u(0,y) = \frac{\delta L_0}{W''(0)} + \sum_{x_i^0 \geqslant 0} \phi\left(y - \frac{x_i^0}{\delta}\right) + \sum_{x_i^0 < 0} \left(\phi\left(y - \frac{x_i^0}{\delta}\right) - 1\right)$$

then

$$u^\delta(t,x) := u\left(\frac{t}{\delta^2}, \frac{x}{\delta}\right) \to u^0(t,x) = \sum_{x_i^0 \geqslant 0} H(x - x_i(t)) + \sum_{x_i^0 < 0} (H(x - x_i(t)) - 1) \quad \text{as} \quad \delta \to 0$$

where H is the Heaviside function and with the dynamics

$$\begin{cases} \dfrac{dx_i}{dt} = \gamma\left(-L_0 + \dfrac{1}{\pi}\displaystyle\sum_{j \neq i} \frac{1}{x_i - x_j}\right) \\ \\ x_i(0) = x_i^0. \end{cases} \tag{4.11}$$

System (4.11) is the (rescaled as here we have $g = 1/\pi$ instead than $g = 1$) system (3.4), with $s = \frac{1}{2}$ and $\zeta_i = 1$ for any i. Moreover for the choice $p = \delta p_0$ with $p_0 > 0$ and $x_i^0 = i/p_0$ that we extend formally for all $i \in \mathbb{Z}$, we see (at least formally) that

$$|u(0,y) - \delta p_0 y| \leqslant C_\delta.$$

This suggests also that the infinite sum in (4.11) should vanish (by antisymmetry) and then the mean velocity should be

$$\frac{dx_i}{dt} \simeq -\gamma L_0$$

i.e., after scaling back

$$u(\tau, y) \simeq \delta p_0 (y - c_1 \tau) + \text{bounded}$$

with the velocity

$$c_1 = \frac{d(x_i/\delta)}{d(t/\delta^2)} \simeq -\gamma L_0 \delta$$

i.e.,

$$u(\tau, y) \simeq \delta p_0 y + \lambda\tau + \text{bounded} \quad \text{with} \quad \lambda \simeq \delta^2 \gamma p_0 L_0.$$

We deduce that we should have

$$\frac{u(\tau,y)}{\tau}\to\lambda\simeq\delta^2\gamma p_0L_0\quad\text{as}\quad\tau\to+\infty.$$

We see that this $\lambda=\overline{H}(\delta p_0,\delta L_0)$ is exactly the one we expect asymptotically in Theorem 4.4 when $p_0>0$.

4.5. **Homogenization and Orowan's law for anisotropic fractional operators of any order.** The results of [28, 29] have been generalized in [36] to Lévy operators of any order $2s$, with $s\in(0,1)$. In [36], for $s>\frac12$ we considered the following homogenization problem:

$$\begin{cases}\partial_t u^\varepsilon=\varepsilon^{2s-1}\mathcal{I}_s[u^\varepsilon(t,\cdot)]-W'\left(\frac{u^\varepsilon}{\varepsilon}\right)+\sigma\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) & \text{in}\quad\mathbb{R}^+\times\mathbb{R}^N\\ u^\varepsilon(0,x)=u_0(x) & \text{on}\quad\mathbb{R}^N,\end{cases}\tag{4.12}$$

and for $s<\frac12$:

$$\begin{cases}\partial_t u^\varepsilon=\mathcal{I}_s[u^\varepsilon(t,\cdot)]-W'\left(\frac{u^\varepsilon}{\varepsilon^{2s}}\right)+\sigma\left(\frac{t}{\varepsilon^{2s}},\frac{x}{\varepsilon}\right) & \text{in}\quad\mathbb{R}^+\times\mathbb{R}^N\\ u^\varepsilon(0,x)=u_0(x) & \text{on}\quad\mathbb{R}^N.\end{cases}\tag{4.13}$$

