WEAKLY COUPLED MEAN-FIELD GAME SYSTEMS

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ABSTRACT. Here, we prove the existence of solutions to first-order mean-field games (MFGs) arising in optimal switching. First, we use the penalization method to construct approximate solutions. Then, we prove uniform estimates for the penalized problem. Finally, by a limiting procedure, we obtain solutions to the MFG problem.

1. INTRODUCTION

The mean-field game (MFG) framework [34, 35, 36, 37, 38, 39] is a class of methods used to study large populations of rational, non-cooperative agents. MFGs have been the focus of intense research, see, for example, the surveys [28, 31]. Here, we investigate MFGs that arise in optimal switching. These games are given by a weakly coupled system of Hamilton-Jacobi equations of the obstacle type and a corresponding system of transport equations.

To simplify the presentation, we use periodic boundary conditions. Thus, the spatial domain is the N-dimensional flat torus, $\mathbb{T}^N$. Our MFG is determined by a value function, $u : \mathbb{T}^N \to \mathbb{R}^d$, a probability density, $\theta : \mathbb{T}^N \to (\mathbb{R}^+)^d$, and a switching current, ν, that together satisfy the following system of variational inequalities:

$$\max\left(H^i(Du^i, x) + u^i - g(\theta^i), \max_j \left(u^i - u^j - \psi^{ij}\right)\right) = 0 \tag{1.1}$$

coupled with the system

$$-\text{div}(D_pH^i(Du^i, x)\theta^i) + \theta^i + \sum_{j\neq i}\left(\nu^{ij} - \nu^{ji}\right) = 1. \tag{1.2}$$

Moreover, for $1 \leq i, j \leq d$, ν^{ij} is a non-negative measure on $\mathbb{T}^N$ supported in the set $u^i - u^j - \psi^{ij} = 0$.

This system models a stationary population of agents. Each agent moves in $\mathbb{T}^N$ and can switch between different modes that are given by the index i. Their actions seek to minimize a certain cost. Agents can change their state by continuously modifying their spatial position, x, and by switching between different modes, i to j, at a cost ψ^{ji}. The function $u^i(x)$ is the value function for an agent whose spatial location is x and whose mode is i. The function $\theta^i(x)$ is the density of the agents on $\mathbb{T}^N \times \{1, \ldots, d\}$. Thus, we require that $\theta^i(x) \geq 0$. We note that θ^i is not a

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probability measure on $\mathbb{T}^N \times \{1, \dots, d\}$ because the source term in the right-hand side of (1.2) is not normalized.

In Section 2, we discuss detailed assumptions on the Hamiltonians H^i, on the nonlinearity g, and on the switching costs ψ^{ij}. A concrete example that satisfies those is

$$H^i(x,p) = \frac{|p|^2}{2} + V^i(x), \qquad g(\theta) = \ln \theta, \quad \text{and} \quad \psi^{ij}(x) = \eta, \tag{1.3}$$

with $V^i : \mathbb{T}^N \to \mathbb{R}$ being a C^∞ function and η being a positive real number. Another case of interest is the polynomial nonlinearity, $g(m) = m^\alpha$ for $\alpha > 0$.

Standard MFGs involve two equations, a Hamilton-Jacobi equation and a transport or Fokker-Planck equation. This latter equation is the adjoint of the linearization of the former. Because the non-linear operator in (1.1) is non-differentiable, (1.2) is obtained by a limiting procedure. In the context of MFGs, this method was first used in [22]. Here, we consider the following penalized problem.

$$H^i(Du^i, x) + u^i + \sum_{j\neq i} \beta_\epsilon(u^i - u^j - \psi^{ij}) = g(\theta^i) \tag{1.4}$$

$$-\text{div}(D_pH^i(Du^i,x)\theta^i) + \theta^i + \sum_{j\neq i} \beta_\epsilon'(u^i - u^j - \psi^{ij})\theta^i - \beta_\epsilon'(u^j - u^i - \psi^{ji})\theta^j = 1, \tag{1.5}$$

where the penalty function, β_ϵ, is an increasing C^∞ function and $\epsilon > 0$. We assume that, as $\epsilon \to 0$, $\beta_\epsilon(s) \to \infty$ for $s > 0$ and $\beta_\epsilon(s) = 0$ for $s \leq 0$, see Assumption 8. The study of optimal switching has a long history that predates viscosity solutions and, certainly, MFGs, see, for example [2, 6, 7, 17]. In those references, the use of a penalty to approximate a non-smooth Hamilton-Jacobi equation is a fundamental tool. The penalty in (1.4) is similar to the ones in the aforementioned references.

More recently, several authors have investigated weakly coupled Hamilton-Jacobi equations [44], the corresponding extension of the weak KAM and Aubry-Mather theories [5, 14, 40], the asymptotic behavior of solutions [4, 3, 42, 41, 45], and homogenization [43]. In these references, the state of the system has different modes, and a random process drives the switching between them. In contrast, here, the switches occur at deterministic times. Thus, our models are the MFG counterpart of the Hamilton-Jacobi systems considered in [18, 32]. MFGs with different populations [12, 13] are a limit case of (1.1)-(1.2). This can be seen by taking the limit $\psi^{ij} \to +\infty$; that is, the case where agents are not allowed to change their state.

