DERIVATION OF THE 1-D GROMA-BALOGH EQUATIONS FROM THE PEIERLS-NABARRO MODEL

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Abstract. We consider a semi-linear integro-differential equation in dimension one associated to the half Laplacian whose solution represents the atom dislocation in a crystal. The equation comprises the evolutive version of the classical Peierls-Nabarro model. We show that for a large number of dislocations, the solution, properly rescaled, converges to the solution of a fully nonlinear integro-differential equation which is a model for the macroscopic crystal plasticity with density of dislocations. This leads to the formal derivation of the 1-D Groma-Balogh equations [14], a popular model describing the evolution of the density of positive and negative oriented parallel straight dislocation lines. This paper completes the work of [28]. The main novelty here is that we allow dislocations to have different orientation and so we have to deal with collisions of them.

1. Introduction

The goal of this paper is to complete the study started by the authors in [28] of the behavior as $\varepsilon \to 0$ of the solution u^ε of the following evolutionary partial-integro-differential equation

$$\begin{cases} \delta \partial_t u^\varepsilon = \mathcal{I}_1[u^\varepsilon] - \dfrac{1}{\delta} W'\left(\dfrac{u^\varepsilon}{\varepsilon}\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ u^\varepsilon(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R} \end{cases} \tag{1.1}$$

where ε, $\delta > 0$ are small scale parameters and $\delta = \delta_\varepsilon \to 0$ as $\varepsilon \to 0$, W is a multi-well periodic potential and we denote by $\mathcal{I}_1$ the so-called fractional Laplacian of order 1, $-(-\Delta)^{\frac{1}{2}}$, defined by

$$\mathcal{I}_1[v](x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{v(y)-v(x)}{(y-x)^2} dy,$$

where PV stands for principal value. We refer to [34] or [7] for a basic introduction to the fractional Laplace operator.

Equation (1.1) arises in the Peierls-Nabarro model to describe at microscopic scale the motion of dislocation lines in crystals. Dislocations are line defects in crystalline materials whose motion is responsible of the plastic behavior of metals. Dislocations can be described at several scales by different models:

a) atomic scale (Frenkel-Kontorova model),
b) microscopic scale (Peierls-Nabarro model),
c) mesoscopic scale (Discrete dislocation dynamics),

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d) macroscopic scale (Elasto-visco-plasticity with density of dislocations).

We refer the reader to the book [15] for a tour in the theory of dislocations. The 1-D Peierls-Nabarro model describes the microscopic effect of an *ensemble of straight edge dislocation lines all lying in the same plane.* After a cross section, dislocation lines can be identified by points on a line. Every dislocation is associated to either a *positive* or a *negative orientation*, depending on the direction of the Burgers' vector (a fixed vector associated to the dislocation). Equation (1.1) with $\varepsilon = 1$, which is obtained after a parabolic rescaling of the original model, has been investigated in a series of papers [13, 6, 5, 29, 31, 32]. The solution here is a phase transition function which represents the atom displacement, in terms of δ, which in turn represents the size of the crystal scale. Starting from an initial configuration where the transitions occurs at some given points, for small δ, the displacement function approaches a piecewise constant function. The plateaus of this asymptotic limit correspond to the periodic sites induced by the crystalline structure, but its jump points evolve in time, according to a singular potential. Roughly speaking, one can imagine that the discontinuity points of this limit displacement function behave like a "particle" system (though no "material" particle is really involved), driven by a system of ordinary differential equations which describe the position of the jump points $y_1(t), \dots, y_N(t)$. The system corresponds to the dynamics of discrete dislocations and the convergence result is a passage from (b) to (c). The physical properties of the singular potential of this ODE system depend on the orientation of the displacement function at the jump points. Namely, if the displacement function has the same spatial monotonicity at y_i and y_{i+1} (i.e., y_i and y_{i+1} have the same orientation), then the potential induces a repulsion between the particles y_i and y_{i+1}. Conversely, when the displacement function has opposite spatial monotonicity at y_i and y_{i+1} (i.e., y_i and y_{i+1} have opposite orientation), then the potential becomes attractive, and the two particles may collide in a finite time. We will give more details in Section 1.1. Collisions create a problem in the analysis as the dynamical system that governs the motion of the dislocation particles ceases to be well-defined at the collision time. The study of the asymptotics of the displacement function after collision time permits to understand how the dynamical law of the interphase points can be continued/extended after collisions, see [32].

Different space/time scales of the original Peierls-Nabarro model also produce homogenization results, whose effective Hamiltonian depends on the scaling properties of the operator see [21, 30]. The model can also be linked to the classical model at the atomic scale which was introduced by Frenkel and Kontorova (see [8]) (from (a) to (b)).

We refer to [3, 9, 10, 11, 12, 18, 19, 22, 33] for further related results.

In [28] the authors considered for the first time the case in which the number of dislocations N goes to ∞. We introduced a second parameter ε such that $N = N_\varepsilon \to \infty$ as $\varepsilon \to 0$. A parabolic rescaling in δ and hyperbolic rescaling in ε of the original model leads to (1.1). In [28] we only considered the case when the dislocation points have all the same orientation, which in the model corresponds to assuming the initial condition u_0 to be monotonic. In the present paper, we remove the monotonicity assumption on u_0, *allowing dislocations to have different orientation.* More precisely, on u_0 we assume

$$\begin{cases} u_0 \in C^{1,1}(\mathbb{R}), \\ \lim\limits_{x\to-\infty} u_0(x) = 0, \\ \lim\limits_{x\to+\infty} u_0(x) = l, \text{ for some } l \in \mathbb{R}. \end{cases} \tag{1.2}$$

For fixed ε, the dislocation points at initial time are approximated by the points in the level sets $\{u_0 = \varepsilon i\}$, $i \in \mathbb{Z}$, while their orientations are determined by the monotonicity of u_0 at the points. The limits in (1.2) guarantee that the dislocation points remain in a compact set for fixed ε. The first limit is just a normalization, 0 could be replace by any real number.

On the potential W we assume

$$\begin{cases} W \in C^{2,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1 \\ W(u+1) = W(u) & \text{for any } u \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ W''(0) > 0. \end{cases} \tag{1.3}$$

Our goal in this paper is to understand the large scale limit of the Peierls-Nabarro model for a *large number of parallel straight edge dislocation lines lying in the same slip plane, with possibly different orientation, moving with self-interactions.* We perform a *direct passage from the model (b) to the model (d)* and show that at macroscopic scale the density of dislocations is governed by the following evolution law:

$$\begin{cases} \partial_t u = c_0 |\partial_x u| \, \mathcal{I}_1[u] & \text{in } (0, +\infty) \times \mathbb{R} \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R} \end{cases} \tag{1.4}$$

where $c_0 > 0$ is defined in the forthcoming (1.14). Our main result is the following:

Theorem 1.1. *Assume* (1.2) *and* (1.3)*. Let u^ε be the solution of* (1.1)*. There exists a number $A_\varepsilon > 0$ depending on ε and u_0 such that if $\delta \to 0$ and $\delta A_\varepsilon \to 0$ as $\varepsilon \to 0$, then u^ε converges locally uniformly in* $(0, +\infty) \times \mathbb{R}$ *to the viscosity solution $\overline{u}$ of* (1.4)*, as $\varepsilon \to 0$.*

Remark 1.2. *The quantity A_ε in Theorem 1.1, will be made explicit later on, see Section 2.2. The condition $\delta A_\varepsilon \to 0$ as $\varepsilon \to 0$ is automatically satisfied by any δ that converges to 0 with ε, if we have some control on the number of dislocation points at time 0 with respect to ε. This is the case, for example, when u_0 is either monotone or goes to* 0 *and l, respectively as $x \to -\infty$ and $x \to +\infty$, faster or equal than respectively c/x and $l + c/x$, for some $c > 0$, see Section 2.2. The latter condition is natural in this setting, see* (2.16)*.*

To prove Theorem 1.1, the idea is to approximate the dislocation particles with points $x_i(t)$ where the limit function $\overline{u}$ attains the value εi at time t, $i \in \mathbb{Z}$. We then show that

$$b_i \dot{x}_i = -\frac{\partial \overline{u}_t(t, x_i(t))}{|\partial_x \overline{u}(t, x_i(t))|} \simeq -c_0 \mathcal{I}_1[\overline{u}(t, \cdot)](x_i(t)),$$

with $b_i = \operatorname{sgn}(\partial_x \overline{u}(t, x_i(t)))$, provided $\partial_x \overline{u}(t, x_i(t)) \neq 0$.

One of the main difficulties in the proof of Theorem 1.1 consists in proving that $\partial_t \overline{u} = 0$ (in the viscosity sense) at points where $\partial_x \overline{u}$ vanishes. This result is also the main novelty with respect to our previous work [28]. Indeed in the monotonic case we could prove, by using an approximation argument, that if u_0, and thus the limit function $\overline{u}$, is monotone, it is enough to test equation (1.4) with test functions for $\overline{u}$ with non vanishing derivative in x. That argument cannot be applied in the present setting and we have to deal with the case of test functions with vanishing derivatives. Roughly speaking, points x where $\partial_x \overline{u}(t, x) = 0$ corresponds to the locations where collisions occur at time t. The proof here is based on a new analysis of how the datum in (1.1) is transported along the characteristics $x_i(t)$ around a collision point. The strategy and the heuristic of the proofs are explained in Section 3.

Differentiating equation (1.4) formally yields the following system of equations for the positive and negative part of $f = \partial_x \overline{u}$,

$$\begin{aligned} \partial_t f^+ &= c_0 \partial_x (f^+ \mathcal{H}(f^+ - f^-)), \\ \partial_t f^- &= -c_0 \partial_x (f^- \mathcal{H}(f^+ - f^-)), \end{aligned} \tag{1.5}$$

with $\mathcal{H}$ the Hilbert transform. Equations (1.5) are the 1-D version of the 2-D Groma-Balogh equations [14], a popular model describing the evolution of the density of positive and negative oriented parallel straight dislocation lines. This is the first time such equations are formally derived from the microscopic Peierls-Nabarro model.

As a by-product of the proof of Theorem 1.1 we also obtain the following asymptotic behavior of the limit function.

Proposition 1.3. *The limit function* $\overline{u}$ *satisfies*

$$\lim_{x \to -\infty} \overline{u}(t,x) = 0, \qquad \lim_{x \to +\infty} \overline{u}(t,x) = l, \tag{1.6}$$

uniformly in $t \in [0,T]$*, for any* $T > 0$*. Moreover, for all* $(t,x) \in (0,+\infty) \times \mathbb{R}$

$$\inf u_0 \leqslant \overline{u}(t,x) \leqslant \sup u_0. \tag{1.7}$$

Remark 1.4. *The limits* (1.6) *can be interpreted as the property of the dislocations particles to remain in a compact set in the interval* $[0,T]$*. Property* (1.7)*, which is an easy consequence of the comparison principle, says that, while dislocations may annihilate, no dislocations are created.*

1.1. **The Peierls-Nabarro model.** The Peierls-Nabarro model [24, 25] is a phase field model for dislocation dynamics incorporating atomic features into continuum framework. In a phase field approach, the dislocations are represented by transition of a continuous field. We refer to [26] for a survey of the Peierls-Nabarro model. See also Section 1.1 in [28] for some basic physical derivation. After a section of the three-dimensional crystal with a plane, a straight dislocation line can be identified with a point on a line. A positive oriented dislocation located at 0 is described by the transition from 0 to 1 of the phase transition function, solution to

$$\begin{cases} \mathcal{I}_1[\phi] = W'(\phi) & \text{in } \mathbb{R} \\ \phi' > 0 & \text{in } \ \mathbb{R} \\ \displaystyle\lim_{z \to -\infty} \phi(z) = 0, \quad \lim_{z \to +\infty} \phi(z) = 1, \quad \phi(0) = \frac{1}{2}, \end{cases} \tag{1.8}$$

while a negative oriented dislocation located at 0 is described by the transition from 0 to -1 of the solution of

$$\begin{cases} \mathcal{I}_1[\phi] = W'(\phi) & \text{in } \mathbb{R} \\ \phi' < 0 & \text{in } \ \mathbb{R} \\ \displaystyle\lim_{z \to -\infty} \phi(z) = 0, \quad \lim_{z \to +\infty} \phi(z) = -1, \quad \phi(0) = \frac{1}{2}. \end{cases} \tag{1.9}$$

Under assumption (1.3), existence of a unique solution of (1.8) has been proven in [2, 27]. Define

$$\phi(z,b) := \begin{cases} \phi(z) & \text{for} \quad b=1 \\ \phi(-z)-1 & \text{for} \quad b=-1. \end{cases} \tag{1.10}$$

Notice that, by the periodicity of W, if ϕ is the solution of (1.8) then $\phi(x,-1)$ is the solution of (1.9).

In the face cubic structured (FCC) observed in many metals and alloys, dislocations move at low temperature on the slip plane. The dynamics for a collection of straight edge dislocations lines, all contained in a single slip plane, moving with self-interactions (no exterior forces) is then described by the evolutive version of the Peierls-Nabarro model (see for instance [23] and [4]):

$$\partial_t u = \mathcal{I}_1[u(t,\cdot)] - W'(u) \quad \text{in} \quad (0,+\infty)\times\mathbb{R}, \tag{1.11}$$

with initial condition

$$u(0,x) = \sum_{i=1}^{N} \phi\left(x - \frac{y_i^0}{\delta}, b_i\right), \tag{1.12}$$

where ϕ is the solution of (1.8), N is the number of dislocations, y_i^0 are the initial locations of the dislocation points and neighboring dislocations are at distance at microscopic scale of order $1/\delta$, that is

$$0 \leqslant y_{i+1}^0 - y_i^0 \sim 1.$$

The number $b_i \in \{-1,1\}$ identifies the orientation of the dislocation: when $b_i = 1$ the dislocation is positive oriented, when $b_i = -1$ it is instead negative oriented.

Let u be the solution of (1.11) with initial condition (1.12). Then, the rescaled function

$$v^\delta(t,x) = u\left(\frac{t}{\delta^2}, \frac{x}{\delta}\right),$$

which is solution to the integro-differential equation in (1.1) with $\varepsilon = 1$ converges as $\delta \to 0$ to a sum of Heaviside functions of the form $\sum_{i=1}^N H(b_i(x - y_i(t)))$, where the interphase (jump) points $y_i(t)$, $i = 1,\dots,N$ evolve in time driven by the following system of ODE:

$$\begin{cases} \dot{y}_i = c_0 \displaystyle\sum_{j\neq i} \frac{b_i b_j}{y_i - y_j} & \text{in } (0,T_c) \\ y_i(0) = y_i^0, \end{cases} \tag{1.13}$$

where c_0 is defined by

$$c_0 = \left(\int_{\mathbb{R}} (\phi')^2 \right)^{-1}, \tag{1.14}$$

see [13, 32]. Here $0 < T_c \leqslant +\infty$ is the first time a collision between opposite oriented interphase points occurs. Indeed, if y_i and y_{i+1} have opposite orientation, that is $b_i b_{i+1} = -1$, the equation for $\dot{y}_i$ contains the term $-1/(y_i - y_{i+1}) > 0$ and the equation for $\dot{y}_{i+1}$ contains the term $-1/(y_{i+1} - y_i) < 0$. Since $y_i(0) < y_{i+1}(0)$, the two points may collide in finite time. Points with same orientation repel each other, thus never collide. System (1.13) can be extended after collision by removing the particles that annihilate at collision, see

[32, 19]. In the physical model, the ODE system (1.13) represents the discrete dynamics of N dislocation points with possibly different orientation.

In the present we want to identify at *large (macroscopic) scale* the evolution model for the dynamics of a density of dislocations. We introduce a further parameter ε and consider a number of dislocations $N = N_\varepsilon$ such that $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and we send both δ and ε to 0 together. We do not specify how N_ε goes to 0 with ε but we only require that

$$\varepsilon^2 N_\varepsilon \delta \to 0$$

as $\varepsilon \to 0$. We consider the following rescaling

$$u^\varepsilon(t,x) = \varepsilon u\left(\frac{t}{\varepsilon\delta^2}, \frac{x}{\varepsilon\delta}\right),$$

with u the solution of (1.11)-(1.12). Then we see that u^ε is solution of (1.1) with initial datum

$$u^\varepsilon(0,x) = \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}, b_i\right), \tag{1.15}$$

with $x_i^0 = \varepsilon y_i^0$.

In general, we consider an initial datum u_0 satisfying (1.2). One can actually prove (see Proposition 4.10) that any function satisfying (1.2), can be approximated by a function of the form (1.15).

1.2. **Organization of the paper.** The paper is organized as follows. In Section 2 we introduce notations and recall some general auxiliary results that will be used in the paper. The strategy and the heuristic of the proof of Theorem 1.1 are presented in Section 3. In Section 4 we prove a discrete approximation formula of the fractional Laplacian $\mathcal{I}_1$ which extends to non monotonic functions the one given in [28]. In Section 5 we construct local in time and global in space supersolutions of (1.1). Sections 6 and 7 are devoted to the proof of our main result, Theorem 1.1. Proposition 1.3 is proven in Section 8. Finally, the proofs of some auxiliary lemmas are given in Section 9.

2. Definitions, Notations and preliminary results

2.1. **Definitions and Notations.** Let v be a function satisfying the following assumptions

$$\begin{cases} v \in C^{1,1}(\mathbb{R}), \\ v \text{ not constant}, \\ \lim\limits_{x\to-\infty} v(x) = 0, \\ \lim\limits_{x\to+\infty} v(x) = l, \text{ for some } l \in \mathbb{R}. \end{cases} \tag{2.1}$$

For a fixed $\varepsilon \in (0,1)$, we define

$$\Lambda_i := \{x \,|\, i\varepsilon < v(x) < \varepsilon(i+1)\}, \quad i = s_\varepsilon, \dots, S_\varepsilon,$$

where $s_\varepsilon := \left\lceil \frac{\inf_{\mathbb{R}} v}{\varepsilon} \right\rceil$ and $S_\varepsilon := \left\lfloor \frac{\sup_{\mathbb{R}} v}{\varepsilon} \right\rfloor$.

Let us denote by $\tilde{\Lambda}_i$ the subset of Λ_i obtained by removing the connected components in which the oscillation of v is smaller than ε. By the limits in (2.1), there exists a compact set $[-K_\varepsilon, K_\varepsilon]$ such that $\tilde{\Lambda}_i \subseteq [-K_\varepsilon, K_\varepsilon]$ for all $i = s_\varepsilon, \dots, S_\varepsilon$. Moreover, any connected component of $\tilde{\Lambda}_i$ has measure bigger or equal than ε/L, where L is the Lipschitz constant of v. Indeed, if A is any connected component of $\tilde{\Lambda}_i$, then there exists a point $x_0 \in A$ such that $v(x_0) = \varepsilon(i + 1/2)$ and by the regularity of v, the interval $(x_0 - \varepsilon/(2L), x_0 + \varepsilon/(2L))$ is contained in A. We infer that the number of connected components of $\tilde{\Lambda}_i$ is finite. In particular, the set $\bigcup_i \partial\tilde{\Lambda}_i$ has a finite number of points, that is

$$\bigcup_{i=s_\varepsilon}^{S_\varepsilon} \partial\tilde{\Lambda}_i = \{x_1, x_2, ..., x_{N_\varepsilon}\}, \tag{2.2}$$

for some positive integer N_ε depending on ε, where the points x_i are ordered such that $x_1 < x_2 < ... < x_{N_\varepsilon}$. For each $i \in \{1, 2, ..., N_\varepsilon\}$, define $v(x_0) = 0$ and

$$b_i = \frac{v(x_i) - v(x_{i-1})}{\varepsilon} \in \{-1, 1\}, \tag{2.3}$$

which also gives the following expression for $v(x_i)$

$$v(x_i) = v(x_{i-1}) + b_i = \sum_{j=1}^{i} b_j \varepsilon. \tag{2.4}$$

We will sometimes refer to the level set points defined in (2.2) as *particles*. By definition, in any interval $[x_i, x_{i+1}]$, the oscillation of v is equal to ε, thus,

$$|v(x) - v(y)| \leqslant \varepsilon, \quad \text{for all } x,\, y \in [x_i, x_{i+1}]. \tag{2.5}$$

For any $x \in (x_1, x_{N_\varepsilon})$ we will call the *closest particle to* x the points x_{i_0} such that $x \in (x_{i_0 - 1}, x_{i_0}]$ and $|x - x_{i_0}| \leqslant |x - x_i|$ for all $i = 1, \ldots, N_\varepsilon$. If $x \leqslant x_1$ the closest particle to x is x_1, while for $x \geqslant x_{N_\varepsilon}$ the closest particle is x_{N_ε}.