Remark that the scalings for $s>\frac12$ and $s<\frac12$ are different. They formally coincide when $s=\frac12$. We proved that the solution u^ε of (4.12) converges as $\varepsilon\to0$ to the solution u^0 of the homogenized problem

$$\begin{cases}\partial_t u=\overline{H}_1(\nabla_x u) & \text{in}\quad\mathbb{R}^+\times\mathbb{R}^N\\ u(0,x)=u_0(x) & \text{on}\quad\mathbb{R}^N,\end{cases}\tag{4.14}$$

with an effective Hamiltonian $\overline{H}_1$ which does not depend on $\mathcal{I}_s$ anymore, while the solution u^ε of (4.13) converges as $\varepsilon\to0$ to u^0 solution of the following

$$\begin{cases}\partial_t u=\overline{H}_2(\mathcal{I}_s[u]) & \text{in}\quad\mathbb{R}^+\times\mathbb{R}^N\\ u(0,x)=u_0(x) & \text{on}\quad\mathbb{R}^N,\end{cases}\tag{4.15}$$

with an effective Hamiltonian $\overline{H}_2$ not depending on the gradient. That is, roughly speaking, for any $s\in(0,1)$, the effective Hamiltonian is an operator of order $\min\{2s,1\}$, which reveals the stronger non-local effects present in the case $s<\frac12$. As before, the functions $\overline{H}_1$ and $\overline{H}_2$ are determined by the following cell problem:

$$\begin{cases}\partial_\tau w=\mathcal{I}_s[w(\tau,\cdot)]+L-W'(w+p\cdot y)+\sigma(\tau,y) & \text{in}\quad\mathbb{R}^+\times\mathbb{R}^N\\ w(0,y)=0 & \text{on}\quad\mathbb{R}^N,\end{cases}\tag{4.16}$$

and we look for some λ such that $w-\lambda\tau$ is bounded. As in the case $s=\frac12$, we proved the following ergodic result.

Theorem 4.5 (Theorem 1.1 [36])**.** *Assume (H1)-(H4). For $L\in\mathbb{R}$ and $p\in\mathbb{R}^N$, there exists a unique viscosity solution $w\in C_b(\mathbb{R}^+\times\mathbb{R}^N)$ of (4.5) and there exists a unique $\lambda\in\mathbb{R}$ such that w satisfies:*

$$\frac{w(\tau,y)}{\tau}\ \textit{converges towards}\ \lambda\ \textit{as}\ \tau\to+\infty,\ \textit{locally uniformly in}\ y.$$

The real number λ is denoted by $\overline{H}(p,L)$. The function $\overline{H}(p,L)$ is continuous on $\mathbb{R}^N\times\mathbb{R}$ and non-decreasing in L.

Once the cell problem was solved, we could prove the following convergence results:

Theorem 4.6 (Theorem 1.2 [36])**.** *Assume (H1)-(H4). The solution u^ε of (4.12) converges towards the solution u^0 of (4.14) locally uniformly in (t,x), where*

$$\overline{H}_1(p) := \overline{H}(p,0)$$

and $\overline{H}(p,L)$ is defined in Theorem 4.5.

Theorem 4.7 (Theorem 1.3 [36])**.** *Assume (H1)-(H4). The solution u^ε of (4.13) converges towards the solution u^0 of (4.15) locally uniformly in (t,x), where*

$$\overline{H}_2(L) := \overline{H}(0,L)$$

and $\overline{H}(p,L)$ is defined in Theorem 4.5.

We point out that the effective Hamiltonians $\overline{H}_1$ and $\overline{H}_2$ represent the speed of propagation of the dislocation dynamics according to (4.14) and (4.15). In particular, due to Theorems 4.6 and 4.7, such speed only depends on the slope of the dislocation in the weakly non-local setting $s>\frac12$ and only on an operator of order s of the dislocation in the strongly non-local setting $s<\frac12$.

Finally, when $N=1$, $\mathcal{I}_s=-(-\Delta)^s$ and $\sigma\equiv 0$, and we make the further assumptions (2.7) on the potential W, we proved the following extension of the Orowan's law:

Theorem 4.8 (Theorem 1.4, [36])**.** *Assume $\mathcal{I}_s=-(-\Delta)^s$, $N=1$, $\sigma\equiv 0$, $W\in C^{4,\alpha}$, for some $0<\alpha<1$, (2.7), and W even when $s\in\left(0,\frac12\right)$. Let p_0, $L_0\in\mathbb{R}$ with $p_0\neq 0$. Then the function $\overline{H}$ defined in Theorem 4.5 satisfies*

$$\frac{\overline{H}(\delta p_0,\delta^{2s}L_0)}{\delta^{1+2s}}\to\gamma|p_0|L_0\quad \text{as } \delta\to 0^+\quad \text{with}\quad \gamma=\left(\int_{\mathbb{R}}(u')^2\right)^{-1}, \tag{4.17}$$

where u is solution of (2.6).