The development of the existence and regularity theory for MFGs has seen substantial progress in recent years. Uniformly elliptic and parabolic MFGs are now well understood, and the existence of smooth and weak solutions has been established in a broad range of problems, see, respectively, [23, 24, 25, 30, 29, 27, 26] and [10, 46, 47]. However, the regularity theory for first-order MFGs is less developed and, in general, only weak solutions are known to exist [8, 9, 11]. Variational inequality methods are at the heart of a new class of techniques to establish the existence of weak solutions, both for first- and second-order problems [19] and for their numerical approximation [1]. Some MFGs that arise in applications, such as congestion [20, 33] or obstacle-type problems [22], feature singularities. Thus, there is keen interest in developing methods for their analysis. To the best of our knowledge, this paper is the first to address MFGs arising in optimal switching.

Moreover, our techniques contribute to better understanding of the regularity of first-order MFGs.

Our main result is the following theorem.

Theorem 1.1. *Suppose that Assumptions 1-4 (see Section 2) hold and that either*

- *Assumption 5* **L** *or*
- *Assumptions 5* **P-**$\frac{2}{N}$*, 6 and 7*

hold. Then, there exists a solution, (u, θ)*, of* (1.1)-(1.2)*, with* $u \in (W^{2,2}(\mathbb{T}^N))^d \cap (C^\gamma(\mathbb{T}^N))^d$ *for some* $\gamma \in (0,1)$ *and* $\theta \in (W^{1,2}(\mathbb{T}^N))^d$*.*

As mentioned before, to prove the existence of solutions for (1.1)-(1.2), we first examine the existence of solutions for (1.4)-(1.5), prove ϵ independent bounds and, subsequently, consider the limit $\epsilon \to 0$. On the existence of solutions, our main result is the following theorem.

Theorem 1.2. *Suppose that Assumptions 1-4 (see Section 2) and 8 hold, and either*

- *Assumption 5* **L** *or*
- *Assumptions 5* **P-**$\frac{2}{N}$*, 6 and 7*

hold. Then, there exists a unique solution, $(u, \theta) \in (C^\infty(\mathbb{T}^N))^d \times (C^\infty(\mathbb{T}^N))^d$*, of* (1.4)-(1.5) *with* $\theta^i \geq \theta_0 > 0$ *for some constant* θ_0 *that does not depend on* ϵ*.*

To prove Theorem 1.1, we establish the existence of solutions for (1.4)-(1.5) in Theorem 1.2 and prove ϵ independent bounds. The analysis of (1.4)-(1.5) begins in Section 3 where we examine various a priori estimates. Next, in Section 4, we consider separately the two different nonlinearities, $g(m) = \ln m$ and $g(m) = m^\alpha$. In these two sections, our estimates are uniform in ϵ. In contrast, in Section 5, we prove L^∞ estimates for θ and Lipschitz bounds for u that depend on ϵ. These are crucial in the proof of Theorem 1.2 that we present in Section 6. This proof combines the a priori estimates with the continuation method. The paper ends with the proof of Theorem 1.1 in Section 7 and a brief discussion of convergence and uniqueness in Section 8.

2. Main assumptions

We begin by discussing the assumptions on H^i, g, and ψ used in the study of (1.1)-(1.2). On the Hamiltonian, H^i, we assume standard hypotheses that hold in a large class of problems. In particular, they are satisfied by the example (1.3). To simplify the presentation, we select assumptions compatible with quadratic growth of the Hamiltonian, see Remark 2.3 below. Regarding the dependence on the measure: for every coordinate, i, we have the same nonlinearity, g, evaluated at the coordinate θ^i. Some of our estimates are valid without substantial changes in the corresponding proofs if g is replaced by a function, g^i, depending on all coordinates of θ or even on x. Naturally, Assumption 2 must be modified in a suitable way. Finally, we work with positive switching costs, ψ^{ij}. The positivity condition is natural in optimal switching because it prevents the occurrence of infinitely many switches. These conditions and the assumptions that follow are unlikely to give the most general case under which our techniques hold. Our choice reflects a balance between generality and simplicity of the proofs.

Assumption 1. *The Hamiltonian, H^i, the nonlinearity, g, and the switching cost, ψ^{ij}, satisfy:*

(1) *For $1 \leq i \leq d$, $H^i : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is C^∞ and positive.*
(2) *$g : \mathbb{R}^+ \to \mathbb{R}$ is C^∞ and strictly increasing; that is, $g' > 0$.*
(3) *For $1 \leq i, j \leq d$, the function $\psi^{ij} : \mathbb{T}^N \to \mathbb{R}$ is of class $C^\infty(\mathbb{T}^N)$. Furthermore, for $x \in \mathbb{T}^N$, $\psi^{ij}(x) > 0$.*

As usual, we identify whenever convenient, functions in $\mathbb{T}^N$ as $\mathbb{Z}^N$-periodic functions in R^N.