Given integers M and N such that $1 \leqslant M \leqslant N \leqslant N_\varepsilon$, we denote the number of particles with $b_i = 1$ and particles with $b_i = -1$ in $[x_M, x_N]$ by $n^+_{M,N}$ and $n^-_{M,N}$ respectively. Precisely,

$$n^+_{M,N} := |\{i \in \{M, ..., N\} \mid b_i = 1\}|,$$
$$n^-_{M,N} := |\{i \in \{M, ..., N\} \mid b_i = -1\}|.$$

Also, we define

$$n_{M,N} := n^+_{M,N} - n^-_{M,N}. \tag{2.6}$$

When $M = 1$ and $N = N_\varepsilon$, we denote

$$N^+_\varepsilon := n^+_{1,N_\varepsilon}, \quad N^-_\varepsilon := n^-_{1,N_\varepsilon}.$$

Note that $N_\varepsilon = N^+_\varepsilon + N^-_\varepsilon$.

Remark 2.1. *Using* (2.4), (2.6) *can also be expressed by*

$$\varepsilon n_{M,N} = \varepsilon n^+_{M,N} - \varepsilon n^-_{M,N} = \varepsilon \sum_{\substack{i=M \\ b_i=1}}^{N} b_i + \varepsilon \sum_{\substack{i=M \\ b_i=-1}}^{N} b_i = \sum_{i=M}^{N} b_i \varepsilon = v(x_N) - v(x_M) + b_M \varepsilon. \tag{2.7}$$

In particular, for $M = 1$, we have $v(x_1) = b_1 \varepsilon$, which yields

$$\varepsilon n_{1,N} = v(x_N).$$

Similarly to Definition (1.10), for the Heaviside function H, we define for any $z \in \mathbb{R}$ and $b \in \{-1, 1\}$,

$$H(z, b) := bH(z). \tag{2.8}$$

To construct sub and supersolution of (1.1) we will often make use of the following ODE's system

$$\begin{cases} \dot{x}_i(t) &= -c_0 b_i L, \\ x_i(0) &= x_i^0, \end{cases} \tag{2.9}$$

where $x_1^0, x_2^0, ..., x_{N_\varepsilon}^0$ are the level set points of the initial condition, $L \in \mathbb{R}$ and c_0 is given by (1.14).

We denote by $B_r(x)$ the ball of radius r centered at x. The cylinder $(t-\tau, t+\tau) \times B_r(x)$ is denoted by $Q_{\tau,r}(t, x)$. $\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the floor and the ceil integer parts of a real number x.

For $r > 0$, we denote

$$\mathcal{I}_1^{1,r}[v](x) = \frac{1}{\pi} PV \int\limits_{|y-x| \leqslant r} \frac{v(y) - v(x)}{(y-x)^2} dy, \tag{2.10}$$

and

$$\mathcal{I}_1^{2,r}[v](x) = \frac{1}{\pi} \int\limits_{|y-x| > r} \frac{v(y) - v(x)}{(y-x)^2} dy. \tag{2.11}$$

Then we can write

$$\mathcal{I}_1[v](x) = \mathcal{I}_1^{1,r}[v](x) + \mathcal{I}_1^{2,r}[v](x).$$

We denote by $USC_b((0, +\infty) \times \mathbb{R})$ (resp., $LSC_b((0, +\infty) \times \mathbb{R})$) the set of upper (resp., lower) bounded semicontinuous functions on $(0, +\infty) \times \mathbb{R}$ which are bounded on $(0, T) \times \mathbb{R}$ for any $T > 0$ and we set $C_b((0, +\infty) \times \mathbb{R}) := USC_b((0, +\infty) \times \mathbb{R}) \cap LSC_b((0, +\infty) \times \mathbb{R})$. We denote by $C_b^2((0, +\infty) \times \mathbb{R})$ the subset of functions of $C_b((0, +\infty) \times \mathbb{R})$ with continuous second derivatives. Finally, $C^{1,1}(\mathbb{R})$ is the set of functions with bounded $C^{1,1}$ norm over $\mathbb{R}$.

Given a sequence $\{u^\varepsilon\}$ we denote

$$\limsup_{\varepsilon \to 0}{}^* u^\varepsilon(x) = \sup \Big\{ \limsup_{\varepsilon \to 0} u^\varepsilon(x_\varepsilon) \,|\, x_\varepsilon \to x \Big\},$$

and

$$\liminf_{\varepsilon \to 0}{}_* u^\varepsilon(x) = \inf \Big\{ \liminf_{\varepsilon \to 0} u^\varepsilon(x_\varepsilon) \,|\, x_\varepsilon \to x \Big\}.$$

Given a quantity $E = E(x)$, we write $E = O(A)$ is there exists a constant $C > 0$ such that, for all x,

$$|E| \leqslant CA.$$

We write $E = o_\varepsilon(1)$ if

$$\lim_{\varepsilon \to 0} E = 0,$$

uniformly in x.

2.2. **Definition of A_ε.** Since u_0 satisfies (1.2), it is easy to see that there exist $C^{1,1}$ functions v_1 and w_1 such that

$$\begin{cases} v_1 \leqslant u_0, \quad v_1(-\infty) = 0, \quad v_1 \text{ is non-increasing} \\ w_1 \leqslant u_0, \quad w_1(+\infty) = l, \quad w_1 \text{ is non-decreasing,} \end{cases} \tag{2.12}$$

and there exist $C^{1,1}$ functions v_2 and w_2 such that

$$\begin{cases} v_2 \geqslant u_0, \quad v_2(-\infty) = 0, \quad v_2 \text{ is non-decreasing} \\ w_2 \geqslant u_0, \quad w_2(+\infty) = l, \quad w_2 \text{ is non-increasing.} \end{cases} \tag{2.13}$$

Let $K_\varepsilon > 0$ be such that for $i = 1, 2$,

$$|v_i(x)| < \frac{\varepsilon}{4} \text{ if } x < -K_\varepsilon \quad \text{and} \quad |w_i(x) - l| < \frac{\varepsilon}{4} \text{ if } x > K_\varepsilon.$$

Then, all the points in the ε level sets of u_0 defined as in (2.2) must belong to the compact set $[-K_\varepsilon, K_\varepsilon]$ and by the forthcoming formula (4.1), if N_ε^0 is the number of such points, then $N_\varepsilon^0 \leqslant CK_\varepsilon/\varepsilon$. Set

$$A_\varepsilon := \varepsilon K_\varepsilon.$$

Then we choose $\delta = o_\varepsilon(1)$ such that

$$\delta A_\varepsilon = o_\varepsilon(1). \tag{2.14}$$

The condition guarantees that

$$\varepsilon^2 N_\varepsilon^0 \delta = o_\varepsilon(1). \tag{2.15}$$

Notice that if u_0 is monotonic then $N_\varepsilon^0 \leqslant (\sup u_0 - \inf u_0)/\varepsilon$, therefore (2.15) is always satisfied and no condition on how δ goes to 0 as $\varepsilon \to 0$ is required. It is easy to see that (2.14) holds true if u_0 satisfies the following asymptotic estimate

$$|u_0(x) - lH(x)| \leqslant \frac{C}{x} \quad \text{if } |x| > 1,$$

for some $C > 0$, with H the Heaviside function.

2.3. **Short and long range interaction.** We start by recalling a basic fact about the operator $\mathcal{I}_1$. Given $v \in C^{1,1}(\mathbb{R})$ and $r > 0$ we can split $\mathcal{I}_1[v]$ into the short and long range interaction as follows,

$$\mathcal{I}_1[v](x) = \mathcal{I}_1^{1,r}[v](x) + \mathcal{I}_1^{2,r}[v](x),$$

where $\mathcal{I}_1^{1,r}[v](x)$, $\mathcal{I}_1^{2,r}[v](x)$ are defined respectively by (2.10) and (2.11). The short range interaction can be rewritten as

$$\mathcal{I}_1^{1,r}[v](x) = \frac{1}{2\pi} \int\limits_{|y|<r} \frac{v(x+y) + v(x-y) - 2v(x)}{y^2} dy,$$

Therefore,

$$|\mathcal{I}_1^{1,r}[v](x)| \leqslant \frac{r}{\pi} \|v\|_{C^{1,1}(\mathbb{R})}.$$

The long range interaction can be bounded as follows

$$|\mathcal{I}_1^{2,r}[v](x)| \leqslant \frac{4}{r\pi} \|v\|_\infty.$$

2.4. **The functions ϕ and ψ.** In what follows we denote by $H(x)$ the Heaviside function. Let $\alpha := W''(0) > 0$.

Lemma 2.2. *Assume that* (1.3) *holds, then there exists a unique solution ϕ of* (1.8)*. Furthermore $\phi \in C^{2,\beta}(\mathbb{R})$ and there exist constants $K_0, K_1 > 0$ such that*

$$\left|\phi(z) - H(z) + \frac{1}{\alpha\pi z}\right| \leqslant \frac{K_1}{z^2}, \quad \text{for } |z| \geqslant 1, \tag{2.16}$$

and for any $z \in \mathbb{R}$

$$0 < \frac{K_0}{1+z^2} \leqslant \phi'(z) \leqslant \frac{K_1}{1+z^2}. \tag{2.17}$$

Proof. The existence of a unique solution of (1.8) and estimate (2.17) are proven in [2, 27]. Estimate (2.2) is proven in [13]. □

Let c_0 be defined as in (1.14). Let us introduce the function ψ to be the solution of

$$\begin{cases} \mathcal{I}_1[\psi] = W''(\phi)\psi + \frac{L}{\alpha}(W''(\phi) - W''(0)) + c_0 L\phi' & \text{in} \quad \mathbb{R} \\ \lim_{z\to\pm\infty} \psi(z) = 0. \end{cases} \tag{2.18}$$

For later purposes, we recall the following decay estimate on the solution of (2.18):

Lemma 2.3. *Assume that* (1.3) *holds, then there exists a unique solution ψ to* (2.18)*. Furthermore $\psi \in C^{1,\beta}(\mathbb{R})$ and for any $L \in \mathbb{R}$ there exist constants K_2 and K_3, with $K_3 > 0$, depending on L such that*

$$\left|\psi(z) - \frac{K_2}{z}\right| \leqslant \frac{K_3}{z^2}, \quad \text{for } |z| \geqslant 1, \tag{2.19}$$

and for any $z \in \mathbb{R}$

$$-\frac{K_3}{1+z^2} \leqslant \psi'(z) \leqslant \frac{K_3}{1+z^2}. \tag{2.20}$$

Proof. The existence of a unique solution of (2.18) is proven in [13]. Estimates (2.19) and (2.20) are shown in [21]. □

The results of Lemmas 2.2 and 2.3 have been generalized in [1, 6, 5, 27, 30] to the case when the fractional operator is $-(-\Delta)^s$ for any $s \in (0,1)$.

For ψ solution of (2.18) and $b \in \{-1, 1\}$, $z \in \mathbb{R}$, we define

$$\psi(z, b) := \psi(bz) \tag{2.21}$$

2.5. **Definition of viscosity solution.** We first recall the definition of viscosity solution for a general first order non-local equation

$$\partial_t u = F(t, x, u, \partial_x u, \mathcal{I}_1[u]) \quad \text{in} \quad (0, +\infty) \times \Omega \tag{2.22}$$

where Ω is an open subset of $\mathbb{R}$ and $F(t, x, u, p, L)$ is continuous and non-decreasing in L.

Definition 2.1. *A function $u \in USC_b((0,+\infty)\times\mathbb{R})$ (resp., $u \in LSC_b((0,+\infty)\times\mathbb{R})$) is a viscosity subsolution (resp., supersolution) of* (2.22) *if for any $(t_0, x_0) \in (0,+\infty)\times\Omega$, and any test function $\varphi \in C^2_b((0,+\infty)\times\mathbb{R})$ such that $u - \varphi$ attains a global maximum (resp., minimum) at the point (t_0, x_0), then*

$$\partial_t\varphi(t_0, x_0) - F(t_0, x_0, u(t_0, x_0), \partial_x\varphi(t_0, x_0), \mathcal{I}_1[\varphi(t_0, \cdot)](x_0)) \leqslant 0$$
$$\textit{(resp., } \geqslant 0).$$

A function $u \in C_b((0,+\infty)\times\mathbb{R})$ *is a viscosity solution of* (2.23) *if it is a viscosity sub and supersolution of* (2.22).

Remark 2.4. *It is classical that the maximum (resp., the minimum) in Definition 2.1 can be assumed to be strict and that*

$$\varphi(t_0,x_0)=u(t_0,x_0).$$

This will be used later.

Next, let us consider the initial value problem

$$\begin{cases}\partial_t u = F(t,x,u,\partial_x u,\mathcal{I}_1[u]) & \text{in} \quad (0,+\infty)\times\mathbb{R}\\ u(0,x)=u_0(x) & \text{on} \quad \mathbb{R},\end{cases} \tag{2.23}$$

where u_0 is a continuous function.

Definition 2.2. *A function* $u \in USC_b((0,+\infty)\times\mathbb{R})$ *(resp.,* $u \in LSC_b((0,+\infty)\times\mathbb{R})$*) is a viscosity subsolution (resp., supersolution) of the initial value problem* (2.23) *if* $u(0,x)\leqslant (u_0)(x)$ *(resp.,* $u(0,x)\geqslant (u_0)(x)$*) and* u *is viscosity subsolution (resp., supersolution) of the equation*

$$\partial_t u = F(t,x,u,\partial_x u,\mathcal{I}_1[u]) \quad \textit{in} \quad (0,+\infty)\times\mathbb{R}.$$

A function $u \in C_b((0,+\infty)\times\mathbb{R})$ *is a viscosity solution of* (2.23) *if it is a viscosity sub and supersolution of* (2.23).

It is a classical result that smooth solutions are also viscosity solutions.

Proposition 2.5. *If* $u \in C^1((0,+\infty);C^{1,\beta}_{loc}(\Omega)\cap L^\infty(\mathbb{R}))$ *for some* $0<\beta\leqslant 1$*, and* u *satisfies pointwise*

$$\partial_t u - F(t,x,u,\partial_x u,\mathcal{I}_1[u]) \leqslant 0 \ \textit{(resp.} \geqslant 0\textit{)} \quad \textit{in} \quad (0,+\infty)\times\Omega,$$

then u *is a viscosity subsolution (resp., supersolution) of* (2.22).

2.6. **Comparison principle and existence results.** In this subsection, we successively give comparison principles and existence results for (1.1) and (1.4). The following comparison theorem is shown in [17] for more general parabolic integro-PDEs.

Proposition 2.6 (Comparison Principle for (1.1))**.** *Consider* $u \in USC_b((0,+\infty)\times\mathbb{R})$ *subsolution and* $v \in LSC_b((0,+\infty)\times\mathbb{R})$ *supersolution of* (1.1)*, then* $u \leqslant v$ *on* $(0,+\infty)\times\mathbb{R}$.

Following [17] it can also be proven the comparison principle for (1.1) in bounded domains. Since we deal with a non-local equation, we need to compare the sub and the supersolution everywhere outside the domain.

Proposition 2.7 (Comparison Principle on bounded domains for (1.1))**.** *Let* Ω *be a bounded domain of* $(0,+\infty)\times\mathbb{R}$ *and let* $u \in USC_b((0,+\infty)\times\mathbb{R})$ *and* $v \in LSC_b((0,+\infty)\times\mathbb{R})$ *be respectively a sub and a supersolution of*

$$\delta\partial_t u = \mathcal{I}_1[u(t,\cdot)] - \frac{1}{\delta}W'\left(\frac{u}{\varepsilon}\right) \quad \textit{in } \Omega.$$

If $u \leqslant v$ *outside* Ω*, then* $u \leqslant v$ *in* Ω.

Proposition 2.8 (Existence for (1.1))**.** *For* ε, $\delta>0$ *there exists* $u^\varepsilon \in C_b([0,+\infty)\times\mathbb{R})$ *(unique) viscosity solution of* (1.1).

Proof. We can construct a solution by Perron's method if we construct sub and supersolutions of (1.1) which are equal to $u_0(x)$ at $t=0$. Since $u_0 \in C^{1,1}(\mathbb{R})$, the two functions $u^{\pm}(t,x) := u_0(x) \pm \frac{C}{\delta^2} t$ are respectively a super and a subsolution of (1.1), if

$$C \geqslant \frac{4\delta}{\pi} \|u_0\|_{C^{1,1}(\mathbb{R})} + \|W'\|_\infty.$$

Moreover $u^+(0,x) = u^-(0,x) = u_0(x)$. □

We next recall the comparison and the existence results for (1.4), see e.g. [16], Proposition 3.

Proposition 2.9. *If* $u \in USC_b([0,+\infty)\times\mathbb{R})$ *and* $v \in LSC_b([0,+\infty)\times\mathbb{R})$ *are respectively a sub and a supersolution of* (1.4), *then* $u \leqslant v$ *on* $(0,+\infty)\times\mathbb{R}$. *Moreover, under assumption* (1.2), *there exists a (unique) viscosity solution of* (1.4).

3. Strategy and heuristic proofs

In this section we explain the steps that we will follow to prove Theorem 1.1 and the heuristics of the main proofs.

3.1. **Approximation of $\mathcal{I}_1$.** The first result is a discrete approximation formula for the fractional Laplace $\mathcal{I}_1$. Let v be any function satisfying (2.1). Let x_i and $b_i \in \{-1,1\}$, $i = 1,\dots,N_\varepsilon$, be defined as in (2.2) and (2.3) respectively. Then, we show (see Proposition 4.4 and Proposition 4.6) that for any fixed $i_0 \in \{1,\dots,N_\varepsilon\}$,

$$\mathcal{I}_1[v](x_{i_0}) \simeq \frac{1}{\pi} \sum_{i \neq i_0} \frac{b_i \varepsilon}{x_i - x_{i_0}}, \tag{3.1}$$

and for any x,

$$\mathcal{I}_1[v](x) \simeq \frac{1}{\pi} \sum_{|x_i - x| > r} \frac{b_i \varepsilon}{x_i - x},$$

for some $r = o_\varepsilon(1)$, where the error goes to 0 as $\varepsilon \to 0$ uniformly over $\mathbb{R}$. On the other hand, the sum

$$\sum_{\substack{i \neq i_0 \\ |x_i - x| \leqslant r}} \frac{b_i \varepsilon}{x_i - x},$$

where x_{i_0} is the closest particle to x may not be zero but depends on the distance of x from x_{i_0} (see Lemma 4.5). For the proof of these results we follow the proof of the analogous results given in [28] in the case v is monotone non-decreasing, i.e. $b_i = 1$ for all i. We refer to Section 2.1 there for the heuristic of (3.1) in the monotone case.

3.2. **Approximation of v.** Let $\phi(x, b_i)$ be defined as in (1.10). Then, we show (see Proposition 4.10) that any function v satisfying (2.1) can be approximated in $L^\infty(\mathbb{R})$ as follows,

$$v(x) \simeq \sum_{i=1}^{N_\varepsilon} \varepsilon \phi\left(\frac{x - x_i}{\varepsilon\delta}, b_i\right), \tag{3.2}$$

with x_i and $b_i \in \{-1,1\}$, $i = 1,\dots,N_\varepsilon$, defined as in (2.2) and (2.3) respectively. We refer to Section 2.2 in [28] for the heuristic proof of (3.2) in the case v monotone non-decreasing.

3.3. **Heuristic of the proof of Theorem 1.1.** Let u be the limit solution (that here we suppose to exist and be smooth). Fix a point $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}$. We need to distinguish two cases: $\partial_x u(t_0, x_0) \neq 0$ and $\partial_x u(t_0, x_0) = 0$.

Case 1: $\partial_x u(t_0, x_0) \neq 0$.