5. Non-local Allen-Cahn equation

Imbert and Souganidis [26] have considered the following rescaled in time and space version of the evolutive Peierl-Nabarro model in dimension $N\geqslant 2$: for $t>0$ and $x\in\mathbb{R}^N$,

$$\partial_t u+\frac{1}{\varepsilon\eta_\varepsilon}\left\{-\varepsilon^{2s}\mathcal{I}_s[u^\varepsilon]+W'(u^\varepsilon)\right\}=0 \tag{5.1}$$

where $\mathcal{I}_s$ is the Lévy operator of order $2s\in(0,2)$, introduced in (2.3), W' is a bistable nonlinearity and the parameter η_ε depends on s and it is defined as follows:

$$\eta_\varepsilon=\begin{cases}\varepsilon & \text{if } s>\frac12,\\ \varepsilon|\log(\varepsilon)| & \text{if } s=\frac12,\\ \varepsilon^{2s} & \text{if } s<\frac12.\end{cases} \tag{5.2}$$

For our purposes, we assume that W satisfies (2.7). Let q be the phase transition function, solution to: for $e \in \mathbb{S}^{N-1}$,

$$\begin{cases} \mathcal{I}_s^e[q] = W'(q), & \text{in } \mathbb{R} \\ q' > 0 \\ \lim_{\xi\to-\infty} q(\xi) = 0, \quad \lim_{\xi\to+\infty} q(\xi) = 1, \end{cases} \tag{5.3}$$

where

$$\mathcal{I}_s^e[q](\xi) := \mathrm{PV} \int_{\mathbb{R}^N} (q(\xi + e\cdot z) - q(\xi))J(z)dz$$

and

$$J(z) := g\left(\frac{z}{|z|}\right)\frac{1}{|z|^{N+2s}}. \tag{5.4}$$

When $g \equiv c_{N,s}$, then $\mathcal{I}_s^e$ is actually independent of e and a solution is providing by $q(\xi) = u(\xi)$, where u is solution of (2.6). In [26] it is proven that when $s \geqslant \frac{1}{2}$, the solution u_ε of the diffusion-reaction equation (4.2) with initial datum

$$u_\varepsilon^0(x) = q\left(\frac{d_0(x)}{\varepsilon}, Dd_0(x)\right), \tag{5.5}$$

where d_0 is the signed distance function to the boundary of a smooth set Ω_0, can only have, as $\varepsilon \to 0$, two limits: the stable equilibria of the bistable non-linearity W' in $[0,1]$, i.e., 0 and 1. The resulting interface, $\partial\Omega_t$, evolves by anisotropic mean curvature. For $s < \frac{1}{2}$ only partial but significant results have been obtained. In the theory of crystal dislocations when $N = 2$, $\partial\Omega_t$ represents the dislocation line at time t moving on the slip plane x_1x_2. In the one dimensional space, moving interfaces are points. Their dynamics is then described by the system of ODE's (3.4). In this section, we want to give a heuristic proof of the results contained in [26]. Assume for simplicity $g \equiv c_{N,s}$, then the phase transition q is independent of the direction e. In this setting we have the following Ansatz for u_ε:

$$u_\varepsilon(t,x) \sim q\left(\frac{d(t,x)}{\varepsilon}\right)$$

where $d(t,x)$ is the signed distance function from the front that propagates starting from the initial configuration Ω_0. Close to the front, the distance function d is smooth in x and $|Dd| = 1$, which implies in particular that $D^2dDd = 0$. Inserting the derivatives of the Ansatz into the equation (5.1), multiplying by ε and using equation (5.3), we get (close to the front):

$$\begin{aligned} \dot{q}\left(\frac{d(t,x)}{\varepsilon}\right)\partial_t d(t,x) &= \frac{1}{\eta_\varepsilon}\left\{\varepsilon^{2s}\mathcal{I}_s\left[q\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - W'\left(q\left(\frac{d(t,x)}{\varepsilon}\right)\right)\right\} \\ &= \frac{1}{\eta_\varepsilon}\left\{\mathcal{I}_s\left[q\left(\frac{d(t,\varepsilon\cdot)}{\varepsilon}\right)\right]\left(\frac{x}{\varepsilon}\right) - \mathcal{I}_s^e[q]\left(\frac{d(t,x)}{\varepsilon}\right)\right\}. \end{aligned} \tag{5.6}$$