Assumption 2. *The function g satisfies the following.*

(1) *For any $C_0 > 0$, there exists C_1 such that*

$$\int_{\mathbb{T}^N} \theta g(\theta) dx \geq -C_1 \tag{2.1}$$

for any non-negative $\theta \in L^1(\mathbb{T}^N)$ with $\int_{\mathbb{T}^N} \theta dx \leq C_0$.

(2) *There exists $C > 0$ such that, for any $\theta > 0$,*

$$g(\theta) \leq \frac{1}{2}\theta g(\theta) + C. \tag{2.2}$$

Remark 2.1. *The functions $g(\theta) = \ln \theta$ and $g(\theta) = \theta^\alpha$, for $\alpha > 0$, satisfy the preceding assumption.*

Assumption 3. *There exist constants, $c, C \geq 0$, such that*

$$H^i(p, x) - D_p H^i(p, x) \cdot p \leq -cH^i(p, x) + C \tag{2.3}$$

for all $p \in \mathbb{R}^N$, $x \in \mathbb{T}^N$, and $1 \leq i \leq d$.

Remark 2.2. *Consider the Lagrangian, L^i, associated with the Hamiltonian H^i given by*

$$L^i(x, v) = \sup_{p \in \mathbb{R}^N} -p \cdot v - H^i(p, x).$$

Because the supremum is achieved for $v = -D_p H^i(p, x)$,

$$L^i(x, v) = D_p H^i(p, x) \cdot p - H^i(p, x).$$

Accordingly, the preceding hypothesis gives a lower bound on L^i.

Assumption 4. *There exists $\gamma > 0$ such that*

$$H^i_{p_k p_j}(p, x)\xi_k \xi_j \geq \gamma |\xi|^2 \tag{2.4}$$

for all $x \in \mathbb{T}^N$ and p, $\xi \in \mathbb{R}^N$.

There exist C, $c > 0$ such that

$$\begin{aligned} &|D^2_{pp} H^i| \leq C, \\ &|D^2_{xp} H^i| \leq C(1 + |p|), \\ &|D^2_{xx} H^i| \leq C(1 + |p|^2). \end{aligned} \tag{2.5}$$

Remark 2.3. *The preceding assumption implies that there exists $C > 0$ such that*

$$\frac{\gamma}{2}|p|^2 - C \leq H^i(p, x) \leq C|p|^2 + C, \tag{2.6}$$

and

$$
\begin{aligned}
&|D_pH^i(p,x)| \leq C(1+|p|),\\
&|D_xH^i(p,x)| \leq C(1+|p|^2)
\end{aligned}
\tag{2.7}
$$

for all $p \in \mathbb{R}^N$ *and* $x \in \mathbb{T}^N$.

Assumption 5. *There exist constants,* $C, \widetilde{C} > 0$, *and* $\alpha \geq 0$ *such that*

$$
C\theta^{\alpha-1} \leq g'(\theta) \leq \widetilde{C}\theta^{\alpha-1} + \widetilde{C}
\tag{2.8}
$$

for any $\theta \geq 0$. *Two specific cases of interest are*

L - $g(\theta) = \ln\theta$;

P - $g(\theta) = \theta^\alpha$, $0 \leq \alpha \leq 1$.

In the **P** *case, the additional constraint,* $\alpha < \frac{2}{N}$, *is denoted by* **P-**$\frac{2}{N}$.

The next two assumptions are of a technical nature and are used in the study of the **P-**$\frac{2}{N}$ case. Assumption 6 is employed in Proposition 4.3 to obtain a lower bound for θ^i. Assumption 7 is fundamental in the proof of Proposition 5.1.

Assumption 6. *For* $1 \leq i \leq d$, *we have*

$$
D^2_{px}H^i(0,x) = D_xH^i(0,x)D_pH^i(0,x) = 0 \quad \textit{for any } x \in \mathbb{T}^N
\tag{2.9}
$$

and

$$
\max_{x\in\mathbb{T}^N} H^i(0,x) < 1.
\tag{2.10}
$$

Remark 2.4. *The preceding assumption is used to prove lower bounds for* θ^i. *Because* $H^i \geq 0$, *the bound* (2.10) *gives an oscillation condition for* $H^i(0,x)$. *This oscillation condition is natural in light of the example considered in* [28], *Chapter 3. In that reference and also in* [21], *various examples of first-order MFGs are shown to have a vanishing density. The oscillation of* $H(0,x)$ *plays an essential role in these examples.*

Remark 2.5. *The number* 1 *on the right-hand side of* (2.10) *corresponds to the source term in the Fokker-Planck equation* (1.5). *Suppose that we modify* (1.5) *and consider a source,* $\upsilon > 0$; *that is,*

$$
-div(D_pH^i(Du^i,x)\theta^i) + \theta^i + \sum_{j\neq i}\beta'_\epsilon(u^i-u^j-\psi^{ij})\theta^i - \beta'_\epsilon(u^j-u^i-\psi^{ji})\theta^j = \upsilon.
$$

Then, (2.10) *becomes*

$$
\max_{x\in\mathbb{T}^N} H^i(0,x) < \upsilon.
$$

Assumption 7. *The value* α *in Assumption 5 satisfies* $\alpha \in [0,\alpha_0)$, *where* α_0 *solves*

$$
2\alpha_0 = (\alpha_0+1)\beta(\beta-1), \qquad \textit{with} \quad \beta = \sqrt{\frac{2^*}{2}},
\tag{2.11}
$$

if $N > 2$, *and* $\alpha_0 = \infty$ *if* $N \leq 2$.