We are going to give an ansatz for u^ε in a small box Q_R of size R centered at (t_0, x_0). For small R, all the derivatives of u can be considered constant in Q_R:

$$\partial_t u(t,x) \simeq \partial_t u(t_0, x_0), \quad \partial_x u(t,x) \simeq \partial_x u(t_0, x_0)$$

and

$$\mathcal{I}_1[u(t,\cdot)](x) \simeq \mathcal{I}_1[u(t_0,\cdot)](x_0) =: L_0.$$

For t close to t_0, we define the points $x_i(t)$ as in (2.2) and for $v = u(t,\cdot)$. Since u is monotone in Q_R, the b_i of the particles inside that box, defined as in (2.3), have all the same value. Moreover, for those points, by differentiating in t the equation

$$u(t, x_i(t)) = const., \tag{3.3}$$

we get

$$\partial_t u(t, x_i(t)) + \partial_x u(t, x_i(t))\dot{x}_i(t) = 0,$$

from which

$$\dot{x}_i(t) = -\frac{\partial_t u(t, x_i(t))}{\partial_x u(t, x_i(t))} \simeq -\frac{\partial_t u(t_0, x_0)}{\partial_x u(t_0, x_0)}. \tag{3.4}$$

Notice that since particles in the box have the same speed, they never collide there. Next we consider as ansatz for u^ε the approximation of u given by (3.2) plus a small correction:

$$\Phi^\varepsilon(t,x) := \sum_{i=1}^{N_\varepsilon} \varepsilon \left(\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right)\right),$$

where $\phi(\cdot, b_i)$ is defined as in (1.10) and $\psi(\cdot, b_i)$ as in (2.21), with ψ the solution of (2.18) with $L = L_0$. For a detailed heuristic motivation of this correction, see Section 3.1 of [13]. By (3.2), $\Phi^\varepsilon(t,x) \to u(t,x)$ as $\varepsilon \to 0$. Fix $(t,x) \in Q_R$ and let $x_{i_0}(t)$ be the closest point among the $x_i(t)$'s to x and $z_i(t) = (x - x_i(t))/(\varepsilon\delta)$. Plugging into (1.1), we get (see proof of (6.38) in Section 6)

$$\begin{aligned} 0 &= \delta\partial_t\Phi^\varepsilon(t,x) - \mathcal{I}_1[\Phi^\varepsilon(t,\cdot)](x) + \frac{1}{\delta}W'\left(\frac{\Phi_\varepsilon(t,x)}{\varepsilon}\right) \\ &\simeq -\phi'(z_{i_0})(b_{i_0}\dot{x}_{i_0}(t) + c_0 L_0) + (W''(\phi(z_{i_0})) - W''(0))\left(\frac{1}{\delta}\sum_{i\neq i_0}\tilde{\phi}(z_i, b_i) - \frac{L_0}{\alpha}\right) \end{aligned}$$

where $\tilde{\phi}(\cdot, b_i) = \phi(\cdot, b_i) - H(\cdot, b_i)$, with $H(\cdot, b_i)$ defined as in (2.8). Suppose for simplicity that $x = x_{i_0}(t)$, then by (2.16) and (3.1)

$$\frac{1}{\delta}\sum_{i\neq i_0}\tilde{\phi}(z_i, b_i) - \frac{L_0}{\alpha} \simeq \frac{1}{\alpha\pi}\sum_{i\neq i_0}\frac{b_i\varepsilon}{x_i - x_{i_0}} - \frac{L_0}{\alpha} \simeq 0.$$

Since $\phi' > 0$, we must have

$$\dot{x}_{i_0}(t) \simeq -c_0 b_{i_0} L_0$$

that is, by (3.4),

$$\partial_t u(t_0, x_0) \simeq c_0 b_{i_0}\partial_x u(t_0, x_0)\mathcal{I}_1[u(t_0,\cdot)](x_0) = c_0|\partial_x u(t_0, x_0)|\mathcal{I}_1[u(t_0,\cdot)](x_0).$$

To formalize the argument we will construct from the ansatz local in space and time sub and supersolutions of (1.1) to compare with u^ε.

Notice that if we define

$$y_i(\tau) := \frac{x_i(\varepsilon\tau)}{\varepsilon}$$

then the y_i's solve

$$\dot{y}_i(\tau) = \dot{x}_i(\varepsilon\tau) \simeq -c_0 b_i L_0 \simeq \frac{c_0}{\pi}\sum_{j\neq i}\frac{b_i b_j \varepsilon}{x_i - x_j} = \frac{c_0}{\pi}\sum_{j\neq i}\frac{b_i b_j}{y_i - y_j},$$

which is the discrete dislocations dynamics given in (1.13).

Case 2: $\partial_x u(t_0, x_0) = 0$.

When $\partial_x u(t_0, x_0) = 0$, we cannot obtain formula (3.4). However the ODE system (1.13) and the approximation formula (3.1) suggests that, at least locally,

$$u^\varepsilon(t,x) \sim \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right),$$

with $x_i(t)$ solution of

$$\dot{x}_i(t) \simeq -c_0 b_{i_0}\mathcal{I}_1[u(t_0,\cdot)](x_i(t)),$$

and $x_i(t_0) = x_i^0$, with x_i^0 the level set points of the function $u(t_0,\cdot)$. Therefore, we proceed as follows. Assume that in a box Q_ρ around (t_0, x_0) u has the form

$$u(t,x) = a(x - x_0)^2 + g(t) \tag{3.5}$$

for some $a > 0$ and g smooth. Notice that level set points of u in Q_ρ which are smaller than x_0 are associated to $b_i = -1$, and those bigger than x_0 are associated to $b_i = 1$. Fix any $0 < \sigma << \rho$ independent of ε. We construct a smooth approximation, u^σ, of u which is constant in x for $|x - x_0| \leqslant \sigma$. Precisely, u^σ is such that

$$\begin{cases} u \leqslant u^\sigma \\ u^\sigma \leqslant u + C\sigma^2 \text{ if } |x - x_0| \leqslant \sigma \\ u^\sigma \text{ is constant in } x \text{ if } |x - x_0| \leqslant \sigma \\ u^\sigma \text{ is non-increasing in } x \text{ if } x \in (-\infty, x_0) \\ u^\sigma \text{ is non-decreasing in } x \text{ if } x \in (x_0, +\infty). \end{cases} \tag{3.6}$$

Next, we set

$$c := \frac{1}{4c_0 L},$$

with $L > 0$ to be determined. We then define x_i^0 and b_i, $i = 1, \dots, N_\varepsilon$, as in (2.2) and (2.3) for the function $u^\sigma(t_0 - c\sigma, \cdot)$. By (3.2), for all $x \in \mathbb{R}$,

$$u^\sigma(t_0 - c\sigma, x) = \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1), \tag{3.7}$$

where the constant $M_\varepsilon := \lceil u^\sigma(t_0 - c\sigma, -\infty)/\varepsilon \rceil$ is a normalization so that $u^\sigma(t_0 - c\sigma, -\infty) - \varepsilon M_\varepsilon = o_\varepsilon(1)$. Define the function

$$H^\varepsilon(t,x) := \sum_{i=1}^{N_\varepsilon} \varepsilon\left(\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right)\right) + \varepsilon M_\varepsilon + \varepsilon\left\lceil\frac{o_\varepsilon(1)}{\varepsilon}\right\rceil, \tag{3.8}$$

with $x_i(t)$ the solution of the ODE system (2.9) with initial condition $x_i(t_0 - c\sigma) = x_i^0$, that is

$$x_i(t) = x_i^0 - b_i c_0 L[t - (t_0 - c\sigma)].$$

Since $\sum_{i=1}^{N_\varepsilon} \varepsilon\delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) = o_\varepsilon(1)$, from (3.7) we can choose $o_\varepsilon(1)$ in (3.8) in such a way

$$u(t_0 - c\sigma, x) \leqslant u^\sigma(t_0 - c\sigma, x) \leqslant H^\varepsilon(t_0 - c\sigma, x).$$

Notice that particles $x_i(t)$ and $x_{i+1}(t)$ with the same orientation ($b_i b_{i+1} = 1$) move in parallel, while opposite oriented particles, ($b_i b_{i+1} = -1$) move each toward the other. However, since u^σ is constant in $x \in [x_0 - \sigma, x_0 + \sigma]$ and monotonic in $(-\infty, x_0)$ and in $(x_0, +\infty)$, particles with opposite orientation are at distance larger than 2σ at time $t_0 - c\sigma$. This guarantees that no collision occurs in the interval $[t_0 - c\sigma, t_0 + c\sigma]$. Then, we are able to show that setting

$$L := \frac{C_0}{\sigma^{\frac{1}{2}}}$$

for some $C_0 > 0$ large enough but independent of ε and σ, H^ε is supersolution of (1.1) in $[t_0 - c\sigma, t_0 + c\sigma] \times \mathbb{R}$, and by the comparison principle,

$$H^\varepsilon(t, x) \geqslant u^\varepsilon(t, x) \quad \text{for any } (t, x) \in [t_0 - c\sigma, t_0 + c\sigma] \times \mathbb{R}.$$

This yields,

$$\begin{aligned}
u^\varepsilon(t_0, x_0) &\leqslant H^\varepsilon(t_0, x_0) \\
&= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_0 - x_i(t_0)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\
&= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_0 + b_i c_0 L c\sigma) - x_i^0)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\
&= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_0 + c_0 L c\sigma) - x_i^0)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1),
\end{aligned}$$

where the last equality needs to be justified (see Lemma 6.1). Then, by (3.7) and the second inequality in (3.6), we obtain

$$\begin{aligned}
u^\varepsilon(t_0, x_0) &\leqslant \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_0 + c_0 L c\sigma) - x_i^0)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\
&= u^\sigma(t_0 - c\sigma, x_0 + c_0 L c\sigma) + o_\varepsilon(1) \\
&\leqslant u(t_0 - c\sigma, x_0 + c_0 L c\sigma) + o_\varepsilon(1) + C\sigma^2.
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and recalling the definitions of c and L we get

$$u(t_0, x_0) - u\left(t_0 - k_0\sigma^{\frac{3}{2}}, x_0 + k_1\sigma\right) \leqslant C\sigma^2,$$

for some k_0, k_1 independent of σ. Dividing both sides by $k_0\sigma^{\frac{3}{2}}$, by (3.5) we finally obtain

$$\partial_t u(t_0, x_0) \leqslant 0.$$

Similarly, one can prove that $\partial u_t(t_0, x_0) \geqslant 0$.

3.4. **Viscosity sub and supersolutions.** To formally prove the convergence result we show the functions $u^+ := \limsup_{\varepsilon\to 0}{}^* u^\varepsilon$ and $u^- := \liminf_{\varepsilon\to 0*} u^\varepsilon$, which are everywhere finite, are respectively sub and supersolution of (1.4). Moreover, $u^+(0,x) \leqslant u_0(x) \leqslant u^-(0,x)$. The comparison principle then implies that $u^+ \leqslant \overline{u} \leqslant u^-$. Since the reverse inequality $u^- \leqslant u^+$ always holds true, we conclude that the two functions coincide with the continuous viscosity solution of (1.4).

4. Approximation Results

In this section, we present several approximation results, which are similar to those in [28]. In this paper, however, we consider a function v satisfying (2.1), which is not necessarily monotonic. Since the proofs of some results are similar, they will be omitted. Readers may consult [28] if necessary. The following lemma is proven in [28], see Lemma 4.1.

Lemma 4.1. *Assume that v satisfies* (2.1)*. Let $\|v_x\|_\infty \leqslant L$, and let x_i be defined as in* (2.2)*. Then,*

$$x_{i+1} - x_i \geqslant \varepsilon L^{-1} \quad \text{for all } i = 1, \dots, N_\varepsilon - 1. \tag{4.1}$$

Moreover, there exists $c > 0$ independent of v such that for any $\overline{x} \in \mathbb{R}$

$$\sum_{\substack{i=1\\ i\neq i_0}}^{N_\varepsilon} \frac{\varepsilon^2}{(x_i - \overline{x})^2} \leqslant cL^2. \tag{4.2}$$

In addition, if $|v_x| \geqslant a > 0$ on an interval I, then for all x_{i+1}, $x_i \in I$, we have

$$x_{i+1} - x_i \leqslant \varepsilon a^{-1}. \tag{4.3}$$

Lemma 4.2 (Short range interaction)**.** *Assume that v satisfies* (2.1) *and let x_i and b_i be defined as in* (2.2) *and* (2.3)*. Let $r = r_\varepsilon = o_\varepsilon(1)$ and $\varepsilon/r = o_\varepsilon(1)$. For any $\rho \geqslant r$ and $\overline{x} \in (x_{M_\varepsilon} + \rho, x_{N_\varepsilon} - \rho)$, then*

$$\frac{1}{\pi} \sum_{\substack{i\neq i_0\\ r\leqslant |x_i - \overline{x}|\leqslant \rho}} \frac{b_i \varepsilon}{x_i - \overline{x}} = \mathcal{I}_1^{1,\rho}[v](\overline{x}) + \frac{1}{\pi}\frac{v(\overline{x}+\rho) + v(\overline{x}-\rho) - 2v(\overline{x})}{\rho} + o_\varepsilon(1). \tag{4.4}$$

Proof. Since $v \in C^{1,1}(\mathbb{R})$ and $r = o_\varepsilon(1)$, there exists $C > 0$ such that

$$|\mathcal{I}_1^{1,r}[v](\overline{x})| \leqslant Cr = o_\varepsilon(1).$$

Therefore, we have

$$\mathcal{I}_1^{1,\rho}[v](\overline{x}) = \frac{1}{\pi}\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x) - v(\overline{x})}{(x-\overline{x})^2}dx + \frac{1}{\pi}\int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(x) - v(\overline{x})}{(x-\overline{x})^2}dx + o_\varepsilon(1). \tag{4.5}$$

We write the first term in (4.5) as

$$\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x) - v(\overline{x})}{(x-\overline{x})^2}dx = \int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2}dx - \int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(\overline{x})}{(x-\overline{x})^2}dx.$$

and notice that we can integrate the second term in (4.5) as follows

$$\int\limits_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(\overline{x})}{(x-\overline{x})^2}dx = v(\overline{x}) \int\limits_{\overline{x}-\rho}^{\overline{x}-r} \frac{1}{(x-\overline{x})^2}dx = \frac{v(\overline{x})}{r} - \frac{v(\overline{x})}{\rho}. \tag{4.6}$$

Let us denote by M_ρ and M_r respectively the smallest and the largest integer i such that $x_i \in [\overline{x}-\rho, \overline{x}-r]$, that is

$$x_{M_\rho - 1} < \overline{x} - \rho \leqslant x_{M_\rho} \leqslant x_{M_r} \leqslant \overline{x} - r < x_{M_r+1}. \tag{4.7}$$

Then, we have that

$$\int\limits_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2}dx = \int\limits_{\overline{x}-\rho}^{x_{M_\rho}} \frac{v(x)}{(x-\overline{x})^2}dx + \sum_{i=M_\rho}^{M_r-1} \int\limits_{x_i}^{x_{i+1}} \frac{v(x)}{(x-\overline{x})^2}dx + \int\limits_{x_{M_r}}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2}dx. \tag{4.8}$$

By (2.5), $v(x) \leqslant v(\overline{x}-\rho) + \varepsilon$ for $x \in [\overline{x}-\rho, x_{M_\rho}]$. Hence, we obtain

$$\int\limits_{\overline{x}-\rho}^{x_{M_\rho}} \frac{v(x)}{(x-\overline{x})^2}dx \leqslant \int\limits_{\overline{x}-\rho}^{x_{M_\rho}} \frac{v(\overline{x}-\rho)+\varepsilon}{(x-\overline{x})^2}dx = \frac{v(\overline{x}-\rho)+\varepsilon}{-\rho} - \frac{v(\overline{x}-\rho)+\varepsilon}{x_{M_\rho}-\overline{x}}. \tag{4.9}$$

Similarly, $v(x) \leqslant v(\overline{x}-r) + \varepsilon$ for $x \in [x_{M_r}, \overline{x}-r]$, which gives us

$$\int\limits_{x_{M_r}}^{\overline{x}-r} \frac{v(x)}{(x-\overline{x})^2}dx \leqslant \int\limits_{x_{M_r}}^{\overline{x}-r} \frac{v(\overline{x}-r)+\varepsilon}{(x-\overline{x})^2}dx = \frac{v(\overline{x}-r)+\varepsilon}{x_{M_r}-\overline{x}} - \frac{v(\overline{x}-r)+\varepsilon}{-r}. \tag{4.10}$$

Also, for $x \in [x_i, x_{i+1}]$, we have $v(x) \leqslant v(x_i) + \varepsilon$. Thus, we obtain

$$\begin{aligned}
\sum_{i=M_\rho}^{M_r-1} \int\limits_{x_i}^{x_{i+1}} \frac{v(x)}{(x-\overline{x})^2}dx &\leqslant \sum_{i=M_\rho}^{M_r-1} \int\limits_{x_i}^{x_{i+1}} \frac{v(x_i)+\varepsilon}{(x-\overline{x})^2}dx \\
&= \sum_{i=M_\rho}^{M_r-1} \left[\frac{v(x_i)+\varepsilon}{x_i-\overline{x}} - \frac{v(x_i)+\varepsilon}{x_{i+1}-\overline{x}} \right] \\
&= \sum_{i=M_\rho}^{M_r-1} \frac{v(x_i)+\varepsilon}{x_i-\overline{x}} - \sum_{i=M_\rho+1}^{M_r} \frac{v(x_{i-1})+\varepsilon}{x_i-\overline{x}} \\
&= -\frac{v(x_{M_r})+\varepsilon}{x_{M_r}-\overline{x}} + \sum_{i=M_\rho}^{M_r} \frac{v(x_i)-v(x_{i-1})}{x_i-\overline{x}} + \frac{v(x_{M_\rho-1})+\varepsilon}{x_{M_\rho}-\overline{x}} \\
&= \frac{v(x_{M_\rho-1})+\varepsilon}{x_{M_\rho}-\overline{x}} - \frac{v(x_{M_r})+\varepsilon}{x_{M_r}-\overline{x}} + \sum_{i=M_\rho}^{M_r} \frac{b_i\varepsilon}{x_i-\overline{x}},
\end{aligned} \tag{4.11}$$

using $v(x_i) = v(x_{i-1}) + b_i\varepsilon$ in the last equality. Finally, we combine (4.6), (4.9), (4.10) and (4.11) to get

$$\begin{aligned}\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \leqslant{}& \frac{v(\overline{x}-\rho)+\varepsilon}{-\rho} - \frac{v(\overline{x}-\rho)+\varepsilon}{x_{M_\rho}-\overline{x}} \\ &+ \frac{v(\overline{x}-r)+\varepsilon}{x_{M_r}-\overline{x}} - \frac{v(\overline{x}-r)+\varepsilon}{-r} \\ &+ \frac{v(x_{M_\rho-1})+\varepsilon}{x_{M_\rho}-\overline{x}} - \frac{v(x_{M_r})+\varepsilon}{x_{M_r}-\overline{x}} + \sum_{i=M_\rho}^{M_r} \frac{b_i\varepsilon}{x_i-\overline{x}} - \frac{v(\overline{x})}{r} + \frac{v(\overline{x})}{\rho},\end{aligned}$$

which simplifies to

$$\begin{aligned}\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \leqslant{}& \sum_{i=M_\rho}^{M_r} \frac{b_i\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x})-v(\overline{x}-\rho)}{\rho} - \frac{\varepsilon}{\rho} + \frac{v(\overline{x}-r)-v(\overline{x})}{r} + \frac{\varepsilon}{r} \\ &+ \frac{v(x_{M_\rho-1})-v(\overline{x}-\rho)}{x_{M_\rho}-\overline{x}} + \frac{v(\overline{x}-r)-v(x_{M_r})}{x_{M_r}-\overline{x}}.\end{aligned}$$

Note that, by (2.5) and (4.7), $|v(x_{M_\rho-1}) - v(\overline{x}-\rho)| \leqslant \varepsilon$, $|v(\overline{x}-r) - v(x_{M_r})| \leqslant \varepsilon$, $|x_{M_\rho} - \overline{x}| \geqslant r$ and $|x_{M_r} - \overline{x}| \geqslant r$. Hence, the following holds:

$$\frac{v(x_{M_\rho-1})-v(\overline{x}-\rho)}{x_{M_\rho}-\overline{x}} \leqslant \frac{\varepsilon}{r} \quad \text{and} \quad \frac{v(\overline{x}-r)-v(x_{M_r})}{x_{M_r}-\overline{x}} \leqslant \frac{\varepsilon}{r}. \tag{4.12}$$

Using (4.12), we therefore obtain

$$\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \leqslant \sum_{i=M_\rho}^{M_r} \frac{b_i\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x})-v(\overline{x}-\rho)}{\rho} + \frac{v(\overline{x}-r)-v(\overline{x})}{r} + \frac{3\varepsilon}{r}. \tag{4.13}$$

Similarly, for the second term in (4.5), one can show that

$$\int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \leqslant \sum_{i=N_r}^{M_\rho} \frac{b_i\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x})-v(\overline{x}+\rho)}{\rho} + \frac{v(\overline{x}+r)-v(\overline{x})}{r} + \frac{3\varepsilon}{r}. \tag{4.14}$$

By (4.5), (4.13) and (4.14), we conclude that

$$\begin{aligned}\mathcal{I}_1^{1,\rho}[v](\overline{x}) \leqslant{}& \frac{1}{\pi} \sum_{\substack{i\neq i_0 \\ r\leqslant|x_i-\overline{x}|\leqslant\rho}} \frac{b_i\varepsilon}{x_i-\overline{x}} + \frac{6\varepsilon}{r} + o_\varepsilon(1) \\ &+ \frac{1}{\pi}\frac{v(\overline{x}+r)-2v(\overline{x})+v(\overline{x}-r)}{r} - \frac{1}{\pi}\frac{v(\overline{x}+\rho)-2v(\overline{x})+v(\overline{x}-\rho)}{\rho}.\end{aligned} \tag{4.15}$$