Let us introduce the notation $\xi = \frac{d(t,x)}{\varepsilon}$, $y = \frac{x}{\varepsilon}$ and $e = Dd(t,x)$. Then, we can write the right-hand side of the previous equation as follows

$$\mathcal{I}_s\left[q\left(\frac{d(t,\varepsilon\cdot)}{\varepsilon}\right)\right](y) - \mathcal{I}_s^e[q](\xi) = \mathrm{PV}\int_{\mathbb{R}^N}\left[q\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right) - q(\xi + e\cdot z)\right]J(z)dz$$
$$= \mathrm{PV}\int_{\mathbb{R}^N}\left[q(\xi + e\cdot z + \varepsilon W_\varepsilon(t,x,z)) - q(\xi + e\cdot z)\right]J(z)dz,$$

where $W_\varepsilon(t,x,z) = \frac{1}{\varepsilon^2}[d(t,x+\varepsilon z) - d(t,x) - \varepsilon Dd(t,x)\cdot z]$. Notice that W_ε is bounded in ε if d is $C^{1,1}$ with respect to the space variable, in a neighborhood of x. Now, if the front is smooth, for (t,x) close to the front, we can assume that the slow variables (t,x) and the fast variable ξ are independent. Therefore, multiplying equation (5.6) by $\dot{q}(\xi)$ and integrating in ξ, we get

$$\gamma^{-1}\partial_t d(t,x) - \frac{1}{\eta_\varepsilon}\overline{a}(t,x,e) = 0, \tag{5.7}$$

where

$$\gamma^{-1} := \int_{\mathbb{R}}(\dot{q})^2(\xi)d\xi, \tag{5.8}$$

$$\overline{a}_\varepsilon(t,x,e) = \int_{\mathbb{R}}\dot{q}(\xi)a_\varepsilon(t,x,\xi,e)d\xi, \tag{5.9}$$

and

$$a_\varepsilon(t,x,\xi,e) = \mathrm{PV}\int_{\mathbb{R}^N}\left[q(\xi + e\cdot z + \varepsilon W_\varepsilon(t,x,z)) - q(\xi + e\cdot z)\right]J(z)dz.$$

From Lemma 4 in [26], we know that when $s \geqslant \frac{1}{2}$, there is a matrix A depending on s and N (but independent of e in the isotropic case), such that, as $\varepsilon \to 0$, $\frac{1}{\eta_\varepsilon}\overline{a}_\varepsilon(t,x,e) \to \mathrm{tr}(AD^2d(t,x))$. Passing to the limit as $\varepsilon \to 0$ in (5.7), we find the following equation for d

$$\partial_t d(t,x) = \gamma\mathrm{tr}(AD^2d) = \gamma\mathrm{tr}\left(\left(I - \frac{Dd\otimes Dd}{|Dd|^2}\right)AD^2d\right), \tag{5.10}$$

since $D^2dDd = 0$. The mean curvature equation just obtained gives the propagation law of the front. When $s < \frac{1}{2}$, the quantity $\frac{1}{\eta_\varepsilon}\overline{a}(t,x,e)$ converges as $\varepsilon \to 0$ to a fractional mean curvature operator, as proven in Lemma 10 of [26]. So in this case one would get (5.10) with the local mean curvature operator replaced by a fractional one.

Let us now state the precise result. To simplify the presentation, we consider the isotropic case. For the anisotropic case we refer to [26].