Remark 2.6. *In the* $N \leq 3$ *case, the value* α_0 *determined by* (2.11) *is larger than* $\frac{2}{N}$. *Whereas, if* $N > 3$ *the opposite inequality holds.*

Our last assumption is required in the study of the penalized problem. For $\epsilon > 0$, we choose a penalty, β_ϵ, satisfying the following assumption.

Assumption 8. *$\beta_\epsilon : \mathbb{R} \to \mathbb{R}$, smooth, with $\beta'_\epsilon \geq 0$, $\beta''_\epsilon \geq 0$ with*

$$\beta_\epsilon(s) = 0 \quad \text{for } s \leq 0, \quad \beta'_\epsilon(s) \leq C\beta''_\epsilon(s) \quad \text{for } s > 0, \tag{2.12}$$

and $\beta_\epsilon(s) \to \infty$ as $\epsilon \to 0$ for $s > 0$.

Remark 2.7. *From the preceding assumption, we get*

$$\beta_\epsilon(s) - s\beta'_\epsilon(s) \leq 0 \quad \text{for } s \in \mathbb{R}. \tag{2.13}$$

The preceding assumption is standard in the setting of variational inequalities and optimal switching. In the context of MFGs, a similar penalty was used in [22] to study the obstacle problem.

3. A priori estimates

Here, we prove a priori estimates for classical solutions of (1.4)-(1.5). The purpose of these estimates is twofold: first, to obtain the existence of solutions; second, to take the limit $\epsilon \to 0$. For that, we seek to prove bounds that are uniform in ϵ. We begin with a simple consequence of the maximum principle for weakly coupled systems.

Proposition 3.1. *Suppose that Assumptions 1 and 8 hold. Let (u, θ) be a C^∞ solution of (1.4)-(1.5). Then, for $i = 1, \ldots, d$, we have $\theta^i \geq 0$.*

Proof. The proof of this Lemma is a straightforward application of the maximum principle to (1.5), see [5] for a similar proof. ☐

As is standard in MFG problems, we can get several estimates by multiplying (1.4)-(1.5) by 1, θ^i or u^i, adding or subtracting, and integrating by parts. We record these in the next lemma.

Lemma 3.2. *Suppose that Assumptions 1-3 and 8 hold. Let (u, θ) be a C^∞ solution of (1.4)-(1.5). Then, there exists a constant, C, that does not depend on the particular solution nor on ϵ, such that, for $i = 1 \ldots d$,*

$$0 \leq \int_{\mathbb{T}^N} \theta^i dx \leq C, \tag{3.1}$$

$$\left| \int_{\mathbb{T}^N} \theta^i g(\theta^i) dx \right| \leq C, \tag{3.2}$$

$$\left| \int_{\mathbb{T}^N} u^i dx \right| \leq C, \tag{3.3}$$

$$\int_{\mathbb{T}^N} |Du^i|^2 \theta^i dx \leq C, \tag{3.4}$$

$$\sum_{i,j=1, i \neq j}^{d} \int_{\mathbb{T}^N} \beta'_\epsilon(u^i - u^j - \psi^{ij}) \psi^{ij} \theta^i dx \leq C, \tag{3.5}$$

and

$$\int_{\mathbb{T}^N} |Du^i|^2 dx \leq C. \tag{3.6}$$

Proof. By summing over i the equations in (1.5), we gather that

$$\sum_{i=1}^{d} -\operatorname{div}(D_pH^i(Du^i,x)\theta^i)+\theta^i=d.$$

Hence, integrating on $\mathbb{T}^N$, we get

$$\int_{\mathbb{T}^N}\theta^i dx\leq\sum_{i=1}^{d}\int_{\mathbb{T}^N}\theta^i dx=d$$

for any $i=1,...,d$. Thus, (3.1) holds. Due to Assumptions 1 and 8, H^i and β_ϵ are non-negative. Consequently, we infer that

$$\int_{\mathbb{T}^N}u^i dx\leq\int_{\mathbb{T}^N}g(\theta^i)dx. \tag{3.7}$$

Next, we multiply (1.4) by θ^i, sum over i, and integrate. Accordingly, we gather the identity

$$\begin{aligned}&\sum_{i=1}^{d}\int_{\mathbb{T}^N}H^i(Du^i,x)\theta^i+u^i\theta^i+\sum_{j\neq i}\beta_\epsilon(u^i-u^j-\psi^{ij})\theta^i dx\\&\quad=\sum_{i=1}^{d}\int_{\mathbb{T}^N}\theta^i g(\theta^i)dx.\end{aligned} \tag{3.8}$$