Since $v \in C^{1,1}(\mathbb{R})$, there exists a constant $C > 0$ such that

$$\left|\frac{v(\overline{x}+r)+v(\overline{x}-r)-2v(\overline{x})}{r}\right| \leqslant Cr = o_\varepsilon(1). \tag{4.16}$$

Therefore, using $\varepsilon/r = o_\varepsilon(1)$, we finally obtain

$$\mathcal{I}_1^{1,\rho}[v](\overline{x}) \leqslant \frac{1}{\pi} \sum_{\substack{i\neq i_0 \\ r\leqslant|x_i-\overline{x}|\leqslant\rho}} \frac{b_i\varepsilon}{x_i-\overline{x}} - \frac{1}{\pi}\frac{v(\overline{x}+\rho)-2v(\overline{x})+v(\overline{x}-\rho)}{\rho} + o_\varepsilon(1). \tag{4.17}$$

For the lower bound, we apply the following inequalities to (4.8).

$$\begin{aligned} v(x) &\geqslant v(x_i)-\varepsilon && \text{for} \quad x\in[x_i,x_{i+1}], i=M_\rho,...,M_r-1.\\ v(x) &\geqslant v(\overline{x}-\rho)-\varepsilon && \text{for} \quad x\in[\overline{x}-\rho, x_{M\rho}]\\ v(x) &\geqslant v(\overline{x}-r)-\varepsilon && \text{for} \quad x\in[x_{M-r},\overline{x}-r]. \end{aligned}$$

Then, we follow the same steps as above to eventually obtain

$$\int_{\overline{x}-\rho}^{\overline{x}-r} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \geqslant \sum_{i=M_\rho}^{M_r} \frac{b_i\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x})-v(\overline{x}-\rho)}{\rho} + \frac{v(\overline{x}-r)-v(\overline{x})}{r} - \frac{3\varepsilon}{r}. \tag{4.18}$$

Similarly, one can show that

$$\int_{\overline{x}+r}^{\overline{x}+\rho} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \geqslant \sum_{i=N_r}^{M_\rho} \frac{b_i\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x})-v(\overline{x}+\rho)}{\rho} + \frac{v(\overline{x}+r)-v(\overline{x})}{r} - \frac{3\varepsilon}{r}. \tag{4.19}$$

By combining (4.18) and (4.19), and using (4.16), we obtain

$$\mathcal{I}_1^{1,\rho}[v](\overline{x}) \geqslant \frac{1}{\pi} \sum_{\substack{i\neq i_0 \\ r\leqslant|x_i-\overline{x}|\leqslant\rho}} \frac{b_i\varepsilon}{x_i-\overline{x}} - \frac{1}{\pi}\frac{v(\overline{x}+\rho)-2v(\overline{x})+v(\overline{x}-\rho)}{\rho} + o_\varepsilon(1), \tag{4.20}$$

where $\varepsilon/r = o_\varepsilon(1)$. By (4.17) and (4.20), we have proven (4.4). $\square$

Lemma 4.3 (Long range interaction)**.** *Under the assumptions of Lemma 4.2 and for r as in the lemma, for any $\rho \geqslant r$ and $\overline{x} \in (x_1+\rho, x_{N_\varepsilon}-\rho)$,*

$$\frac{1}{\pi}\sum_{|x_i-\overline{x}|>\rho} \frac{b_i\varepsilon}{x_i-\overline{x}} = \mathcal{I}_1^{2,\rho}[v](\overline{x}) - \frac{1}{\pi}\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho} + o_\varepsilon(1). \tag{4.21}$$

Proof. First, we consider the following decomposition of $\mathcal{I}_1^{2,\rho}[v](\overline{x})$.

$$\mathcal{I}_1^{2,\rho}[v](\overline{x}) = \int_{-\infty}^{x_1} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \int_{x_1}^{\overline{x}-\rho} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \int_{\overline{x}+\rho}^{x_{N_\varepsilon}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx + \int_{x_{N_\varepsilon}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx. \tag{4.22}$$

Define the following integrals.

$$T_1 := \int_{-\infty}^{x_1} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \qquad T_2 := \int_{x_1}^{\overline{x}-\rho} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx$$

$$T_3 := \int_{\overline{x}+\rho}^{x_{N_\varepsilon}} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx \qquad T_4 := \int_{x_{N_\varepsilon}}^{+\infty} \frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx.$$

Proceeding as in the proof of Lemma 4.2 with $\overline{x}-\rho,\overline{x}-r,\overline{x}+r,\overline{x}+\rho$ replaced by $x_1,\overline{x}-\rho,\overline{x}-\rho,x_{N_\varepsilon}$ respectively in (4.15), we obtain

$$\begin{aligned}T_2+T_3 \leqslant &\sum_{|x_i-\overline{x}|\geqslant\rho}\frac{b_i\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+\rho)-2v(\overline{x})+v(\overline{x}-\rho)}{\rho}\\ &+\frac{v(\overline{x})-v(x_1)}{\overline{x}-x_1}-\frac{v(\overline{x})-v(x_{N_\varepsilon})}{\overline{x}-x_{N_\varepsilon}}+\frac{6\varepsilon}{\rho}.\end{aligned}\tag{4.23}$$

Similarly, by applying the same changing of variables to (4.20), one can show that

$$\begin{aligned}T_2+T_3 \geqslant &\sum_{|x_i-\overline{x}|\geqslant\rho}\frac{b_i\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+\rho)-2v(\overline{x})+v(\overline{x}-\rho)}{\rho}\\ &+\frac{v(\overline{x})-v(x_1)}{\overline{x}-x_1}-\frac{v(\overline{x})-v(x_{N_\varepsilon})}{\overline{x}-x_{N_\varepsilon}}-\frac{6\varepsilon}{\rho}.\end{aligned}\tag{4.24}$$

Next, using that

$$\begin{aligned}\inf_{(-\infty,x_1]}v\leqslant v(x)\leqslant v(x_1)+\varepsilon \quad &\text{for } x\in(-\infty,x_1]\\ v(x_{N_\varepsilon})-\varepsilon\leqslant v(x)\leqslant \sup_{[x_{N_\varepsilon},+\infty)}v \quad &\text{for } x\in[x_{N_\varepsilon},+\infty),\end{aligned}$$

we have the following estimates

$$T_1=\int_{-\infty}^{x_1}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\leqslant\int_{-\infty}^{x_1}\frac{v(x_1)+\varepsilon-v(\overline{x})}{(x-\overline{x})^2}dx=-\frac{v(x_1)-v(\overline{x})}{x_1-\overline{x}}-\frac{\varepsilon}{x_1-\overline{x}},\tag{4.25}$$

$$T_1=\int_{-\infty}^{x_1}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\geqslant\int_{-\infty}^{x_1}\frac{\inf_{(-\infty,x_1]}v-v(\overline{x})}{(x-\overline{x})^2}dx=-\frac{\inf_{(-\infty,x_1]}v-v(\overline{x})}{x_1-\overline{x}},\tag{4.26}$$

and

$$T_4=\int_{x_{N_\varepsilon}}^{+\infty}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\leqslant\frac{\sup_{[x_{N_\varepsilon},+\infty)}v-v(\overline{x})}{x_{N_\varepsilon}-\overline{x}},\tag{4.27}$$

$$T_4=\int_{x_{N_\varepsilon}}^{+\infty}\frac{v(x)-v(\overline{x})}{(x-\overline{x})^2}dx\geqslant\frac{v(\overline{x})-v(x_{N_\varepsilon})}{\overline{x}-x_{N_\varepsilon}}-\frac{\varepsilon}{x_{N_\varepsilon}-\overline{x}}.\tag{4.28}$$

Combining (4.23), (4.25) and (4.27), we obtain

$$\begin{aligned}\mathcal{I}_1^{2,\rho}[v](\overline{x})\leqslant &\sum_{|x_i-\overline{x}|\geqslant\rho}\frac{b_i\varepsilon}{x_i-\overline{x}}+\frac{v(\overline{x}+\rho)-2v(\overline{x})+v(\overline{x}-\rho)}{\rho}\\ &+\frac{\sup_{[x_{N_\varepsilon},+\infty)}v-v(x_{N_\varepsilon})}{x_{N_\varepsilon}-\overline{x}}+\frac{\varepsilon}{\overline{x}-x_1}+\frac{6\varepsilon}{\rho}.\end{aligned}\tag{4.29}$$

Similarly, combining (4.24), (4.26) and (4.28), we obtain

$$
\begin{aligned}
\mathcal{I}_1^{2,\rho}[v](\overline{x}) \geqslant & \sum_{|x_i-\overline{x}|\geqslant\rho} \frac{b_i\varepsilon}{x_i-\overline{x}} + \frac{v(\overline{x}+\rho)-2v(\overline{x})+v(\overline{x}-\rho)}{\rho} \\
& - \frac{v(x_1)-\inf_{(-\infty,x_1]} v}{\overline{x}-x_1} - \frac{\varepsilon}{x_{N_\varepsilon}-\overline{x}} - \frac{6\varepsilon}{\rho}.
\end{aligned} \tag{4.30}
$$

Since $\overline{x}-x_1 \geqslant \rho$, $x_{N_\varepsilon}-\overline{x} \geqslant \rho$, and

$$
0 \leqslant \sup_{[x_{N_\varepsilon},+\infty)} v - v(x_{N_\varepsilon}) \leqslant \varepsilon \quad\text{and}\quad 0 \leqslant v(x_1) - \inf_{(-\infty,x_1]} v \leqslant \varepsilon,
$$

we conclude that

$$
\begin{aligned}
& \frac{\sup_{[x_{N_\varepsilon},+\infty)} v - v(x_{N_\varepsilon})}{x_{N_\varepsilon}-\overline{x}} + \frac{\varepsilon}{\overline{x}-x_1} \leqslant \frac{\varepsilon}{\rho} + \frac{\varepsilon}{\rho} = \frac{2\varepsilon}{\rho} \\
& -\frac{v(x_1)-\inf_{(-\infty,x_1]} v}{\overline{x}-x_1} - \frac{\varepsilon}{x_{N_\varepsilon}-\overline{x}} \geqslant -\frac{\varepsilon}{\rho} - \frac{\varepsilon}{\rho} = -\frac{2\varepsilon}{\rho}.
\end{aligned} \tag{4.31}
$$

Finally, combining (4.29), (4.30) and (4.31) gives (4.21), which completes the proof. □

The following proposition is an immediate consequence of Lemma 4.2 and Lemma 4.3.

Proposition 4.4. *Assume that v satisfies* (2.1) *and let x_i and b_i be defined as in* (2.2) *and* (2.3). *Let $r = r_\varepsilon = o_\varepsilon(1)$ and $\varepsilon/r = o_\varepsilon(1)$. Then, for any $\overline{x} \in (x_{M_\varepsilon}+r, x_{N_\varepsilon}-r)$,*

$$
\frac{1}{\pi}\sum_{|x_i-\overline{x}|\geqslant r} \frac{\varepsilon b_i}{x_i-\overline{x}} = \mathcal{I}_1[v](\overline{x}) + o_\varepsilon(1).
$$

Lemma 4.5. *Under the assumptions of Lemma 4.2, let $\overline{x} = x_{i_0} + \varepsilon\gamma$, where x_{i_0} is the closest point to $\overline{x}$. Then, there exists $r = r_\varepsilon$ satisfying $\varepsilon^{\frac{5}{8}} \leqslant r \leqslant c\varepsilon^{\frac{1}{2}}$, with c depending on the $C^{1,1}$ norm of v, such that*

$$
\frac{1}{\pi}\sum_{\substack{i\neq i_0 \\ |x_i-\overline{x}|<r}} \frac{\varepsilon b_i}{x_i-\overline{x}} = O(\varepsilon^{\frac{1}{8}}) + O(\gamma). \tag{4.32}
$$

Proof. First, we want to show that

$$
\frac{1}{\pi}\sum_{\substack{i\neq i_0 \\ |x_i-\overline{x}|<r}} \frac{\varepsilon b_i}{x_i-x_{i_0}} = O(\varepsilon^{\frac{1}{8}}). \tag{4.33}
$$

Let $K>0$ be such that $\|v_{xx}\|_{L^\infty(\mathbb{R})} \leqslant K$. We consider three cases.

Case 1: $|v_x(x_{i_0})| \leqslant 12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$.

By Taylor expansion and Young's inequality, we have

$$
\begin{aligned}
\varepsilon = |b_{i_0+1}\varepsilon| = |v(x_{i_0+1}) - v(x_{i_0})| &\leqslant |v_x(x_{i_0})||x_{i_0+1}-x_{i_0}| + \frac{K}{2}(x_{i_0+1}-x_{i_0})^2 \\
&\leqslant \frac{v_x(x_{i_0})^2}{2(12)^2K} + \left(\frac{12^2K}{2} + \frac{K}{2}\right)(x_{i_0+1}-x_{i_0})^2 \\
&\leqslant \frac{\varepsilon}{2} + \frac{12^2+1}{2}K(x_{i_0+1}-x_{i_0})^2,
\end{aligned}
$$

which implies

$$x_{i_0+1} - x_{i_0} \geqslant c\varepsilon^{\frac{1}{2}},$$

for some $c > 0$ independent of ε. Similarly, one can prove that

$$x_{i_0} - x_{i_0-1} \geqslant c\varepsilon^{\frac{1}{2}}.$$

Since x_{i_0} is the closest point to $\overline{x}$, we must have that $\overline{x} - x_{i_0-1} \geqslant c\varepsilon^{\frac{1}{2}}/2$ and $x_{i_0+1} - \overline{x} \geqslant c\varepsilon^{\frac{1}{2}}/2$. Therefore, if we choose $r = r_\varepsilon = c\varepsilon^{\frac{1}{2}}/4$, there is no index $i \neq i_0$ for which $|\overline{x} - x_i| \leqslant r$ and thus (4.32) is trivially true.

Case 2: $12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}} \leqslant |v_x(x_{i_0})| \leqslant \varepsilon^{\frac{1}{2}-\tau}$, *for some* $\tau \in (0, 1/4)$.

If $|\overline{x} - x_{i_0}| \geqslant \varepsilon^{\frac{1}{2}}/(4K^{\frac{1}{2}})$, then we choose $r = \varepsilon^{\frac{1}{2}}/(8K^{\frac{1}{2}})$ and, as in Case 1, there is no index $i \neq i_0$ for which $|\overline{x} - x_i| \leqslant r$. Thus (4.32) holds true.

Now, assume $|\overline{x} - x_{i_0}| \leqslant \varepsilon^{\frac{1}{2}}/(4K^{\frac{1}{2}})$ and define

$$r := \frac{\varepsilon^{\frac{1}{2}}}{2K^{\frac{1}{2}}} \geqslant 2|\overline{x} - x_{i_0}|. \tag{4.34}$$

We claim that v is monotone in $(\overline{x} - r, \overline{x} + r)$, where r is defined in (4.34). To show this, suppose that $v_x(x_{i_0}) \geqslant 12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$. Then, for $x \in (\overline{x} - r, \overline{x} + r)$,

$$|x - x_{i_0}| \leqslant |x - \overline{x}| + |\overline{x} - x_{i_0}| < r + \frac{r}{2} = \frac{3r}{2}.$$

Now, we have that

$$v_x(x) - v_x(x_{i_0}) \geqslant -K|x - x_{i_0}| \geqslant -\frac{3rK}{2},$$

which gives us

$$v_x(x) \geqslant v_x(x_{i_0}) - \frac{3rK}{2} \geqslant 12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}} - \frac{3rK}{2} = \frac{45K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}}{4} > 0.$$

Hence, $v_x(x) > 0$ if $|x - \overline{x}| < r$.

One can similarly show that $v_x(x) < 0$ if $|x - \overline{x}| < r$ when $v_x(x_{i_0}) \leqslant -12K^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}$. Therefore, the claim is proved. Since the monotonicity is obtained, we can apply the same proof of Lemma 4.6 in [28] to conclude that

$$\left| \frac{1}{\pi} \sum_{\substack{i \neq i_0 \\ |x_i - \overline{x}| < r}} \frac{\varepsilon}{x_i - x_{i_0}} \right| \leqslant C\varepsilon^{\frac{1}{2}-\tau} \leqslant C\varepsilon^{\frac{1}{4}}, \tag{4.35}$$

which gives (4.33).

Case 3: $|v_x(x_{i_0})| \geqslant \varepsilon^{\frac{1}{2}-\tau}$, *for some* $\tau \in (0, 1/4)$.

Arguing as before, we can assume that $|\overline{x} - x_{i_0}| \leqslant \varepsilon^{\frac{1+\tau}{2}}$. Then, we define

$$r := 2\varepsilon^{\frac{1+\tau}{2}} \geqslant 2|\overline{x} - x_{i_0}|. \tag{4.36}$$

As in Case 2, we can show that v is monotone in the interval $(\overline{x} - r, \overline{x} + r)$. Thus, the same proof of Lemma 4.6 in [28] applies, and gives us (4.33).

Finally, (4.32) follows by showing that

$$\left| \sum_{\substack{i\neq i_0\\ |x_i-\overline{x}|<r}} \frac{\varepsilon}{x_i-\overline{x}} - \sum_{\substack{i\neq i_0\\ |x_i-x_{i_0}|<r}} \frac{\varepsilon}{x_i-x_{i_0}} \right| \leqslant C\gamma, \tag{4.37}$$

which has been proven again in Lemma 4.6 in [28]. Combining (4.37) with (4.33), we have (4.32), which completes the proof. □

The following proposition is an immediate consequence of Lemma 4.2, Proposition 4.4 and Lemma 4.5.

Proposition 4.6. *Assume that v satisfies* (2.1) *and let x_i and b_i be defined as in* (2.2) *and* (2.3)*. Then, there exists $c>0$ depending on the $C^{1,1}$ norm of v such that if $\rho \geqslant c\varepsilon^{\frac{1}{2}}$ and $\overline{x}\in(x_{M_\varepsilon}+\rho, x_{N_\varepsilon}-\rho)$, $\overline{x}=x_{i_0}+\varepsilon\gamma$, where x_{i_0} is the closest point to $\overline{x}$, then*

$$\frac{1}{\pi}\sum_{\substack{i\neq i_0\\ |x_i-\overline{x}|\leqslant\rho}} \frac{\varepsilon b_i}{x_i-\overline{x}} = \mathcal{I}_1^{1,\rho}[v](\overline{x}) + O(\gamma) + \frac{1}{\pi}\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho} + o_\varepsilon(1), \tag{4.38}$$

and

$$\frac{1}{\pi}\sum_{i\neq i_0} \frac{\varepsilon b_i}{x_i-\overline{x}} = \mathcal{I}_1[v](\overline{x}) + o_\varepsilon(1) + O(\gamma). \tag{4.39}$$

Proof. The result follows by choosing r such that $\varepsilon^{\frac{5}{8}} \leqslant r \leqslant c\varepsilon^{\frac{1}{2}} \leqslant \rho$, and then applying Lemmas 4.2, 4.5, and 4.4. □

Remark 4.7. *If $\varepsilon|\gamma| = |\overline{x}-x_{i_0}| > c\varepsilon^{\frac{1}{2}} \geqslant r$, then $|\overline{x}-x_i|>r$ for all i and*

$$\begin{aligned}\frac{1}{\pi}\sum_{\substack{i\neq i_0\\ |x_i-\overline{x}|\leqslant\rho}} \frac{\varepsilon}{x_i-\overline{x}} &= \frac{1}{\pi}\sum_{r<|x_i-\overline{x}|\leqslant\rho} \frac{\varepsilon}{x_i-\overline{x}}\\ &= \mathcal{I}_1^{1,\rho}[v](\overline{x}) + \frac{1}{\pi}\frac{v(\overline{x}+\rho)+v(\overline{x}-\rho)-2v(\overline{x})}{\rho} + o_\varepsilon(1).\end{aligned}$$

Therefore, we can assume

$$\varepsilon^{\frac{1}{2}}|\gamma| \leqslant C \tag{4.40}$$

in (4.38) *and* (4.39)*.*

Lemma 4.8. *Assume that v satisfies* (2.1) *and let x_i be defined as in* (2.2)*. Let ϕ be defined by* (1.8)*. Let $1\leqslant M<N\leqslant N_\varepsilon$ and $R\geqslant c\varepsilon^{\frac{1}{2}}$, with $c>0$ given by Proposition 4.6. Then, for all $x\in(x_M+R, x_N-R)$,*

$$\left|\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + v(x_M) - v(x)\right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}, \tag{4.41}$$

with $o_\varepsilon(1)$ independent of R and x.