Theorem 5.1 (Theorem 1, [26])**.** *Let J be given by* (5.4) *with $g \equiv c_{N,s}$ and $s \in \left[\frac{1}{2}, 1\right)$. Let u^ε be the unique solution of* (5.1) *with initial datum* (5.5)*, where $q(x,e) = q(x) = u(x)$ is the solution of* (2.6) *and d_0 is the signed distance function to the boundary of a smooth set Ω_0. Then, there exists a symmetric matrix $A \in \mathcal{S}(n)$ depending on q, s and N, such that if u is the unique (generalized flow) solution of the geometric equation* (5.10) *with*

initial condition $u(0,x) = d_0(x)$*, where* γ *is defined by* (5.8)*, the function* u^ε *satisfies, for* $t > 0$*,* $x \in \mathbb{R}^N$*,*

$$\begin{cases} u_\varepsilon(t,x) \to 1 & \text{in } \{u(t,x) > 0\} \\ u_\varepsilon(t,x) \to 0 & \text{in } \{u(t,x) < 0\} \end{cases} \qquad \text{as } \varepsilon \to 0.$$

Moreover both limits are local uniform.

6. Some open problems

A first open problem, to the best of our knowledge, is the extension of Theorem 5.1 to the case $s \in \left(0, \frac{1}{2}\right)$. Indeed, in [26] it is proven that, for a smooth function d, such that $|\nabla d| = 1$, then

$$\frac{1}{\eta_\varepsilon} \overline{a}_\varepsilon\,(t,x,e) \to k[d](x)$$

where k is a fractional operator and $\overline{a}_\varepsilon$ is defined in (5.9). This suggests that the front moves according the following fractional mean-curvature equation:

$$\partial_t d(t,x) = \gamma k[d(t,\cdot)](x)|Dd|$$

with μ and k depending on the gradient variable in the anisotropic case. The proof of Theorem 5.1 relies on the construction of barriers, i.e., sub and supersolution of (5.1), which are suitable correction of the Ansatz. In the case $s < \frac{1}{2}$, the authors are not able to construct a barrier far from the front. Indeed, in the case $s < \frac{1}{2}$ the contributions from far from the front are not negligible. This is somehow expected, see for instance [33].

A further interesting problem in this direction, with important applications in the theory of crystal dislocations, consists in extending the result of Section 3 to higher dimensions, i.e., proving analogous results to those presented in Section 5 in the case of two or more dislocation lines. To provide a concrete example, suppose that there are two closed dislocation lines in the slip plane x_1x_2. This situation can be modeled by equation (5.1), with associated initial condition

$$u^0_\varepsilon(x) = q\left(\frac{d^1_0(x)}{\varepsilon}, Dd^1_0(x)\right) + q\left(\frac{d^2_0(x)}{\varepsilon}, Dd^2_0(x)\right), \tag{6.1}$$

where d^i_0 is the signed distance function to the boundary of a smooth set Ω^i_0, $i = 1,2$, $\Omega^1_0 \subset \Omega^2_0$ and the two sets are at positive distance one from each other. Here q is solution of (5.3). Existence of such layer solutions needs to be proven as well. In the isotropic case, solutions of (5.3) are given by $q(x) = u(x - x_0)$, for any $x_0 \in \mathbb{R}$, where u is the unique solution of (2.6).

The solution u_ε of (5.1)-(6.1), will convergence as $\varepsilon \to 0$ to the following stable equilibria of W': 0,1 and 2. Let $\Omega_1(t)$ and $\Omega_2(t)$ be the resulting interphases at time t starting from the initial configurations Ω^1_0 and Ω^2_0 respectively. We expect a double behavior for the motion of these fronts. Indeed, we think that in the case $s \geqslant \frac{1}{2}$, the fronts move by local mean curvature and the interaction between the two fronts is negligible at a first level of approximation, i.e., the mean curvature motion is predominant. On the other hand, when $s < \frac{1}{2}$, we expect that the interaction is not negligible anymore and that the equations for the motion of the fronts involve a fractional mean curvature operator and a local term taking into account the interaction of the fronts.

If the expected results hold true, this would suggest to investigate a higher order of asymptotics for the solution of (5.1)-(6.1) to gather how dislocation lines interact, having in mind that the case $s = \frac{1}{2}$ has a physical interest in view of its applications to the theory of crystal dislocations.

Finally, one could also consider dislocation lines with different orientations. In this case, the initial datum, with two of them would be

$$u_\varepsilon^0(x) = q\left(\frac{d_0^1(x)}{\varepsilon}, Dd_0^1(x)\right) + q\left(-\frac{d_0^2(x)}{\varepsilon}, -Dd_0^2(x)\right).$$

In this situation, we expect some sort of collision in finite time.

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