Next, we multiply (1.5) by u^i, add over i, and integrate by parts to conclude that

$$\begin{aligned}&\sum_{i=1}^{d}\int_{\mathbb{T}^N}\Big[D_pH^i(Du^i,x)\cdot Du^i\theta^i+u^i\theta^i\\&\quad+\sum_{j\neq i}\beta_\epsilon'(u^i-u^j-\psi^{ij})\theta^i u^i-\beta_\epsilon'(u^j-u^i-\psi^{ji})\theta^j u^i\Big]dx\\&\quad=\sum_{i=1}^{d}\int_{\mathbb{T}^N}u^i dx.\end{aligned} \tag{3.9}$$

Subtracting equations (3.8) and (3.9), we get

$$\begin{aligned}&\sum_{i=1}^{d}\int_{\mathbb{T}^N}\theta^i g(\theta^i)dx\\&=\sum_{i=1}^{d}\int_{\mathbb{T}^N}H^i(Du^i,x)\theta^i+u^i\theta^i+\sum_{j\neq i}\beta_\epsilon(u^i-u^j-\psi^{ij})\theta^i dx\\&=\sum_{i=1}^{d}\int_{\mathbb{T}^N}(H^i(Du^i,x)-D_pH^i(Du^i,x)\cdot Du^i)\theta^i+u^i dx\\&+\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}\beta_\epsilon(u^i-u^j-\psi^{ij})\theta^i dx\\&+\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}-\beta_\epsilon'(u^i-u^j-\psi^{ij})\theta^i u^i+\beta_\epsilon'(u^j-u^i-\psi^{ji})\theta^j u^i dx.\end{aligned}$$

According to Assumption 3, we have

$$\begin{aligned}&\sum_{i=1}^{d}\int_{\mathbb{T}^N}(H^i(Du^i,x)-D_pH^i(Du^i,x)\cdot Du^i)\theta^i dx\\&\quad\leq -c\sum_{i=1}^{d}\int_{\mathbb{T}^N}H^i(Du^i,x)\theta^i dx+C\end{aligned}$$

using (3.1). Moreover, we have

$$\begin{aligned}&\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}\beta_\epsilon(u^i-u^j-\psi^{ij})\theta^i dx\\&\quad+\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}-\beta_\epsilon'(u^i-u^j-\psi^{ij})\theta^i u^i+\beta_\epsilon'(u^j-u^i-\psi^{ji})\theta^j u^i dx\\&=\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}[\beta_\epsilon(u^i-u^j-\psi^{ij})-\beta_\epsilon'(u^i-u^j-\psi^{ij})(u^i-u^j-\psi^{ij})]\theta^i dx\\&\quad-\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}\beta_\epsilon'(u^i-u^j-\psi^{ij})\psi^{ij}\theta^i dx\\&\leq-\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}\beta_\epsilon'(u^i-u^j-\psi^{ij})\psi^{ij}\theta^i dx\end{aligned}$$

by (2.13) in Remark 2.7. Gathering the previous estimates, we conclude that

$$\begin{aligned}&\sum_{i=1}^{d}\int_{\mathbb{T}^N}\theta^i g(\theta^i)+cH^i(Du^i,x)\theta^i dx+\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}\beta_\epsilon'(u^i-u^j-\psi^{ij})\psi^{ij}\theta^i dx\\&\quad\leq\sum_{i=1}^{d}\int_{\mathbb{T}^N}u^i dx+C\leq\sum_{i=1}^{d}\int_{\mathbb{T}^N}g(\theta^i)dx+C\end{aligned}\tag{3.10}$$

using (3.7). Then, Assumption 2 implies

$$\int_{\mathbb{T}^N}\theta^i g(\theta^i)dx\leq C.$$

On the other hand, (2.1) in Assumption 2 and (3.1) give

$$\int_{\mathbb{T}^N}\theta^i g(\theta^i)dx\geq -C.$$

Therefore, (3.2) holds. Using (3.2) and the bound (2.2) from Assumption 2, we get (3.3). In addition, for any $i=1,...,d$,

$$\int_{\mathbb{T}^N}H^i(Du^i,x)\theta^i dx\leq C.$$

The last estimate combined with (2.4) implies (3.4). A similar argument yields (3.5).

Finally, the bound (3.6) follows from (1.4) by combining (2.4), the non-negativity of β_ϵ, and the previous results with the estimate

$$\int_{\mathbb{T}^N} |Du^i|^2 dx \leq C + \int_{\mathbb{T}^N} (g(\theta^i) - u^i) dx \leq C.$$

□

Lemma 3.3. *Suppose that Assumptions 1-5 and 8 hold. Let (u, θ) be a C^∞ solution of (1.4)-(1.5). Then, there exists a constant, C, that does not depend on the particular solution nor on ϵ, such that, for $i = 1, \ldots, d$,*

$$\int_{\mathbb{T}^N} |D^2 u^i|^2 \theta_i dx \leq C, \tag{3.11}$$

$$\int_{\mathbb{T}^N} g'(\theta^i) |D\theta^i|^2 dx \leq C, \tag{3.12}$$

and

$$\|(\theta^i)^{\frac{\alpha+1}{2}}\|_{W^{1,2}(\mathbb{T}^N)} \leq C. \tag{3.13}$$