Proof. Fix $x\in(x_M+R, x_N-R)$, and let x_{i_0} be the closest point to x. Then, we have

$$x-x_i>0 \quad \text{for} \quad i\leqslant i_0-1, \qquad x-x_i<0 \quad \text{for} \quad i\geqslant i_0+1 \tag{4.42}$$

and by (2.5)

$$v(x_{i_0})-\varepsilon\leqslant v(x)\leqslant v(x_{i_0})+\varepsilon. \tag{4.43}$$

By splitting the sum and using (4.43), we obtain

$$\sum_{i=M}^{N}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta},b_i\right)-v(x)\leqslant\sum_{\substack{i=M\\ i\neq i_0}}^{N}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta},b_i\right)+\varepsilon\phi\left(\frac{x-x_{i_0}}{\varepsilon\delta},b_{i_0}\right)-v(x_{i_0})+\varepsilon.$$

Using the asymptotic estimate (2.16), (2.4), $\phi\leqslant 1$, and (4.42), we have that

$$\begin{aligned}
&\sum_{i=M}^{N}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta},b_i\right)-v(x)\\
&\qquad\leqslant\sum_{\substack{i=M\\ i\neq i_0}}^{N}\varepsilon\left(H\left(\frac{x-x_i}{\varepsilon\delta},b_i\right)+\frac{\varepsilon\delta}{\alpha\pi}\frac{b_i\varepsilon}{x_i-x}+\frac{K_1\varepsilon^2\delta^2}{(x_i-x)^2}\right)+2\varepsilon-v(x_{i_0})\\
&\qquad=\varepsilon(n^+_{M,i_0-1}-n^-_{M,i_0-1})-v(x_{i_0})+2\varepsilon+\frac{\varepsilon\delta}{\alpha\pi}\sum_{\substack{i=M\\ i\neq i_0}}^{N}\frac{b_i\varepsilon}{x_i-x}+\sum_{\substack{i=M\\ i\neq i_0}}^{N}\frac{K_1\varepsilon^2\delta^2}{(x_i-x)^2}.
\end{aligned}$$

Notice that by (2.7),

$$\varepsilon n^+_{M,i_0-1}-\varepsilon n^-_{M,i_0-1}-v(x_{i_0})=-v(x_M)+b_M\varepsilon-b_{i_0}\varepsilon.$$

We conclude that

$$\sum_{i=M}^{N}\varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta},b_i\right)+v(x_M)-v(x)\leqslant\frac{\varepsilon\delta}{\alpha\pi}\sum_{\substack{i=M\\ i\neq i_0}}^{N}\frac{b_i\varepsilon}{x_i-x}+C\varepsilon, \tag{4.44}$$

where we also used (4.2). By decomposing the sum in the right-hand side of the above inequality as follows

$$\sum_{\substack{i=M\\ i\neq i_0}}^{N}\frac{b_i\varepsilon}{x_i-x}=\sum_{\substack{i\neq i_0\\ |x_i-x|\leqslant R}}\frac{b_i\varepsilon}{x_i-x}+\sum_{\substack{i=M\\ |x_i-x|>R}}^{N}\frac{b_i\varepsilon}{x_i-x} \tag{4.45}$$

and applying (4.38) and (4.40), we obtain

$$\frac{\varepsilon\delta}{\alpha\pi}\left|\sum_{\substack{i\neq i_0\\ |x_i-x|\leqslant R}}\frac{b_i\varepsilon}{x_i-x}\right|\leqslant C\varepsilon^{\frac{1}{2}}\delta=o_\varepsilon(1). \tag{4.46}$$

For the second term in (4.45), we have

$$\frac{\varepsilon\delta}{\alpha\pi}\left|\sum_{\substack{i=M\\ |x_i-x|>R}}^{N}\frac{b_i\varepsilon}{x_i-x}\right|\leqslant\frac{\varepsilon\delta}{\alpha\pi}\sum_{\substack{i=M\\ |x_i-x|>R}}^{N}\frac{\varepsilon}{R}\leqslant\frac{\varepsilon\delta}{\alpha\pi}\frac{\varepsilon N_\varepsilon}{R}=\frac{C\varepsilon^2\delta N_\varepsilon}{R}. \tag{4.47}$$

Combining (4.44), (4.45), (4.46) and (4.47), we get

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + v(x_M) - v(x) \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}. \tag{4.48}$$

Similarly, one can show that

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + v(x_M) - v(x) \geqslant o_\varepsilon(1) - \frac{C\varepsilon^2\delta N_\varepsilon}{R}. \tag{4.49}$$

This completes the proof. □

Lemma 4.9. *Under the assumptions of Lemma 4.8, there exists $C > 0$ independent of ε and R such that for all $x > x_N + R$,*

$$\left|\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + v(x_M) - v(x_N)\right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}. \tag{4.50}$$

Furthermore, for all $x < x_M - R$,

$$\left|\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right)\right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}. \tag{4.51}$$

Proof. First, suppose that $x > x_N + R$. Then, $x - x_i > R$ for all $i = M, \ldots, N$. The sign of $b_i(x - x_i)$ depends on the sign of b_i. Then by decomposing the sum and applying (2.16), we have

$$\begin{aligned}
&\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x_N) \\
&\quad\leqslant \sum_{i=M}^{N} \varepsilon\left(H\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + \frac{\varepsilon\delta}{\alpha\pi}\frac{b_i\varepsilon}{x_i - x} + \frac{K_1\varepsilon^2\delta^2}{(x_i-x)^2}\right) - v(x_N) \\
&\quad= \varepsilon n^+_{M,N} - \varepsilon n^-_{M,N} + \frac{\varepsilon\delta}{\alpha\pi}\sum_{i=M}^{N}\frac{b_i\varepsilon}{x_i - x} + K_1\varepsilon\delta^2\sum_{i=M}^{N}\frac{\varepsilon^2}{(x_i-x)^2} - v(x_N).
\end{aligned}$$

Notice that by (2.7),

$$\varepsilon n^+_{M,N} - \varepsilon n^-_{M,N} - v(x_N) = -v(x_M) + b_M\varepsilon.$$

Hence, we have

$$\begin{aligned}
\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + v(x_M) - v(x_N) &\leqslant \frac{\varepsilon\delta}{\alpha\pi}\sum_{i=M}^{N}\frac{b_i\varepsilon}{x_i - x} + K_1\varepsilon\delta^2\sum_{i=M}^{N}\frac{\varepsilon^2}{(x_i-x)^2} + \varepsilon \\
&\leqslant \frac{\varepsilon\delta}{\alpha\pi}\sum_{i=M}^{N}\frac{\varepsilon}{R} + C\varepsilon\delta^2 + \varepsilon \\
&\leqslant C\varepsilon^2 N_\varepsilon\frac{\delta}{R} + C\varepsilon\delta^2 + \varepsilon \\
&= o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R},
\end{aligned}$$

using (4.2). One can similarly show that

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + v(x_M) - v(x_N) \geqslant o_\varepsilon(1) - \frac{C\varepsilon^2\delta N_\varepsilon}{R},$$

which completes the proof of (4.50).

Next, suppose that $x < x_M - R$. Notice that $x - x_i < -R$ for all $i = M, \dots, N$ and by (2.16), we obtain

$$\begin{aligned}\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) &\leqslant \sum_{i=M}^{N} \varepsilon\left(H\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + \frac{\varepsilon\delta}{\alpha\pi}\frac{b_i\varepsilon}{x_i - x} + \frac{K_1\varepsilon^2\delta^2}{(x_i-x)^2}\right)\\ &= \frac{\varepsilon\delta}{\alpha\pi}\sum_{i=M}^{N}\frac{b_i\varepsilon}{x_i-x} + K_1\varepsilon\delta^2\sum_{i=M}^{N}\frac{\varepsilon^2}{(x_i-x)^2}.\end{aligned}$$

Thus, as before, we get

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}.$$

Similarly, we can prove that

$$\sum_{i=M}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) \geqslant o_\varepsilon(1) - \frac{C\varepsilon^2\delta N_\varepsilon}{R},$$

which gives (4.51). □

Proposition 4.10. *Assume that v satisfies* (2.1) *and let x_i and b_i be defined as in* (2.2) *and* (2.3). *Let ϕ be defined by* (1.8) *and let $\delta = \delta(\varepsilon)$ be such that $\varepsilon^2 N_\varepsilon\delta = o_\varepsilon(1)$. Then, for all $x \in \mathbb{R}$,*

$$\left|\sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x)\right| \leqslant o_\varepsilon(1), \tag{4.52}$$

where $o_\varepsilon(1)$ is independent of x.

Proof. Estimate (4.52) is a consequence of the following inequality

$$\left|\sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x)\right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}, \tag{4.53}$$

for some $C > 0$ independet of ε.

Let us denote $a_\varepsilon := \varepsilon^2 N_\varepsilon\delta = o_\varepsilon(1)$. Let $R = R_\varepsilon := \max\{a_\varepsilon^{\frac{1}{2}}, c\varepsilon^{\frac{1}{2}}\} = o_\varepsilon(1)$, with c given in Proposition 4.6. If $x \in (x_1 + R, x_{N_\varepsilon} - R)$, then (4.53) follows from Lemma 4.8 with $M = 1$ and $N = N_\varepsilon$. Next, let us assume $x > x_{N_\varepsilon} + R$. Then, by (4.50), we have

$$\left|\sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x)\right| \leqslant |v(x_{N_\varepsilon}) - v(x)| + o_\varepsilon(1).$$

In the interval $[x_{N_\varepsilon}, +\infty)$, the oscillation of v is less or equal than ε, therefore

$$|v(x_{N_\varepsilon}) - v(x)| \leqslant \varepsilon,$$

from which (4.53) for $x > x_{N_\varepsilon} + R$ follows. By using (4.51), one can similarly prove (4.53) when $x < x_1 - R$.

Next, assume $x_{N_\varepsilon} - R \leqslant x \leqslant x_{N_\varepsilon} + R$. Define N to be an index such that

$$x_N \leqslant x_{N_\varepsilon} - 2R < x_{N+1} \leqslant x_{N_\varepsilon},$$

Using that $v(x) = v(x_{N_\varepsilon}) + O(R) = v(x_N) + O(R)$, we get

$$\begin{aligned}
&\left|\sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x)\right| \\
&\leqslant \left|\sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x_{N_\varepsilon})\right| + O(R) \\
&= \left|\sum_{\substack{i=1 \\ x_i \leqslant x_{N_\varepsilon}-2R}}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + \sum_{\substack{i=1 \\ x_i > x_{N_\varepsilon}-2R}}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x_{N\varepsilon})\right| + O(R) \\
&= \left|\sum_{i=1}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) + \sum_{i=N+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x_{N\varepsilon})\right| + O(R) \\
&\leqslant \left|\sum_{i=1}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x_N)\right| + \left|\sum_{i=N+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right)\right| + O(R).
\end{aligned}$$

By (4.50) with $M = 1$, we obtain

$$\left|\sum_{i=1}^{N} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right) - v(x_N)\right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}. \tag{4.54}$$

Since $0 < \phi < 1$, we have

$$\left|\sum_{i=N+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i}{\varepsilon\delta}, b_i\right)\right| \leqslant \sum_{i=N+1}^{N_\varepsilon} \varepsilon = \varepsilon(n^+_{N+1,N_\varepsilon} + n^-_{N+1,N_\varepsilon}) = O(R), \tag{4.55}$$

where $n^+_{N+1,N_\varepsilon} + n^-_{N+1,N_\varepsilon}$ is the total number of particles lying inside the interval

$$[x_{N+1}, x_{N_\varepsilon}] \subset [x_{N_\varepsilon} - 2R, x_{N_\varepsilon} + R],$$

which by (4.1) can be estimated by CR/ε for some constant $C > 0$.

By (4.54) and (4.55), we have proven (4.53) when $x_{N_\varepsilon} - R \leqslant x \leqslant x_{N_\varepsilon} + R$. Similarly, one can prove (4.53) when $x_1 - R \leqslant x \leqslant x_1 + R$, which concludes the proof of the proposition. $\square$

We conclude this section with the following result that will be used in Section 5. The proof is an easy adaptation of the proof of Lemma 4.13 in [28].

Lemma 4.11. *Assume that v satisfies* (2.1) *and let x_i and b_i be defined as in* (2.2) *and* (2.3)*. Then, there exists $C > 0$ such that for all $x \in \mathbb{R}$,*

$$\left|\sum_{i \neq i_0} \frac{\varepsilon b_i}{x_i - x}\right| \leqslant C.$$

Remark 4.12. *Proposition 4.10 and Lemma 4.11 hold true if in* (2.1) *the assumption* $v(-\infty) = 0$ *is replaced by* $v(-\infty) = m$*, for any* $m \in \mathbb{R}$*. In this case formula* (4.12) *is replaced by*

$$\left| \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i}{\varepsilon\delta}, b_i\right) - v(x) + \varepsilon M_\varepsilon \right| \leqslant o_\varepsilon(1),$$

with $M_\varepsilon = \lceil m/\varepsilon \rceil$.

5. Supersolutions of (1.1)

In this section, we construct global is space and local in time supersolutions of the equation (1.1).

Lemma 5.1. *Let* v *be a* $C^{1,1}$ *function which is monotonic non-increasing in* $(-\infty, x_0)$*, non-decreasing in* (x_0, ∞)*, constant in* $(x_0 - \sigma, x_0 + \sigma)$*, for some* $x_0 \in \mathbb{R}$ *and* $0 < \sigma < 1$*. Let* $x_i(t)$ *be the solution of the ODE system* (2.9) *with* $L > 0$ *and* x_i^0 *and* b_i *defined as in* (2.2) *and* (2.3)*. Then, there exists* $C > 0$ *depending on* v *and* W *such that if* $L = C/\sigma^{\frac{1}{2}}$*, the function*

$$H^\varepsilon(t, x) = \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \sum_{i=1}^{N_\varepsilon} \varepsilon\delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \frac{\varepsilon\delta L}{\alpha} \tag{5.1}$$

is a supersolution of (1.1) *in* $(0, \sigma/(2c_0L)] \times \mathbb{R}$*, where* ϕ *is the solution of* (1.8) *and* ψ *is the solution of* (2.18)*.*

Proof. To simplify the notation, we write $N = N_\varepsilon$. Since v is monotonic in $(-\infty, x_0)$ and (x_0, ∞), the number of particles in each of the intervals is bounded by $(\sup v - \inf v)/\varepsilon$ so that $\varepsilon N \leqslant C$. Notice that the points x_i^0 such that $x_i^0 < x_0$ are associated to $b_i = -1$ while points $x_i^0 > x_0$ to $b_i = -1$. Moreover, since v is constant in $(x_0 - \sigma, x_0 + \sigma)$ we have that

$$x_{i+1}^0 - x_i^0 \geqslant 2\sigma, \quad \text{if } b_i = -1,\ b_{i+1} = 1. \tag{5.2}$$

Since $L > 0$ the particles on the left of x_0 move to the right, while particles on the right of x_0 move to the left. However, since the particles with opposite sign are far enough, there is no collision for small times. More precisely, by the ODE system (2.9) and (5.2), we see that for any $t \in [0, \sigma/(2c_0L)]$.

$$x_{i+1}(t) - x_i(t) \geqslant \sigma \quad \text{if } b_i = -1,\ b_{i+1} = 1. \tag{5.3}$$

Particles with same orientation move in parallel so they never collide.

Now, for a fixed $(t, x) \in [0, \sigma/(2c_0L)] \times \mathbb{R}$, define

$$\Lambda(t, x) = \delta\partial_t H^\varepsilon(t, x) - \mathcal{I}_1[H^\varepsilon(t, \cdot)](x) + \frac{1}{\delta}W'\left(\frac{H^\varepsilon(t, x)}{\varepsilon}\right).$$

Define also $z_i(t) = \dfrac{x - x_i(t)}{\varepsilon\delta}$. Let i_0 be the index such that $x_{i_0}(t)$ is the closest point to x. By direct computation, we obtain

$$\begin{aligned}\Lambda(t,x) = &-\sum_{i=1}^{N} b_i \dot{x}_i \phi'(z_i,b_i) - \sum_{i=1}^{N} \delta b_i \dot{x}_i \psi'(z_i,b_i) - \frac{1}{\delta}\sum_{i=1}^{N} \mathcal{I}_1[\phi](z_i,b_i) - \sum_{i=1}^{N} \mathcal{I}_1[\psi](z_i,b_i)\\ &+ \frac{1}{\delta} W'\left(\sum_{i=1}^{N}[\tilde{\phi}(z_i,b_i) + \delta\psi(z_i,b_i)] + \frac{\delta L}{\alpha}\right),\end{aligned}$$

where $\tilde{\phi}(z_i,b_i) := \phi(z_i,b_i) - H(z_i,b_i)$. Notice that by the periodicity of W, $W'(\phi(z_i,b_i)) = W'(\tilde{\phi}(z_i,b_i))$. Using $\dot{x}_i = -c_0 b_i L$ and $\mathcal{I}_1[\phi](z_i,b_i) = W'(\tilde{\phi}(z_i,b_i))$, we have

$$\begin{aligned}\Lambda(t,x) &= \sum_{i=1}^{N} c_0 L \phi'(z_i,b_i) + \sum_{i=1}^{N} \delta c_0 L \psi'(z_i,b_i) - \frac{1}{\delta}\sum_{i=1}^{N} W'(\tilde{\phi}(z_i,b_i)) - \sum_{i=1}^{N} \mathcal{I}_1[\psi](z_i,b_i)\\ &\quad + \frac{1}{\delta} W'\left(\sum_{i=1}^{N}[\tilde{\phi}(z_i,b_i) + \delta\psi(z_i,b_i)] + \frac{\delta L}{\alpha}\right)\\ &= c_0 L\phi'(z_{i_0},b_{i_0}) - \frac{1}{\delta}W'(\tilde{\phi}(z_{i_0},b_{i_0})) - \mathcal{I}_1[\psi](z_{i_0},b_{i_0}) - \frac{1}{\delta}\sum_{\substack{i=1\\ i\neq i_0}}^{N} W'(\tilde{\phi}(z_i,b_i))\\ &\quad + \frac{1}{\delta} W'\left(\tilde{\phi}(z_{i_0},b_{i_0}) + \sum_{\substack{i=1\\ i\neq i_0}}^{N} \tilde{\phi}(z_i,b_i) + \sum_{i=1}^{N} \delta\psi(z_i,b_i) + \frac{\delta L}{\alpha}\right) + E_0,\end{aligned}$$

where

$$E_0 := \sum_{\substack{i=1\\ i\neq i_0}}^{N} c_0 L\phi'(z_i,b_i) + \sum_{i=1}^{N} \delta c_0 L \psi'(z_i,b_i) - \sum_{\substack{i=1\\ i\neq i_0}}^{N} \mathcal{I}_1[\psi](z_i,b_i).$$

By Taylor expansion of W' around $\tilde{\phi}(z_{i_0},b_{i_0})$, we can write

$$\begin{aligned}\Lambda(t,x) = &\, c_0 L\phi'(z_{i_0},b_{i_0}) - \mathcal{I}_1[\psi](z_{i_0},b_{i_0}) - \frac{1}{\delta}\sum_{\substack{i=1\\ i\neq i_0}}^{N} W'(\tilde{\phi}(z_i,b_i))\\ &+ \frac{1}{\delta}W''(\tilde{\phi}(z_{i_0},b_{i_0}))\left(\sum_{\substack{i=1\\ i\neq i_0}}^{N} \tilde{\phi}(z_i,b_i) + \sum_{i=1}^{N} \delta\psi(z_i,b_i) + \frac{\delta L}{\alpha}\right) + E_0 + E_1,\end{aligned}$$

where

$$E_1 := O\left(\sum_{\substack{i=1\\ i\neq i_0}}^{N} \tilde{\phi}(z_i,b_i) + \sum_{i=1}^{N} \delta\psi(z_i,b_i) + \frac{\delta L}{\alpha}\right)^2.$$

Now, by Taylor expansion of W' around 0 and using that $\alpha = W''(0)$, we obtain

$$\begin{aligned}\Lambda(t,x) &= c_0L\phi'(z_{i_0},b_{i_0}) - \mathcal{I}_1[\psi](z_{i_0},b_{i_0}) + (W''(\tilde{\phi}(z_{i_0},b_{i_0})) - W''(0))\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{\tilde{\phi}(z_i,b_i)}{\delta} + L\\
&\quad + (W''(\tilde{\phi}(z_{i_0},b_{i_0})) - W''(0))\frac{L}{\alpha} + W''(\tilde{\phi}(z_{i_0},b_{i_0}))\sum_{i=1}^{N}\psi(z_i,b_i) + E_0 + E_1 + E_2\\
&= c_0L\phi'(z_{i_0},b_{i_0}) - \mathcal{I}_1[\psi](z_{i_0},b_{i_0}) + (W''(\tilde{\phi}(z_{i_0},b_{i_0})) - W''(0))\frac{L}{\alpha}\\
&\quad + W''(\tilde{\phi}(z_{i_0},b_{i_0}))\psi(z_{i_0},b_{i_0}) + W''(\tilde{\phi}(z_{i_0},b_{i_0}))\sum_{\substack{i=1\\i\neq i_0}}^{N}\psi(z_i,b_i)\\
&\quad + (W''(\tilde{\phi}(z_{i_0},b_{i_0})) - W''(0))\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{\tilde{\phi}(z_i,b_i)}{\delta} + L + E_0 + E_1 + E_2,\end{aligned}$$

where

$$E_2 := \sum_{\substack{i=1\\i\neq i_0}}^{N} O(\tilde{\phi}(z_i,b_i))^2.$$

By using equation (2.18), we obtain

$$\Lambda(t,x) = (W''(\tilde{\phi}(z_{i_0},b_{i_0})) - W''(0))\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{\tilde{\phi}(z_i,b_i)}{\delta} + L + E_0 + E_1 + E_2 + E_3, \tag{5.4}$$

where

$$E_3 := W''(\tilde{\phi}(z_{i_0},b_{i_0}))\sum_{\substack{i=1\\i\neq i_0}}^{N}\psi(z_i,b_i).$$

Similarly to Lemma 5.3 in [28], one can show using the estimates of Lemma 2.2 and Lemma 2.3 that

$$E_0 + E_1 + E_2 + E_3 = o_\varepsilon(1). \tag{5.5}$$

Now, by (2.16) and (4.2), we get

$$\left|\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{\tilde{\phi}(z_i,b_i)}{\delta}\right| \leqslant \frac{1}{\delta\alpha\pi}\left|\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{b_i\varepsilon\delta}{x-x_i(t)}\right| + \frac{K_1}{\delta}\left|\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{\varepsilon^2\delta^2}{(x-x_i(t))^2}\right| \leqslant \frac{1}{\alpha\pi}\left|\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{b_i\varepsilon}{x-x_i(t)}\right| + C\delta, \tag{5.6}$$

for some constant $C > 0$. Recall that $x_i(t) = x_i^0 - c_0b_iLt$ and that $x_{i_0}(t)$ is the closest point to x among the particles $x_i(t)$. Going back at time 0, it is easy to see that $x_{i_0}^0$ is the closest point to $x + c_0b_{i_0}Lt$. We write

$$\sum_{\substack{i=1\\i\neq i_0}}^{N}\frac{b_i\varepsilon}{x-x_i(t)} = \sum_{\substack{i=1\\i\neq i_0, x_i^0>x_0}}^{N}\frac{\varepsilon}{(x+c_0Lt)-x_i^0} - \sum_{\substack{i=1\\i\neq i_0, x_i^0<x_0}}^{N}\frac{\varepsilon}{(x-c_0Lt)-x_i^0}. \tag{5.7}$$

Assume $x \geqslant x_0$ so that $b_{i_0} = 1$. The case $x < x_0$ can be treated similarly. Denote $y(t) := x + c_0 Lt$. Then, by Lemma 4.11 we have that

$$\left| \sum_{\substack{i=1 \\ i\neq i_0, x_i^0 > x_0}}^{N} \frac{\varepsilon}{y(t) - x_i^0} \right| = \left| \sum_{\substack{i=1 \\ i\neq i_0}}^{N} \frac{\varepsilon b_i}{y(t) - x_i^0} - \sum_{\substack{i=1 \\ x_i^0 < x_0}}^{N} \frac{\varepsilon b_i}{y(t) - x_i^0} \right| \leqslant C + \left| \sum_{\substack{i=1 \\ x_i^0 < x_0}}^{N} \frac{\varepsilon}{y(t) - x_i^0} \right|.$$

Notice that by (5.2) and (5.3), if $x_i^0 < x_0$ then $x_i^0 < y(t) - \sigma$. Now, fix $\rho > \sigma$. By (4.1) the number of particles in $(y(t) - \rho, y(t) - \sigma)$ is bounded by $C\rho/\varepsilon$. This, together with $N\varepsilon \leqslant C$ yields

$$\left| \sum_{\substack{i=1 \\ x_i^0 < x_0}}^{N} \frac{\varepsilon}{y(t) - x_i^0} \right| \leqslant \left| \sum_{\substack{i=1 \\ x_i^0 \leqslant y(t)-\rho}}^{N} \frac{\varepsilon}{y(t) - x_i^0} \right| + \left| \sum_{\substack{i=1 \\ y(t)-\rho < x_i^0 < y(t)-\sigma}}^{N} \frac{\varepsilon}{y(t) - x_i^0} \right| \leqslant \frac{C}{\rho} + \frac{C\rho}{\sigma}.$$

Choosing $\rho = \sigma^{\frac{1}{2}}$ the two last estimates give

$$\left| \sum_{\substack{i=1 \\ i\neq i_0, x_i^0 > x_0}}^{N} \frac{\varepsilon}{(x + c_0 Lt) - x_i^0} \right| \leqslant \frac{C}{\sigma^{\frac{1}{2}}}. \tag{5.8}$$

If $x_i^0 < x_0$, then by (5.2) and (5.3), $x_i^0 < (x - c_0 Lt) - \sigma/2$ for any $t \in [0, \sigma/(2c_0 L)]$, so that similar computations as above yield

$$\left| \sum_{\substack{i=1 \\ i\neq i_0, x_i^0 < x_0}}^{N} \frac{\varepsilon}{(x - c_0 Lt) - x_i^0} \right| \leqslant \frac{C}{\sigma^{\frac{1}{2}}}. \tag{5.9}$$

Combining (5.4), (5.5), (5.6), (5.7), (5.8) and (5.9), we finally obtain

$$\Lambda(t, x) \geqslant -\frac{C}{\sigma^{\frac{1}{2}}} + L \geqslant 0,$$

which implies that H^ε is a supersolution of (1.1) by choosing $L = C/\sigma^{\frac{1}{2}}$ with $C > 0$ sufficiently large. $\square$

6. Proof of Theorem 1.1

In this section, we prove our main Theorem 1.1. We first show that the functions u^ε are bounded uniformly in ε. Since $W'(z) = 0$ for any $z \in \mathbb{Z}$, integers are stationary solutions to (1.1). Let $\lambda_1, \lambda_2 \in \mathbb{Z}$ be such that $\lambda_1 \leqslant \inf_{\mathbb{R}} u_0 \leqslant \sup_{\mathbb{R}} u_0 \leqslant \lambda_2$. Then by the comparison principle we have that for any $\varepsilon > 0$

$$\lambda_1 \leqslant u^\varepsilon(t, x) \leqslant \lambda_2 \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{R}.$$

In particular, $u^+ := \limsup^*_{\varepsilon\to 0} u^\varepsilon$ is everywhere finite. We will prove that u^+ is a viscosity subsolution of (1.4). Similarly, we can prove that $u^- := \liminf_{*\varepsilon\to 0} u^\varepsilon$ is a supersolution of (1.4). We will then show that

$$u^+(0, x) \leqslant u_0(x) \leqslant u^-(0, x), \tag{6.1}$$

with u_0 the initial condition in (1.4). The proof of (6.1) is postponed to Section 7. Then, if $\overline{u}$ is the viscosity solution of (1.4), by the comparison principle,

$$u^+ \leqslant \overline{u} \leqslant u^-. \tag{6.2}$$

Since the reverse inequality $u^- \leqslant u^+$ always holds true, we conclude that the two functions coincide with $\overline{u}$ and that $u^\varepsilon \to \overline{u}$ as $\varepsilon \to 0$, uniformly on compact sets.

Let us start by proving that u^+ is a viscosity subsolution of (1.4). Let η be a smooth bounded function such that

$$u^+(t,x) - \eta(t,x) < u^+(t_0,x_0) - \eta(t_0,x_0) = 0 \quad \text{for all } (t,x) \neq (t_0,x_0). \tag{6.3}$$

We separate the proof into two cases.

6.1. Case 1: Test function η with $\partial_x \eta(t_0,x_0) = 0$.

In this case, we want to show that

$$\partial_t \eta(t_0,x_0) \leqslant 0. \tag{6.4}$$

Without loss of generality, we may assume that η has the form

$$\eta(t,x) = h(x) + g(t) \tag{6.5}$$

with g any smooth function and h satisfying

$$\begin{cases} h(x) = a(x-x_0)^2 \text{ for } |x-x_0| < \rho \\ h \text{ is non-increasing in } (-\infty, x_0) \\ h \text{ is non-decreasing in } (x_0, +\infty) \end{cases} \tag{6.6}$$

for some $a, \rho > 0$. Fix $\sigma > 0$ such that $4\sigma < \rho$.

We are going to construct a global in space supersolution of (1.1) in an interval around t_0 by using Lemma 5. We cannot apply the lemma to the function η as the required flat condition is not satisfied by η. Therefore, we consider the function $\eta^\sigma(t,x) := h^\sigma(x) + g(t)$, where $h^\sigma(x) = \max\{h(x), a(2\sigma)^2\}$. Notice that $\eta^\sigma \geqslant \eta$, $\eta^\sigma \leqslant \eta + a(2\sigma)^2$ and η^σ is constant in the interval $(x_0 - 2\sigma, x_0 + 2\sigma)$. However, η^σ is not of class $C^{1,1}$. To overcome this problem, we consider any $C^{1,1}$ function $\tilde{\eta}^\sigma$ such that

$$\tilde{\eta}^\sigma(t,x) = \tilde{h}^\sigma(x) + g(t),$$

with $\tilde{h}^\sigma$ satisfying

$$\begin{cases} \tilde{h}^\sigma \geqslant h^\sigma \geqslant h & \\ \tilde{h}^\sigma(x) = h^\sigma(x) = a(2\sigma)^2 & \text{if } |x-x_0| \leqslant \sigma \\ \tilde{h}^\sigma \text{ is non-increasing in } (-\infty, x_0) & \\ \tilde{h}^\sigma \text{ is non-decreasing in } (x_0, +\infty). & \end{cases} \tag{6.7}$$

For example, the function $\tilde{h}^\sigma(x)$ defined for $|x-x_0| < \rho$ by

$$\tilde{h}^\sigma(x) = \begin{cases} 2a(x-x_0-\sigma)^2 + 4a\sigma^2 & \text{if } x > x_0 + \sigma \\ 4a\delta^2 & \text{if } |x-x_0| \leqslant \sigma \\ 2a(x-x_0+\sigma)^2 + 4a\sigma^2 & \text{if } x < x_0 - \sigma \end{cases}$$

and extended to be monotonic, bounded and above h outside $(x_0-\rho, x_0+\rho)$ would work. Moreover, since u^ε is bounded uniformly in ε, without loss of generality we can assume that for all $\varepsilon > 0$ and all $t > 0$,

$$\tilde{\eta}^\sigma(t,x) \geqslant u^\varepsilon(t,x) \quad \text{if } |x-x_0| > 1. \tag{6.8}$$

Finally, since $u^+ - \eta$ attains a strict maximum at (t_0, x_0) and $\tilde{\eta}^\sigma \geqslant \eta$, for all $R > 0$ there exists $\varepsilon_0 = \varepsilon_0(R)$ such that for $\varepsilon < \varepsilon_0$

$$u^\varepsilon(t,x) - \tilde{\eta}^\sigma(t,x) < 0 \quad \text{for all } (t,x) \in Q_{1,1}(t_0,x_0) \setminus Q_{R,R}(t_0,x_0). \tag{6.9}$$

Next, we set

$$c := \frac{1}{4c_0 L}, \tag{6.10}$$

with L to be determined. Define x_i^0 and b_i, $i = 1, \dots, N_\varepsilon$ as in as in (2.2) and (2.3) for the function $\tilde{\eta}^\sigma(t_0 - c\sigma, \cdot)$. Since $\tilde{\eta}^\sigma(t, \cdot)$ is monotonic in $(-\infty, x_0)$ and (x_0, ∞), the number of particles in each interval is bounded by $(\sup \tilde{\eta}^\sigma - \inf \tilde{\eta}^\sigma)/\varepsilon$ so that $\varepsilon N_\varepsilon \leqslant C$. In particular the condition $\varepsilon^2 N_\varepsilon \delta = o_\varepsilon(1)$ is satisfied and we are in position to apply Proposition 4.10 (recall Remark 4.12) to get

$$\tilde{\eta}^\sigma(t_0 - c\sigma, x) = \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1), \tag{6.11}$$

where $M_\varepsilon := \lceil \tilde{\eta}^\sigma(t_0 - c\sigma, -\infty)/\varepsilon \rceil$. By (6.8), (6.9) with $R = c\sigma$, and (6.11) we also have

$$u^\varepsilon(t_0 - c\sigma, x) \leqslant \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1). \tag{6.12}$$

Let $x_i(t)$ be the solution of the ODE system (2.9) with initial condition $x_i(t_0 - c\sigma) = x_i^0$, that is

$$x_i(t) = x_i^0 - b_i c_0 L[t - (t_0 - c\sigma)].$$

Define

$$H^\varepsilon(t,x) := \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \sum_{i=1}^{N_\varepsilon} \varepsilon\delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + \frac{\varepsilon\delta L}{\alpha} + \varepsilon\left\lceil \frac{o_\varepsilon(1)}{\varepsilon} \right\rceil,$$

with ϕ and ψ the solutions of (1.8) and (2.18) respectively. Notice that

$$\left| \sum_{i=1}^{N_\varepsilon} \varepsilon\delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) \right| \leqslant C\varepsilon N_\varepsilon \delta \leqslant C\delta = o_\varepsilon(1). \tag{6.13}$$

By (6.12) and (6.13) we can choose $o_\varepsilon(1)$ in the definition of H^ε such that

$$H^\varepsilon(t_0 - c\sigma, x) \geqslant u^\varepsilon(t_0 - c\sigma, x).$$

Now, by Lemma 5.1 if

$$L = \frac{C_0}{\sigma^{\frac{1}{2}}} \tag{6.14}$$

with C_0 large enough, the function H^ε is supersolution of (1.1) in $[t_0 - c\sigma, t_0 + c\sigma] \times \mathbb{R}$. Therefore, by the comparison principle, we obtain

$$H^\varepsilon(t,x) \geqslant u^\varepsilon(t,x) \quad \text{for any } (t,x) \in [t_0 - c\sigma, t_0 + c\sigma] \times \mathbb{R}. \tag{6.15}$$

Consider a sequence $(t_\varepsilon, x_\varepsilon)$ converging to (t_0, x_0) as $\varepsilon \to 0$. By (6.15) and (6.13) we have that

$$\begin{aligned} u^\varepsilon(t_\varepsilon, x_\varepsilon) &\leqslant H^\varepsilon(t_\varepsilon, x_\varepsilon) \\ &= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_\varepsilon - x_i(t_\varepsilon)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\ &= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_\varepsilon + b_i c_0 L(t_\varepsilon - t_0 + c\sigma)) - x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1). \end{aligned} \tag{6.16}$$

Next we use the following result.

Lemma 6.1. *We have that,*

$$\begin{aligned} &\sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_\varepsilon + b_i c_0 L(t_\varepsilon - t_0 + c\sigma)) - x_i^0}{\varepsilon\delta}, b_i\right) \\ &= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_\varepsilon + c_0 L(t_\varepsilon - t_0 + c\sigma)) - x_i^0}{\varepsilon\delta}, b_i\right) + o_\varepsilon(1). \end{aligned} \tag{6.17}$$

We postpone the proof of Lemma 6.1 to Section 9.

Now, from (6.16), Lemma 6.1, Proposition 4.10, the definition of x_i^0 and using that $\tilde{\eta}^\sigma(t,x) \leqslant \eta(t,x) + C\sigma^2$ if $|x - x_0| \leqslant \sigma$, we infer that

$$\begin{aligned} u^\varepsilon(t_\varepsilon, x_\varepsilon) &\leqslant \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_\varepsilon + b_i c_0 L(t_\varepsilon - t_0 + c\sigma)) - x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\ &= \tilde{\eta}^\sigma(t_0 - c\sigma, x_\varepsilon + c_0 L(t_\varepsilon - t_0 + c\sigma)) + o_\varepsilon(1) \\ &\leqslant \eta(t_0 - c\sigma, x_\varepsilon + c_0 L(t_\varepsilon - t_0 + c\sigma)) + o_\varepsilon(1) + C\sigma^2. \end{aligned}$$

By passing to $\limsup^*$ as $\varepsilon \to 0$, we obtain

$$u^+(t_0, x_0) \leqslant \eta(t_0 - c\sigma, x_0 + cc_0 L\sigma) + C\sigma^2.$$

Since $u^+(t_0, x_0) = \eta(t_0, x_0)$, we also have

$$\eta(t_0, x_0) - \eta(t_0 - c\sigma, x_0) \leqslant \eta(t_0 - c\sigma, x_0 + cc_0 L\sigma) - \eta(t_0 - c\sigma, x_0) + C\sigma^2,$$

by subtracting $\eta(t_0 - c\sigma, x_0)$ on both sides. Now, recalling the expression of η with h as in (6.6), (6.10) and (6.14), we see that the inequality above yields

$$\eta(t_0, x_0) - \eta(t_0 - k_0\sigma^{\frac{3}{2}}, x_0) \leqslant a\left(\frac{\sigma}{4}\right)^2 + C\sigma^2,$$

where $k_0 := 1/(4c_0 C_0)$. By dividing both sides by $k_0\sigma^{\frac{3}{2}}$ and taking the limit as $\sigma \to 0^+$, we finally get (6.4).

6.2. **Case 2: Test function η with $\partial_x \eta(t_0, x_0) \neq 0$.**

Without loss of generality we assume that

$$\partial_x \eta(t_0, x_0) > 0. \tag{6.18}$$

The goal is to show that

$$\partial_t \eta(t_0, x_0) \leqslant c_0 \partial_x \eta(t_0, x_0)\, \mathcal{I}_1[\eta(t_0, \cdot)](x_0). \tag{6.19}$$

We start with the following asymptotic result whose proof is postponed to Section 8.

Lemma 6.2. *Let* v_1, v_2, w_1, w_2 *be defined as in* (2.12) *and* (2.13). *Then, there exists* $L > 0$ *such that for all* $(t,x) \in (0,+\infty) \times \mathbb{R}$

$$\max\{v_1(x+c_0Lt), w_1(x-c_0Lt)\} \leqslant u^-(t,x) \leqslant u^+(t,x) \leqslant \min\{v_2(x+c_0Lt), w_2(x-c_0Lt)\}.$$

Without loss of generality we may assume that η satisfies

$$\eta(t,x) = v_2(x+c_0Lt) \quad \text{if } x < -K, \qquad \eta(t,x) = w_2(x-c_0Lt) \quad \text{if } x > K, \tag{6.20}$$

for K large enough and $L > 0$ given in Lemma 6.2. Indeed, assume that (6.19) holds true for any test function satisfying (6.20). If $\tilde{\eta}$ is any test function satisfying (6.3), by Lemma 6.2 we can always build a function η such that $\eta = \tilde{\eta}$ in a neighborhood of (t_0,x_0), $\eta \leqslant \tilde{\eta}$, and η satisfies (6.20). By $\partial_t\eta(t_0,x_0) = \partial_t\tilde{\eta}(t_0,x_0)$, $\partial_x\eta(t_0,x_0) = \partial_x\tilde{\eta}(t_0,x_0)$ and $\mathcal{I}_1[\eta(t_0,\cdot)](x_0) \leqslant \mathcal{I}_1[\tilde{\eta}(t_0,\cdot)](x_0)$ and (6.19) we infer that

$$\partial_t\tilde{\eta}(t_0,x_0) \leqslant c_0\partial_x\tilde{\eta}(t_0,x_0)\,\mathcal{I}_1[\tilde{\eta}(t_0,\cdot)](x_0),$$

as desired. Condition (6.20) implies that for any $T > 0$ the points $x_i = x_i(t)$ defined as in (2.2) for the function $\eta(t,\cdot)$ with $t \in [0,T]$ belong to the set $[-K_\varepsilon - c_0LT, K_\varepsilon + c_0LT]$ with K_ε defined as in Section 2.2. Therefore, the number of such particles $N_\varepsilon = N_\varepsilon(t)$ satisfies $N_\varepsilon \leqslant C(K_\varepsilon + c_0LT)/\varepsilon$. In particular, by (2.14),

$$\varepsilon^2 N_\varepsilon \delta = o_\varepsilon(1). \tag{6.21}$$

This will allow us to apply Proposition 4.10 to $v(x) = \eta(t,x)$ with t close to t_0.

Next, the proof of (6.19) is an adaptation of the proof given in [28] in the monotonic case, therefore we will skip some details and refer to the corresponding results in [28].