Proof. We begin by differentiating (1.4) twice with respect to x_k and then summing over k. In this way, we get

$$\begin{aligned} &D_p H^i \cdot D(\Delta u^i) + H^i_{x_k x_k} + 2H^i_{x_k p_l} u^i_{x_l x_k} + H^i_{p_l p_m} u^i_{x_l x_k} u^i_{x_m x_k} + \Delta u^i \\ &+ \sum_{j \neq i} \beta'_\epsilon(u^i - u^j - \psi^{ij}) \Delta(u^i - u^j - \psi^{ij}) + \beta''_\epsilon(u^i - u^j - \psi^{ij}) |D(u^i - u^j - \psi^{ij})|^2 \\ &\quad = \Delta(g(\theta^i)). \end{aligned}$$

Next, we multiply the previous equation by θ^i, add in the index i, and integrate by parts to conclude that

$$\begin{aligned} &\sum_{i=1}^d \int_{\mathbb{T}^N} \Delta(g(\theta^i)) \theta^i dx \\ &\qquad = \sum_{i=1}^d \int_{\mathbb{T}^N} (D_p H^i \cdot D(\Delta u^i) + \Delta u^i + \sum_{j \neq i} \beta'_\epsilon(u^i - u^j - \psi^{ij}) \Delta u^i) \theta^i dx \\ &\qquad - \sum_{i=1}^d \sum_{j \neq i} \int_{\mathbb{T}^N} \beta'_\epsilon(u^i - u^j - \psi^{ij}) \Delta(u^j + \psi^{ij}) \theta^i dx \\ &\qquad + \int_{\mathbb{T}^N} (H^i_{x_k x_k} + 2H^i_{x_k p_l} u^i_{x_l x_k} + H^i_{p_l p_m} u^i_{x_l x_k} u^i_{x_m x_k}) \theta^i dx \\ &\qquad + \sum_{i=1}^d \sum_{j \neq i} \int_{\mathbb{T}^N} \beta''_\epsilon(u^i - u^j - \psi^{ij}) |D(u^i - u^j - \psi^{ij})|^2 \theta^i dx. \end{aligned} \tag{3.14}$$

Multiplying (1.5) by Δu^i and integrating by parts results in

$$\sum_{i=1}^{d}\int_{\mathbb{T}^N}(D_pH^i\cdot D(\Delta u^i)+\Delta u^i+\sum_{j\neq i}\beta'_\epsilon(u^i-u^j-\psi^{ij})\Delta u^i)\theta^i dx$$
$$=\sum_{i=1}^{d}\sum_{j\neq i}\int_{\mathbb{T}^N}(\beta'_\epsilon(u^j-u^i-\psi^{ji})\Delta u^i\theta^j+\Delta u^i dx$$
$$=\sum_{i=1}^{d}\sum_{j\neq i}\int_{\mathbb{T}^N}(\beta'_\epsilon(u^i-u^j-\psi^{ij})\Delta u^j\theta^i dx.$$

Using the previous identity in (3.14) gives

$$\sum_{i=1}^{d}\int_{\mathbb{T}^N}\Delta(g(\theta^i))\theta^i dx=-\sum_{i=1}^{d}\sum_{j\neq i}\int_{\mathbb{T}^N}\beta'_\epsilon(u^i-u^j-\psi^{ij})\Delta\psi^{ij}\theta^i dx$$
$$+\int_{\mathbb{T}^N}(H^i_{x_kx_k}+2H^i_{x_kp_l}u^i_{x_lx_k}+H^i_{p_lp_m}u^i_{x_lx_k}u^i_{x_mx_k})\theta^i dx$$
$$+\sum_{i=1}^{d}\sum_{j\neq i}\int_{\mathbb{T}^N}\beta''_\epsilon(u^i-u^j-\psi^{ij})|D(u^i-u^j-\psi^{ij})|^2\theta^i dx.$$

Taking into account that $\Delta\psi^{ij}$ is bounded and $\psi^{ij}>0$, estimate (3.5) implies that

$$\left|\sum_{i,j=1,i\neq j}^{d}\int_{\mathbb{T}^N}\beta'_\epsilon(u^i-u^j-\psi^{ij})\Delta\psi^{ij}\theta^i dx\right|\leq C.$$

Because $\beta''_\epsilon\geq 0$ and because of the uniform convexity from (2.4), (2.5) and (3.4), we obtain

$$\begin{aligned}&\sum_{i=1}^{d}\int_{\mathbb{T}^N}g'(\theta^i)|D\theta^i|^2dx+C\sum_{i=1}^{d}\int_{\mathbb{T}^N}|D^2u^i|^2\theta^i dx\\&\leq C\sum_{i=1}^{d}\int_{\mathbb{T}^N}(1+|Du^i|^2)\theta^i dx\leq C.\end{aligned}\tag{3.15}$$

Hence, we have (3.11) and (3.12). Moreover, from (3.12) and (2.8), we infer that

$$\int_{\mathbb{T}^N}(\theta^i)^{\alpha-1}|D\theta^i|^2dx\leq C\int_{\mathbb{T}^N}g'(\theta^i)|D\theta^i|^2dx\leq C;$$

that is, $|D(\theta^i)^{\frac{\alpha+1}{2}}|\in L^2(\mathbb{T}^N)$. By (3.1), $\theta^i\in L^1(\mathbb{T}^N)$. Thus, the previous inequality and the Poincaré inequality imply (3.13). $\square$

4. Further a priori estimates

In this section, we prove additional a priori estimates for logarithmic (Assumption 5 **L**) and power-like nonlinearities (Assumptions 5 **P** and **P-**$\frac{2}{N}$). These two cases are examined separately. Nevertheless, for both the logarithmic nonlinearity and for the power case, if $\alpha<\frac{2}{N}$ (Assumption 5 **P-**$\frac{2}{N}$), we obtain similar, ϵ-independent lower bounds on θ, on $\|u\|_{W^{2,2}(\mathbb{T}^N)}$, and on $\|u\|_{W^{1,p}(\mathbb{T}^N)}$ for any $p\geq 1$.

4.1. **The logarithmic case.** Here, we consider the logarithmic nonlinearity $g(\theta) = \ln\theta$.

Proposition 4.1. *Suppose that Assumptions 1-4, 5* **L** *and 8 hold. Let* (u, θ) *be a* C^∞ *solution of* (1.4)-(1.5)*. Then, there exist constants* C, C_p, $\theta_0 > 0$ *that do not depend on the particular solution nor on* ϵ*, such that, for* $i = 1, \dots, d$ *and for any* $p \in [1, +\infty)$*,*

$$\|u^i\|_{W^{1,p}(\mathbb{T}^N)} \leq C_p. \tag{4.1}$$

Moreover, for any $\gamma \in (0,1)$*,*

$$\|u^i\|_{C^\gamma(\mathbb{T}^N)} \leq C. \tag{4.2}$$

In addition,

$$\theta^i \geq \theta_0 \quad \text{in } \mathbb{T}^N, \tag{4.3}$$

$$\|\theta^i\|_{W^{1,2}(\mathbb{T}^N)} \leq C, \tag{4.4}$$

and

$$\|u^i\|_{W^{2,2}(\mathbb{T}^N)} \leq C. \tag{4.5}$$

Proof. In what follows, we use C and C_p to denote any of several constants, possibly depending on p but independent of ϵ. We remark that, for any $p \geq 1$, there exists a constant, $C_p > 0$, such that

$$\log(\theta^i) \leq (\theta^i)^{\frac{1}{p}} + C_p.$$

Therefore, from (1.4), using (2.6) in Remark 2.3 and the positivity of β_ϵ, we infer that

$$C|Du^i|^2 \leq (\theta^i)^{\frac{1}{p}} + C_p - u^i = (\theta^i)^{\frac{1}{p}} + C_p - \left(u^i - \int_{\mathbb{T}^N} u^i dx\right) - \int_{\mathbb{T}^N} u^i dx.$$

Combining the previous inequality with (3.3) yields

$$|Du^i|^{2p} \leq C\theta^i + C\left|u^i - \int_{\mathbb{T}^N} u^i dx\right|^p + C_p.$$

Then, integrating, using (3.1) and the Poincaré inequality, we get

$$\begin{aligned}
\int_{\mathbb{T}^N} |Du^i|^{2p} dx &\leq C\int_{\mathbb{T}^N} \theta^i dx + C\int_{\mathbb{T}^N}\left|u^i - \int_{\mathbb{T}^N} u^i dx\right|^p dx + C_p \\
&\leq C_p \int_{\mathbb{T}^N} |Du^i|^p dx + C_p \\
&\leq \frac{1}{2}\int_{\mathbb{T}^N} |Du^i|^{2p} dx + C_p.
\end{aligned}$$

We conclude that, for any $p \geq 1$,

$$\int_{\mathbb{T}^N} |Du^i|^{2p} dx \leq C.$$

This bound, together with (3.3) and the Poincaré inequality, gives (4.1). The Sobolev Embedding Theorem then implies (4.2). In particular, we have

$$\|u^i\|_{L^\infty(\mathbb{T}^N)} \leq C.$$

From (1.4), the previous estimate and the positivity of H and β, we infer that

$$\log(\theta^i) \geq -C,$$

from which (4.3) follows.

Estimate (4.4) is a consequence of (3.1), (3.12), (4.3) and the Poincaré inequality.

Finally, estimate (4.5) is a consequence of (3.3), (3.6), (3.11), (4.3) and the Poincaré inequality.

□

4.2. **Power case.** We devote this section to the study of power nonlinearities. We begin by examining the general case, Assumption 5 **P**. Then, we obtain additional results by considering Assumption 5 **P-**$\frac{2}{N}$. As in the previous section, our estimates are uniform in ϵ.