Suppose by contradiction that

$$\partial_t\eta(t_0,x_0) > c_0\partial_x\eta(t_0,x_0)\,\mathcal{I}_1[\eta(t_0,\cdot)](x_0). \tag{6.22}$$

Denote

$$L_0 := \mathcal{I}_1[\eta(t_0,\cdot)](x_0).$$

By (6.18) and (6.22), there exist $0 < \rho < 1$ and $L_1 > 0$ such that

$$\partial_x\eta(t,x) \geqslant \frac{\partial_x\eta(t_0,x_0)}{2} > 0 \quad \text{for all } (t,x) \in Q_{2\rho,2\rho}(t_0,x_0), \tag{6.23}$$

and

$$\partial_t\eta(t,x) \geqslant c_0\partial_x\eta(t,x)(L_0+L_1) \quad \text{for all } (t,x) \in Q_{2\rho,2\rho}(t_0,x_0). \tag{6.24}$$

Define $x_i^0 = x_i(t_0)$ and b_i, $i = 1,\dots,N_\varepsilon$, as in (2.2) and (2.3) for the function η at $t = t_0$. For $0 < R \ll \rho$ to be determined, let $x^0_{M_\rho}$ be the biggest point which is smaller than $x_0 - (\rho+R)$, and $x^0_{N_\rho}$ the lowest point bigger than $x_0 + (\rho+R)$, that is

$$x^0_{M_\rho} < x_0 - (\rho+R) \leqslant x^0_{M_\rho+1} \tag{6.25}$$

and

$$x^0_{N_\rho-1} \leqslant x_0 + (\rho+R) < x^0_{N_\rho}. \tag{6.26}$$

In other words, $\{x^0_{M_\rho}, x^0_{M_\rho+1}, ..., x^0_{N_\rho-1}, x^0_{N_\rho}\}$ are the particle points in the interval $(x_0-(\rho+R), x_0+(\rho+R))$. By definition, there exists $J_0 \in \{1,...,N_\varepsilon\}$ such that $\eta(t_0,x^0_{M_\rho}) = J_0\varepsilon$, and since $\eta(t_0,\cdot)$ is increasing in $(x_0-(\rho+R), x_0+(\rho+R))$, we have that

$$\eta(t_0,x^0_{M_\rho+i}) = (i+J_0)\varepsilon, \quad \text{for } i = 0,1,...,N_\rho - M_\rho := K_\rho.$$

Define $B_0 := \partial_x\eta(t_0,x_0)/(2\|\partial_t\eta\|_\infty)$. Now, for any time t such that $|t-t_0|<B_0R$, we define a set

$$X_i(t) := \{x \in (x_0-(\rho+3R), x_0+(\rho+3R)) \ : \ \eta(t,x)=(i+J_0)\varepsilon\}, \tag{6.27}$$

for $i=0,1,...,K_\rho$.

Lemma 6.3. *Let $B_0 := \partial_x\eta(t_0,x_0)/(2\|\partial_t\eta\|_\infty)$ and $X_i(t)$ be defined by (6.27), $i=0,\dots,K_\rho$. Then, there exists $\varepsilon_0=\varepsilon_0(\rho)$ such that for $\varepsilon<\varepsilon_0$ and $R<\rho/3$, $X_i(t)$ is a singleton, that is, $X_i(t)=\{\zeta^i(t)\}$, and $\zeta^i\in C^1(t_0-B_0R,t_0+B_0R)$ and for $|t-t_0|<B_0R$,*

$$|\dot{\zeta}^i(t)| \leqslant B_0^{-1}, \tag{6.28}$$

$$x_0+\rho<\zeta^{K_\rho}(t)<x_0+\rho+3R, \tag{6.29}$$

$$x_0-(\rho+3R)<\zeta^0(t)<x_0-\rho. \tag{6.30}$$

In particular $(t,\zeta^i(t))\in Q_{2\rho,2\rho}(t_0,x_0)$.

Proof. By the monotonicity of η, $X_i(t)$ is a singleton. The rest of the proof of Lemma 6.3 directly follows the proof of Lemma 5.1 in [28]. ☐

Therefore, by choosing $R<\rho/3$, we have that $(t,\zeta^i(t))\in Q_{2\rho,2\rho}(t_0,x_0)$ and

$$\eta(t,\zeta^i(t))=(i+J_0)\varepsilon, \tag{6.31}$$

for $i=0,1,...,K_\rho$. By Lemma 6.3, $\zeta^i(t)$ is of class $C^1(t_0-B_0R,t_0+B_0R)$, allowing us to differentiate (6.31) in t, which yields

$$\partial_t\eta(t,\zeta^i(t))+\partial_x\eta(t,\zeta^i(t))\dot{\zeta}^i(t)=0.$$

Using (6.24), for $|t-t_0|<B_0R$, we obtain

$$-\dot{\zeta}_i(t)\geqslant c_0(L_0+L_1), \quad i=0,1,\dots,K_\rho. \tag{6.32}$$

Next, we will construct a supersolution of (1.1) in $Q_{B_0R,R}(t_0,x_0)$ for $R\ll\rho<1$. Since the maximum of $u^+-\eta$ is strict, there exists $\gamma_R>0$ such that

$$u^+-\eta\leqslant -2\gamma_R<0 \quad \text{in } Q_{2\rho,2\rho}(t_0,x_0)\setminus Q_{B_0R,R}(t_0,x_0). \tag{6.33}$$

Then, we define

$$\Phi^\varepsilon(t,x):=\begin{cases} h^\varepsilon(t,x)+\frac{\varepsilon\delta L_1}{\alpha}-\varepsilon\left\lfloor\frac{\gamma_R}{\varepsilon}\right\rfloor & \text{for } (t,x)\in Q_{B_0R,\frac{\rho}{2}}(t_0,x_0)\\ u^\varepsilon(t,x) & \text{outside,}\end{cases} \tag{6.34}$$

where

$$\begin{aligned} h^\varepsilon(t,x)=&\sum_{i=0}^{K_\rho}\varepsilon\left(\phi\left(\frac{x-\zeta^i(t)}{\varepsilon\delta},1\right)+\delta\psi\left(\frac{x-\zeta^i(t)}{\varepsilon\delta},1\right)\right)\\ &+\sum_{i=1}^{M_\rho-1}\varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta},b_i\right)+\sum_{i=N_\rho+1}^{N_\varepsilon}\varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta},b_i\right)\end{aligned} \tag{6.35}$$

with ϕ a solution the (1.8) and ψ the solution of (2.18) with $L=L_0+L_1$.

Lemma 6.4. *There exist* $0 < R \ll \rho$ *and* $\varepsilon_0 = \varepsilon_0(R, \rho) > 0$ *such that for any* $\varepsilon < \varepsilon_0$*, the function* Φ^ε *defined by* (6.34) *satisfies*

$$\Phi^\varepsilon \geqslant u^\varepsilon \quad \textit{outside } Q_{B_0R,R}(t_0, x_0), \tag{6.36}$$

$$\Phi^\varepsilon \leqslant \eta + o_\varepsilon(1) - \varepsilon \left\lfloor \frac{\gamma_R}{\varepsilon} \right\rfloor \quad \textit{in } Q_{B_0R,R}(t_0, x_0), \tag{6.37}$$

and

$$\delta \partial_t \Phi^\varepsilon \geqslant \mathcal{I}_1[\Phi^\varepsilon] - \frac{1}{\delta} W' \left(\frac{\Phi_\varepsilon}{\varepsilon} \right) \quad \textit{in } Q_{B_0R,R}(t_0, x_0), \tag{6.38}$$

We are now in position to conclude the proof for Case 2. By (6.36) and (6.38) and the comparison principle, Proposition 2.7, we have

$$u^\varepsilon(t, x) \leqslant \Phi^\varepsilon(t, x) \quad \text{for all } (t, x) \in Q_{B_0R,R}(t_0, x_0).$$

Passing to the upper limit as $\varepsilon \to 0$ and using (6.37) and that $u^+(t_0, x_0) = \eta(t_0, x_0)$, we obtain

$$0 \leqslant -\gamma_R,$$

which is a contradiction. This completes the proof of (6.19).

Proof of Lemma 6.4. We divide the proof of Lemma 6.4 in several steps. To prove (6.36) and (6.37), we will need the following lemma whose proof is postponed to Section 9.

Lemma 6.5. *There exists* $\varepsilon_0 = \varepsilon_0(R, \rho) > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *and for any* $(t, x) \in Q_{B_0R,\rho-R}(t_0, x_0)$*, we have*

$$|h^\varepsilon(t, x) - \eta(t, x)| \leqslant o_\varepsilon(1).$$

Proof of (6.36). By definition (6.34) of Φ^ε, $\Phi^\varepsilon(t, x) = u^\varepsilon(t, x)$ outside of $Q_{B_0R,\frac{\rho}{2}}(t_0, x_0)$. Next, by Lemma 6.5 and (6.33), for $(t, x) \in Q_{B_0R,\frac{\rho}{2}}(t_0, x_0) \setminus Q_{B_0R,R}(t_0, x_0)$,

$$\begin{aligned} \Phi^\varepsilon(t, x) &= h^\varepsilon(t, x) + \frac{\varepsilon \delta L_1}{\alpha} - \varepsilon \left\lfloor \frac{\gamma_R}{\varepsilon} \right\rfloor \\ &\geqslant \eta(t, x) + o_\varepsilon(1) - \varepsilon \left\lfloor \frac{\gamma_R}{\varepsilon} \right\rfloor \\ &\geqslant u^\varepsilon(t, x). \end{aligned}$$

This concludes the proof of (6.36).

Proof of (6.37). By Lemma 6.5, for $(t, x) \in Q_{B_0R,R}(t_0, x_0)$

$$\Phi^\varepsilon(t, x) = h^\varepsilon(t, x) + \frac{\varepsilon \delta L_1}{\alpha} - \varepsilon \left\lfloor \frac{\gamma_R}{\varepsilon} \right\rfloor \leqslant \eta(t, x) + o_\varepsilon(1) - \varepsilon \left\lfloor \frac{\gamma_R}{\varepsilon} \right\rfloor,$$

which gives (6.37).

For $|x - x_0| \geqslant \rho - R$ we obtain a worse approximation result than the one in Lemma 6.5 as shown below. This is due to the fact that we have choosen the particles x_i to be constant in time, equal to x_i^0, for $i < M_\rho$ and $i > N_\rho$.

Lemma 6.6. *There exists* $\varepsilon_0 = \varepsilon_0(R, \rho) > 0$ *such that for any* $\varepsilon < \varepsilon_0$*, if* $|t - t_0| < B_0R$*, and* $|x - x_0| \geqslant \rho - R$*, then*

$$|h^\varepsilon(t, x) - \eta(t, x)| \leqslant o_\varepsilon(1) + O(R).$$

We postpone the proof of Lemma 6.6 to Section 9.

Corollary 6.7. *There exists $\varepsilon_0 = \varepsilon_0(R,\rho) > 0$ such that for any $\varepsilon < \varepsilon_0$, $R < \rho/4$, and any $(t,x) \in Q_{B_0R,R}(t_0,x_0)$, we have*

$$\mathcal{I}_1[\Phi^\varepsilon(t,\cdot)](x) \leqslant \mathcal{I}_1[h^\varepsilon(t,\cdot)](x) + o_\varepsilon(1) + \frac{o_R(1)}{\rho}. \tag{6.39}$$

Proof. The corollary is a consequence of Lemma 6.5, Lemma 6.6 and the definition (6.34) of Φ^ε. For details, we refer to the proof of Corollary 5.7 in [28]. □

Now, we are ready to prove (6.38).

Proof of (6.38). Denote

$$\Lambda := \delta\partial_t\Phi^\varepsilon - \mathcal{I}_1[\Phi^\varepsilon] + \frac{1}{\delta}W'\left(\frac{\Phi_\varepsilon}{\varepsilon}\right).$$

We want to show that $\Lambda(t,x) \geqslant 0$ for all $(t,x) \in Q_{B_0R,R}(t_0,x_0)$. Fix $(\bar t,\bar x) \in Q_{B_0R,R}(t_0,x_0)$. Let i_0 be such that $\zeta^{i_0}(\bar t)$ is the closest point to $\bar x$. Then, $\bar x = \zeta^{i_0}(\bar t) + \varepsilon\gamma$, with $|\gamma| \leqslant 2/|\partial_x\eta(t_0,x_0)|$ by (4.3) and (6.23). Define

$$z_i(t) := \frac{x-\zeta^i(t)}{\varepsilon\delta}, \qquad z_i^0 := \frac{x-x_i^0}{\varepsilon\delta} \quad \text{and} \quad \tilde\phi(z,b_i) := \phi(z,b_i) - H(z,b_i),$$

with $H(z,b)$ defined as in (2.8). Using Corollary 6.7, equations (1.8) and (2.18), performing Taylor expansions, as in the proof of Lemma 5.3 in [28], we obtain

$$\begin{aligned}\Lambda(\bar t,\bar x) \geqslant{}& (W''(\phi(z_{i_0}),1) - W''(0))\left(\frac{1}{\delta}\sum_{\substack{i=0\\ i\neq i_0}}^{K_\rho}\tilde\phi(z_i,1) + \frac{1}{\delta}\sum_{i=1}^{M_\rho-1}\tilde\phi(z_i^0,b_i) + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon}\tilde\phi(z_i^0,b_i) - \frac{L_0}{\alpha}\right)\\ &+ L_1 + E_0 + E_1 + E_2 + E_3 + E_4,\end{aligned} \tag{6.40}$$

where

$$E_0 = o_\varepsilon(1) + \frac{o_R(1)}{\rho}$$

$$E_1 = -\sum_{\substack{i=0\\ i\neq i_0}}^{K_\rho}\dot\zeta^i(\bar t)\phi'(z_i,1) - \delta\sum_{\substack{i=0\\ i\neq i_0}}^{K_\rho}\dot\zeta^i(\bar t)\psi'(z_i,1) - \delta\dot\zeta^{i_0}(\bar t)\psi'(z_{i_0},1)$$

$$E_2 = \frac{1}{\delta}O\left(\sum_{\substack{i=0\\ i\neq i_0}}^{K_\rho}[\tilde\phi(z_i,1)+\delta\psi(z_i,1)] + \delta\psi(z_{i_0},1) + \sum_{i=1}^{M_\rho-1}\tilde\phi(z_i^0,b_i) + \sum_{i=N_\rho+1}^{N_\varepsilon}\tilde\phi(z_i^0,b_i) + \frac{\delta L_1}{\alpha}\right)^2$$

$$E_3 = \frac{1}{\delta}\sum_{\substack{i=0\\ i\neq i_0}}^{K_\rho}O(\tilde\phi(z_i,1))^2 + \frac{1}{\delta}\sum_{i=1}^{M_\rho-1}O(\tilde\phi(z_i^0,b_i))^2 + \frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon}O(\tilde\phi(z_i^0,b_i))^2$$

$$E_4 = W''(\tilde\phi(z_{i_0},1))\sum_{\substack{i=0\\ i\neq i_0}}^{K_\rho}\psi(z_i,1) - \sum_{\substack{i=0\\ i\neq i_0}}^{K_\rho}\mathcal{I}_1[\psi](z_i,1).$$

We have estimates for the error terms E_1, E_2, E_3 and E_4 as stated in the following lemma.

Lemma 6.8. *For $i \geqslant 1$, the error E_i defined as above satisfies*

$$E_i = O(\delta).$$

Proof. The proof follows directly the proof of Lemma 5.9 in [28]. $\square$

Furthermore, we claim the following.

Lemma 6.9.
(6.41)

$$(W''(\phi(z_{i_0},1)) - W''(0)) \left(\frac{1}{\delta} \sum_{\substack{i=0 \\ i \neq i_0}}^{K_\rho} \tilde{\phi}(z_i, 1) - \frac{1}{\alpha} \mathcal{I}_1^{1,\rho}[\eta(t_0,\cdot)](x_0) \right) = o_\varepsilon(1) + o_R(1) + o_\rho(1) + O\left(\frac{R}{\rho}\right),$$

and

$$\frac{1}{\delta} \sum_{i=1}^{M_\rho - 1} \tilde{\phi}(z_i^0, b_i) + \frac{1}{\delta} \sum_{i=N_\rho+1}^{N_\varepsilon} \tilde{\phi}(z_i^0, b_i) - \frac{1}{\alpha} \mathcal{I}_1^{2,\rho}[\eta(t_0,\cdot)](x_0) = o_\varepsilon(1) + o_\rho(1) + O\left(\frac{R}{\rho}\right). \tag{6.42}$$

Proof. By the monotonicity of η in $Q_{B_0 R, R}(t_0, x_0)$, the proof of (6.41) directly follows the proof of Lemma 5.8 in [28]. With a slight modification of the proof of (5.32) in Lemma 5.8 in [28], using (4.2) and Lemma 4.3 presented in this paper, we now show the estimate (6.42). By (2.16) and (4.2), we have that

(6.43)

$$\begin{aligned} \frac{1}{\delta} \sum_{i=1}^{M_\rho - 1} \tilde{\phi}(z_i^0, b_i) + \frac{1}{\delta} \sum_{i=N_\rho+1}^{N_\varepsilon} \tilde{\phi}(z_i^0, b_i) &\leqslant \frac{1}{\alpha\pi} \sum_{i=1}^{M_\rho-1} \frac{\varepsilon b_i}{x_i^0 - x} + K_1 \sum_{i=1}^{M_\rho-1} \frac{\varepsilon^2 \delta}{(x_i^0 - x)^2} \\ &\quad + \frac{1}{\alpha\pi} \sum_{i=N_\rho+1}^{N_\varepsilon} \frac{\varepsilon b_i}{x_i^0 - x} + K_1 \sum_{i=N_\rho+1}^{N_\varepsilon} \frac{\varepsilon^2 \delta}{(x_i^0 - x)^2} \\ &\leqslant \frac{1}{\alpha\pi} \sum_{i=1}^{M_\rho-1} \frac{\varepsilon b_i}{x_i^0 - x} + \frac{1}{\alpha\pi} \sum_{i=N_\rho+1}^{N_\varepsilon} \frac{\varepsilon b_i}{x_i^0 - x} + O(\delta). \end{aligned}$$

Since $|x - x_0| < R$ and $|x_i^0 - x_0| > \rho + R$ for $i \leqslant M_\rho - 1$ and $i \geqslant N_\rho + 1$, we have that for those indices $|x - x_i^0| \geqslant \rho$. However, there may be particles x_i^0 with $i = M_\rho, \dots, N_\rho$ for which $|x - x_i^0| \geqslant \rho$. Therefore, we can write

$$\frac{1}{\alpha\pi} \sum_{i=1}^{M_\rho-1} \frac{\varepsilon b_i}{x_i^0 - x} + \frac{1}{\alpha\pi} \sum_{i=N_\rho+1}^{N_\varepsilon} \frac{\varepsilon b_i}{x_i^0 - x} = \frac{1}{\alpha\pi} \sum_{\substack{i=M_\varepsilon \\ |x - x_i^0| \geqslant \rho}}^{N_\varepsilon} \frac{\varepsilon b_i}{x_i^0 - x} - \frac{1}{\alpha\pi} \sum_{\substack{i=M_\rho \\ |x - x_i^0| \geqslant \rho}}^{N_\rho} \frac{\varepsilon b_i}{x_i^0 - x}. \tag{6.44}$$

Notice that, by (4.1), (6.25), (6.26), and $|x - x_0| < R$, the number of particles x_i^0, $i = M_\rho, \dots, N_\rho$ such that $|x - x_i^0| \geqslant \rho$ is bounded by CR/ε. Therefore,

$$\left| \sum_{\substack{i=M_\rho \\ |x - x_i^0| \geqslant \rho}}^{N_\rho} \frac{\varepsilon b_i}{x_i^0 - x} \right| \leqslant \frac{CR}{\rho}.$$

By Lemma 4.3 and the estimate above, then (6.44) becomes

(6.45)
$$\frac{1}{\alpha\pi}\sum_{i=1}^{M_\rho-1}\frac{\varepsilon b_i}{x_i^0-x}+\frac{1}{\alpha\pi}\sum_{i=N_\rho+1}^{N_\varepsilon}\frac{\varepsilon b_i}{x_i^0-x}=\mathcal{I}_1^{2,\rho}[\eta(t_0,\cdot)](x_0)+o_\varepsilon(1)+o_\rho(1)+O\left(\frac{R}{\rho}\right).$$

Combining (6.43) and (6.45) yields

$$\frac{1}{\delta}\sum_{i=1}^{M_\rho-1}\tilde{\phi}(z_i^0,b_i)+\frac{1}{\delta}\sum_{i=N_\rho+1}^{N_\varepsilon}\tilde{\phi}(z_i^0,b_i)-\frac{1}{\alpha}\mathcal{I}_1^{2,\rho}[\eta(t_0,\cdot)](x_0)\leqslant o_\varepsilon(1)+o_\rho(1)+O\left(\frac{R}{\rho}\right).$$

Similarly, one can prove the opposite inequality. This proves (6.42).

□

As a consequence of Lemma 6.8 and Lemma 6.9, the inequality (6.40) becomes

$$\Lambda(\bar{t},\bar{x})\geqslant L_1+o_\varepsilon(1)+o_R(1)+o_\rho(1)+\frac{o_R(1)}{\rho}.$$

We choose $R\ll\rho\ll 1$ and ε_0 so small that for any $\varepsilon<\varepsilon_0$,

$$\left|o_\varepsilon(1)+o_R(1)+o_\rho(1)+\frac{o_R(1)}{\rho}\right|<\frac{L_1}{2}.$$

Then,

$$\Lambda(\bar{t},\bar{x})>\frac{L_1}{2}>0.$$

This completes the proof of (6.38).

7. Proof of (6.1)

To prove (6.1) we are going to build supersolutions of (1.1) for small times to compare to u^ε. Fix any point $x_0\in\mathbb{R}$. Since u_0 is a $C^{1,1}$ function, there exists a parabola $a(x-y_0)^2+b$ touching from above u_0 at x_0, for some $y_0,b\in\mathbb{R}$ and $a>0$. Since u_0 is bounded, there exists a bounded smooth function g touching u_0 from above such that

$$\begin{cases} g\geqslant u_0, \quad g(x_0)=u_0(x_0)\\ g=a(x-y_0)^2+b \quad \text{in } (x_0-1,x_0+1)\\ g \text{ is non-increasing in } (-\infty,y_0)\\ g \text{ is non-decreasing in } (y_0,+\infty).\end{cases}$$

Finally, following the construction of Section 6.1, for $\sigma>0$ small enough it is easy to see that there exists a $C^{1,1}$ function g_σ such that

$$\begin{cases} g_\sigma\geqslant g, \quad g_\sigma(x_0)\to g(x_0) \text{ as } \sigma\to 0\\ g_\sigma \quad \text{is constant in } (y_0-\sigma,y_0+\sigma)\\ g_\sigma \text{ is non-increasing in } (-\infty,y_0)\\ g_\sigma \text{ is non-decreasing in } (y_0,+\infty).\end{cases}$$

Let x_i^0 and b_i, $i=1,\dots,N_\varepsilon$ be defined as in (2.2) and (2.3) for the function g_σ.

Let $M_\varepsilon := \lceil g_\sigma(-\infty)/\varepsilon \rceil$. Then, by Lemma 5.1 there exists $L = C/\sigma^{\frac{1}{2}}$ such that if $x_i(t)$ is the solution of the ODE system (2.9) with $x_i(0) = x_i^0$, the function

$$H^\varepsilon(t,x) = \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \sum_{i=1}^{N_\varepsilon} \varepsilon\delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + \varepsilon\left\lceil \frac{o_\varepsilon(1)}{\varepsilon} \right\rceil + \frac{\varepsilon\delta L}{\alpha}$$

is supersolution of (1.1) in $(0, \sigma/(2c_0L)] \times \mathbb{R}$. Since $u^\varepsilon(0,x) = u_0(x) \leqslant g_\sigma(x)$, by Proposition 4.10 (recall Remark 4.12) and the fact that

$$\sum_{i=1}^{N_\varepsilon} \varepsilon\delta\psi\left(\frac{x - x_i(t)}{\varepsilon\delta}, b_i\right) = o_\varepsilon(1),$$

we can choose $o_\varepsilon(1)$ in the definition of H^ε such that $u^\varepsilon(0,x) \leqslant H^\varepsilon(0,x)$. Then, by the comparison principle,

$$u^\varepsilon(t,x) \leqslant H^\varepsilon(t,x) \quad \text{for all } (t,x) \in (0, \sigma/(2c_0L)] \times \mathbb{R}.$$

Consider any sequence $(t_\varepsilon, x_\varepsilon)$ converging to $(0, x_0)$ as $\varepsilon \to 0$. As in the proof of Theorem 1.1, we have

$$\begin{aligned}
u^\varepsilon(t_\varepsilon, x_\varepsilon) &\leqslant H^\varepsilon(t_\varepsilon, x_\varepsilon) \\
&= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x_\varepsilon - x_i(t)}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\
&= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_\varepsilon + b_i c_0 L t_\varepsilon) - x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\
&= \sum_{i=1}^{N_\varepsilon} \varepsilon\phi\left(\frac{(x_\varepsilon + c_0 L t_\varepsilon) - x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon M_\varepsilon + o_\varepsilon(1) \\
&= g^\sigma(x_\varepsilon + c_0 L t_\varepsilon) + o_\varepsilon(1).
\end{aligned}$$

Passing to the $\limsup^*$ we get

$$u^+(0, x_0) \leqslant g^\sigma(x_0).$$

Finally, letting $\sigma \to 0$ and using that $g^\sigma(x_0) \to g(x_0) = u_0(x_0)$ as $\sigma \to 0$, we get

$$u^+(0, x_0) \leqslant u_0(x_0)$$

as desired.

8. Asymptotic behavior of the limit function: proof of Proposition 1.3

In this section, we investigate the asymptotic behavior as $x \to \pm\infty$ of the limit function $\overline{u}$. We first prove Lemma 6.2 and then Proposition 1.3.

The following result is proven in [28, Section 6].

Lemma 8.1. *Let v_0 be a $C^{1,1}$ non-decreasing function and let v^ε be the solution of* (1.1) *with $v^\varepsilon(0,x) = v_0(x)$. Then, there exists $L > 0$ independent of ε such that for all $(t,x) \in (0, +\infty) \times \mathbb{R}$,*

$$v_0(x - c_0 L t) + o_\varepsilon(1) \leqslant v^\varepsilon(t,x) \leqslant v_0(x + c_0 L t) + o_\varepsilon(1).$$

Similarly, one can prove

Lemma 8.2. *Let w_0 be a $C^{1,1}$ non-increasing function and let w^ε be the solution of* (1.1) *with $w^\varepsilon(0,x) = w_0(x)$. Then, there exists $L > 0$ independent of ε such that for all $(t,x) \in (0,+\infty) \times \mathbb{R}$,*

$$w_0(x + c_0 Lt) + o_\varepsilon(1) \leqslant w^\varepsilon(t,x) \leqslant w_0(x - c_0 Lt) + o_\varepsilon(1).$$

8.1. **Proof of Lemma 6.2.** Let v_2 and w_2 defined as in (2.13). Let v^ε be the solution of (1.1) with initial condition $v^\varepsilon(0,x) = v_2(x)$ and let w^ε be the solution of (1.1) with initial condition $w^\varepsilon(0,x) = w_2(x)$. By the comparison principle, $u^\varepsilon(t,x) \leqslant v^\varepsilon(t,x)$ and $u^\varepsilon(t,x) \leqslant w^\varepsilon(t,x)$ for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. The inequality

$$u^-(t,x) \leqslant u^+(t,x) \leqslant \min\{v_2(x + c_0 Lt), w_2(x - c_0 Lt)\}$$

then follows from Lemma 8.1 and Lemma 8.2. Similarly, one can prove that

$$u^-(t,x) \geqslant \max\{v_1(x + c_0 Lt), w_1(x - c_0 Lt)\},$$

and this concluded the proof of the lemma.

8.2. **Proof of Proposition 1.3.** By Theorem 1.1 we know that $u^- = u^+ = \overline{u}$ with $\overline{u}$ the solution of (1.4). Then, the limits in (1.6) immediately follow from Lemma 6.2. Finally, estimate (1.7) is a consequence of the comparison principle and the fact that constants are solutions to the equation $\partial_t u = c_0 |\partial_x u| \, \mathcal{I}_1[u]$.

9. Appendix

Lemma 9.1. *There exists $C > 0$ independent of ε and ρ such that, for any $x \in \mathbb{R}$,*

$$\left| \sum_{i=0}^{K_\rho} \varepsilon\delta\psi\left(\frac{x - \zeta^i(t)}{\varepsilon\delta}, 1\right) \right| \leqslant C\delta.$$

Proof. Using (6.31) and $\|\psi\|_\infty \leqslant C$ for some $C > 0$, we have

$$\begin{aligned} \left| \sum_{i=0}^{K_\rho} \varepsilon\delta\psi\left(\frac{x - \zeta^i(t)}{\varepsilon\delta}, 1\right) \right| &\leqslant \delta\|\psi\|_\infty \varepsilon(K_\rho + 1) \\ &= \delta\|\psi\|_\infty \varepsilon(K_\rho + J_0 - J_0 + 1) \\ &= \delta\|\psi\|_\infty (\eta(t, \zeta^{K_\rho}(t)) - \eta(t, \zeta^0(t)) + \varepsilon) \\ &\leqslant C\delta. \end{aligned}$$

□

9.1. **Proof of Lemma 6.5.** To prove the lemma, we will show the following claims.

Claim 1: $\left| \sum_{i=0}^{K_\rho} \varepsilon\phi\left(\frac{x-\zeta^i(t)}{\varepsilon\delta}, 1\right) + \varepsilon J_0 - \eta(t,x) \right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}$.

Proof of Claim 1. If $(t,x) \in Q_{B_0 R, \rho - R}(t_0, x_0)$, then by Lemma 6.3 $x \in (\zeta^0(t) + R, \zeta^{K_\rho}(t) - R)$. Then, Claim 1 follows from Lemma 4.8 and the fact that $\eta(t, \zeta^0(t)) = J_0 \varepsilon$.

Claim 2: $\left| \sum_{i=1}^{M_\rho - 1} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}, b_i\right) - J_0\varepsilon \right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}$.

Proof of Claim 2. By using (6.25), if $(t,x) \in Q_{B_0 R, \rho - R}(t_0, x_0)$, then $x > x^0_{M_\rho - 1} + R$. Claim 2 then follows from (4.50) and the fact that $\eta(t_0, x_1^0) = \varepsilon$ and $\eta(t_0, x^0_{M_\rho - 1}) = J_0\varepsilon - \varepsilon$.

Claim 3: $\left| \sum_{i=N_\rho + 1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x - x_i^0}{\varepsilon\delta}, b_i\right) \right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}$.

Proof of Claim 3. By using (6.26), if $(t,x) \in Q_{B_0R,\rho-R}(t_0,x_0)$, then $x < x^0_{N_\rho+1} - R$. Claim 3 then immediately follows from (4.51).

Finally, the lemma is a consequence of Claims 1-3, Lemma 9.1 and (6.21).

9.2. **Proof of Lemma 6.6.** We first consider $|x - x_0| > \rho + 4R$. Let us assume that $x > x_0 + \rho + 4R$. One can similarly prove the lemma for $x < x_0 - (\rho + 4R)$. We divide the proof into three claims as follows.

Claim 1: $\left|\sum_{i=0}^{K_\rho} \varepsilon\phi\left(\frac{x-\zeta^i(t)}{\varepsilon\delta}, 1\right) - \varepsilon K_\rho\right| \leqslant o_\varepsilon(1) + \dfrac{C\varepsilon^2\delta N_\varepsilon}{R}.$

Proof of Claim 1. If $|t - t_0| < B_0R$ and $x > x_0 + \rho + 4R$, then by Lemma 6.3 $x > \zeta^{K_\rho}(t) + R$. Therefore, Claim 1 follows immediately by (4.50) and the fact that $\eta(t,\zeta^{K_\rho}(t)) - \eta(t,\zeta^0(t)) = \varepsilon K_\rho$.

Claim 2: $\left|\sum_{i=1}^{M_\rho-1} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) - \varepsilon J_0\right| \leqslant o_\varepsilon(1) + \dfrac{C\varepsilon^2\delta N_\varepsilon}{R}.$

Proof of Claim 2. By (6.25), if $x > x_0 + \rho + 4R$, then $x > x^0_{M_\rho} + R$. Claim 2 then follows from (4.50) and the fact that $\eta(t_0, x^0_{M_\rho-1}) = J_0\varepsilon - \varepsilon$.

Claim 3: $\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t,x)\right| \leqslant o_\varepsilon(1) + \dfrac{C\varepsilon^2\delta N_\varepsilon}{R} + O(R).$

Proof of Claim 3. By (6.26), if $x > x_0 + \rho + 4R$ and in addition $x < x^0_{N_\varepsilon} - R$, then $x \in (x^0_{N_\rho} + R, x^0_{N_\varepsilon} - R)$. By Lemma 4.8 and the fact that $\eta(t_0, x^0_{N_\rho+1}) = \varepsilon(K_\rho + J_0 + 1)$, we obtain

$$\begin{aligned}
&\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t,x)\right| \\
&\leqslant \left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t_0,x)\right| + |\eta(t_0,x) - \eta(t,x)| \\
&\leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R} + O(R),
\end{aligned}$$

using $|\eta(t_0,x) - \eta(t,x)| \leqslant O(R)$. This proves Claim 3 with $x < x^0_{N_\varepsilon} - R$.

Next, suppose that $x > x^0_{N_\varepsilon} + R$. In this case, we apply (4.50) to obtain

$$\begin{aligned}
&\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t,x)\right| \\
&\leqslant \left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t_0,x^0_{N_\varepsilon})\right| + |\eta(t_0,x^0_{N_\varepsilon}) - \eta(t,x)| \\
&\leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R} + O(R),
\end{aligned}$$

where, in the last inequality, we used that

$$|\eta(t_0,x^0_{N_\varepsilon}) - \eta(t,x)| \leqslant |\eta(t_0,x^0_{N_\varepsilon}) - \eta(t_0,x)| + |\eta(t_0,x) - \eta(t,x)| \leqslant \varepsilon + O(R). \tag{9.1}$$

Finally, suppose $x^0_{N_\varepsilon} - R \leqslant x \leqslant x^0_{N_\varepsilon} + R$. Define N to be an index such that

$$x^0_N \leqslant x^0_{N_\varepsilon} - 2R < x^0_{N+1} \leqslant x^0_{N_\varepsilon}.$$

We have

$$\begin{aligned}
&\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t,x)\right| \\
&\leqslant \left|\sum_{i=N_\rho+1}^{N} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t_0,x^0_N)\right| \\
&+ \left|\sum_{i=N+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right)\right| + o_\varepsilon(1) + O(R).
\end{aligned}$$

By (4.50)

$$\left|\sum_{i=N_\rho+1}^{N} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) + \varepsilon(K_\rho + J_0) - \eta(t_0,x^0_N)\right| \leqslant o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}.$$

By using that $0<\phi<1$ and that $\{x_{N+1},\dots,x_{N_\varepsilon}\} \subset (x_{N_\varepsilon} - 2R, x_{N_\varepsilon})$ so that $|\{x_{N+1},\dots,x_{N_\varepsilon}\}| \leqslant CR/\varepsilon$, we have

$$\left|\sum_{i=N+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right)\right| \leqslant O(R).$$

This concludes the proof of Claim 3.

The lemma for $|x-x_0| > \rho + 4R$ follows as a consequence of Claims 1-3, Lemma 9.1 and (6.21).

Finally, let us consider the case $\rho - R \leqslant |x - x_0| \leqslant \rho + 4R$. Assume without loss of generality that $\rho - R \leqslant x - x_0 \leqslant \rho + 4R$. We will divide the proof into three claims.

Claim 4: $\left|\sum_{i=1}^{M_\rho} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right) - J_0\varepsilon\right| \leqslant o_\varepsilon(1) + \dfrac{C\varepsilon^2\delta N_\varepsilon}{R}.$

Proof of Claim 4. By (6.25) and $x_0 + \rho - R < x$, we have that $x > x^0_{M_\rho} + R$. Therefore, using (4.50), the claim immediately follows.

Claim 5: $\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right)\right| \leqslant o_\varepsilon(1) + \dfrac{C\varepsilon^2\delta N_\varepsilon}{R} + O(R).$

Proof of Claim 5. Define an index N_1 such that

$$x_0 + \rho + 5R \leqslant x^0_{N_1} < x^0_{N_\rho} + 6R,$$

so that $x < x_0 + \rho + 4R \leqslant x^0_{N_1} - R$. By using (4.51), $0 < \phi < 1$ and $|\{x_{N_\rho+1},\dots,x_{N_1-1}\}| \leqslant CR/\varepsilon$, we obtain

$$\begin{aligned}
\left|\sum_{i=N_\rho+1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right)\right| &\leqslant \left|\sum_{i=N_\rho+1}^{N_1-1} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right)\right| + \left|\sum_{i=N_1}^{N_\varepsilon} \varepsilon\phi\left(\frac{x-x_i^0}{\varepsilon\delta}, b_i\right)\right| \\
&= O(R) + o_\varepsilon(1) + \frac{C\varepsilon^2\delta N_\varepsilon}{R}.
\end{aligned}$$

This completes the proof of Claim 5.

Claim 6: $\left|\sum_{i=0}^{K_\rho}\varepsilon\phi\left(\frac{x-\zeta^i(t)}{\varepsilon\delta},1\right)+J_0\varepsilon-\eta(t,x)\right|\leqslant O(R)$.

Proof of Claim 6. By Lemma 6.3, $|x-\zeta^0(t)|$, $|x-\zeta^{K_\rho}(t)|$=O(R). Then, by using that $0<\phi<1$, $\eta(t,\zeta^0(t))=\varepsilon J_0$ and $\eta(t,\zeta^{K_\rho}(t))=\varepsilon(J_0+K_\rho)$, we get

$$\begin{aligned}\sum_{i=0}^{K_\rho}\varepsilon\phi\left(\frac{x-\zeta^i(t)}{\varepsilon\delta},1\right)+J_0\varepsilon-\eta(t,x)&\leqslant\varepsilon(J_0+K_\rho+1)-\eta(t,x)\\&=\eta(t,\zeta^{K_\rho}(t))-\eta(t,x)+\varepsilon\\&=O(R),\end{aligned}$$

and

$$\sum_{i=0}^{K_\rho}\varepsilon\phi\left(\frac{x-\zeta^i(t)}{\varepsilon\delta},1\right)+J_0\varepsilon-\eta(t,x)\geqslant J_0\varepsilon-\eta(t,x)=\eta(t,\zeta^0(t))-\eta(t,x)=O(R),$$

which proves the claim.

The lemma for $\rho-R\leqslant|x-x_0|\leqslant\rho+4R$ follows from Claims 4-6, Lemma 9.1 and (6.21).

9.3. **Proof of Lemma 6.1.** Recalling that if $x_i^0<x_0$ then $b_i=-1$, while if $x_i^0>x_0$ then $b_i=1$, we write

$$\begin{aligned}&\sum_{i=1}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon+b_ic_0L(t_\varepsilon-t_0+c\sigma)-x_i^0}{\varepsilon\delta},b_i\right)\\&=\sum_{\substack{i=1\\x_i^0>x_0}}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon+c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0}{\varepsilon\delta},1\right)\\&+\sum_{\substack{i=1\\x_i^0<x_0}}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon-c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0}{\varepsilon\delta},-1\right).\end{aligned}\tag{9.2}$$

Let us show that

$$\sum_{\substack{i=1\\x_i^0<x_0}}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon\pm c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0}{\varepsilon\delta},-1\right)=o_\varepsilon(1)-\varepsilon N_\varepsilon^-,\tag{9.3}$$

where N_ε^- is the number of negative oriented particles. By (6.10),

$$x_\varepsilon\pm c_0L(t_\varepsilon-t_0+c\sigma)=x_0\pm c_0Lc\sigma+o_\varepsilon(1)=x_0\pm\frac{\sigma}{4}+o_\varepsilon(1).\tag{9.4}$$

Since $\tilde\eta^\sigma$ is constant in x for $|x-x_0|\leqslant\sigma$, if $x_i^0<x_0$ then

$$x_0-x_i^0\geqslant\sigma,$$

which combined with (9.4) gives

$$x_\varepsilon\pm c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0\geqslant\frac{\sigma}{2}.$$

Therefore by (2.16),

$$\begin{aligned}\sum_{\substack{i=1\\x_i^0<x_0}}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon\pm c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0}{\varepsilon\delta},-1\right)\\ \leqslant\sum_{\substack{i=1\\x_i^0<x_0}}^{N_\varepsilon}\frac{\varepsilon^2\delta C}{\sigma}-\varepsilon N_\varepsilon^-\\ =o_\varepsilon(1)-\varepsilon N_\varepsilon^-,\end{aligned}$$

where we used that $\varepsilon N_\varepsilon\leqslant C$. Similarly, one can show that

$$\sum_{\substack{i=1\\x_i^0<x_0}}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon\pm c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0}{\varepsilon\delta},-1\right)\geqslant o_\varepsilon(1)-\varepsilon N_\varepsilon^-.$$

This concludes the proof of (9.3). From (9.3) in particular we infer that

$$\begin{aligned}\sum_{\substack{i=1\\x_i^0<x_0}}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon-c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0}{\varepsilon\delta},-1\right)\\ =\sum_{\substack{i=1\\x_i^0<x_0}}^{N_\varepsilon}\varepsilon\phi\left(\frac{x_\varepsilon+c_0L(t_\varepsilon-t_0+c\sigma)-x_i^0)}{\varepsilon\delta},-1\right)+o_\varepsilon(1),\end{aligned}$$

which combined with (9.2) yields (6.17).

This concludes the proof of the lemma.

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