Proposition 4.2. *Suppose that Assumptions 1-4, 5* **P** *and 8 hold. Let (u,θ) be a C^∞ solution of (1.4)-(1.5). Then, there exist constants, $C>0$ and $\gamma\in(0,1)$, that do not depend on the particular solution nor on ϵ, such that, for $i=1,\dots,d$,*

$$\|u^i\|_{W^{1,\frac{2}{\alpha}}(\mathbb{T}^N)} \leq C \tag{4.6}$$

and

$$\int_{\mathbb{T}^N} |D((\theta^i)^{\frac{\alpha+1}{2}}Du^i)|^{\frac{2}{\alpha+1}}dx \leq C. \tag{4.7}$$

If, in addition, Assumption 5 **P-**$\frac{2}{N}$ *holds, then there exists $\gamma=\gamma(\alpha)$ such that*

$$\|u^i\|_{C^\gamma(\mathbb{T}^N)} \leq C. \tag{4.8}$$

Proof. In what follows, we denote by C several constants that are independent of ϵ and δ. From (1.4), (2.6), and $\beta_\epsilon\geq 0$, we infer that

$$C|Du^i|^2 \leq (\theta^i)^\alpha + C - u^i = (\theta^i)^\alpha + C - \left(u^i - \int_{\mathbb{T}^N} u^i dx\right) - \int_{\mathbb{T}^N} u^i dx.$$

Consequently, from (3.3),

$$|Du^i|^{\frac{2}{\alpha}} \leq C\theta^i + C\left|u^i - \int_{\mathbb{T}^N} u^i dx\right|^{\frac{1}{\alpha}} + C.$$

Then, integrating and using (3.1) and the Poincaré inequality, we get

$$\begin{aligned}
\int_{\mathbb{T}^N} |Du^i|^{\frac{2}{\alpha}}dx &\leq C\int_{\mathbb{T}^N}\theta^i dx + C\int_{\mathbb{T}^N}\left|u^i - \int_{\mathbb{T}^N} u^i dx\right|^{\frac{1}{\alpha}}dx + C\\
&\leq C\int_{\mathbb{T}^N}|Du^i|^{\frac{1}{\alpha}}dx + C\\
&\leq \frac{1}{2}\int_{\mathbb{T}^N}|Du^i|^{\frac{2}{\alpha}}dx + C.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{T}^N}|Du^i|^{\frac{2}{\alpha}}dx \leq C,$$

which gives, together with (3.3) and the Poincaré inequality, the bound (4.6). If $\alpha\in\left(0,\frac{2}{N}\right)$, then $\frac{2}{\alpha}>N$. Therefore, estimate (4.8) is a consequence of (4.6) combined with the Sobolev Embedding Theorem.

Next, to prove (4.7), we compute

$$D((\theta^i)^{\frac{\alpha+1}{2}} Du^i) = \frac{\alpha+1}{2}\theta^{\frac{\alpha-1}{2}} Du^i \otimes D\theta^i + \theta^{\frac{\alpha+1}{2}} D^2u^i. \tag{4.9}$$

Now, using Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{T}^N} \left[\theta^{\frac{\alpha-1}{2}}|D\theta^i||Du^i|\right]^{\frac{2}{\alpha+1}} dx \\
&= \int_{\mathbb{T}^N} (\theta^i)^{\frac{\alpha-1}{\alpha+1}}|D\theta^i|^{\frac{2}{\alpha+1}}|Du^i|^{\frac{2}{\alpha+1}} dx \\
&\leq \left[\int_{\mathbb{T}^N} \left[(\theta^i)^{\frac{\alpha-1}{\alpha+1}}|D\theta^i|^{\frac{2}{\alpha+1}}\right]^{\alpha+1} dx\right]^{\frac{1}{\alpha+1}} \left[\int_{\mathbb{T}^N} |Du^i|^{\frac{2}{\alpha+1}(\alpha+1)'} dx\right]^{\frac{1}{(\alpha+1)'}} \\
&= \left[\int_{\mathbb{T}^N} (\theta^i)^{\alpha-1}|D\theta^i|^2 dx\right]^{\frac{1}{\alpha+1}} \left[\int_{\mathbb{T}^N} |Du^i|^{\frac{2}{\alpha}} dx\right]^{\frac{\alpha}{\alpha+1}}.
\end{aligned}$$

From (3.13) and (4.6), we infer that

$$\int_{\mathbb{T}^N} \left[\theta^{\frac{\alpha-1}{2}}|D\theta^i||Du^i|\right]^{\frac{2}{\alpha+1}} dx \leq C. \tag{4.10}$$

Next, using Hölder's inequality again, we have

$$\begin{aligned}
\int_{\mathbb{T}^N} \left[|D^2u^i|(\theta^i)^{\frac{\alpha+1}{2}}\right]^{\frac{2}{\alpha+1}} dx &= \int_{\mathbb{T}^N} |D^2u^i|^{\frac{2}{\alpha+1}}\theta^i dx = \int_{\mathbb{T}^N} \left[|D^2u^i|^2\theta^i\right]^{\frac{1}{\alpha+1}} (\theta^i)^{\frac{\alpha}{\alpha+1}} dx \\
&\leq \left[\int_{\mathbb{T}^N} |D^2u^i|^2\theta^i dx\right]^{\frac{1}{\alpha+1}} \left[\int_{\mathbb{T}^N} \theta^i dx\right]^{\frac{\alpha}{\alpha+1}}.
\end{aligned}$$

From (3.1) and (3.11), we gather the bound

$$\int_{\mathbb{T}^N} \left[|D^2u^i|(\theta^i)^{\frac{\alpha+1}{2}}\right]^{\frac{2}{\alpha+1}} dx \leq C. \tag{4.11}$$

Estimate (4.7) is then a consequence of (4.9), (4.10) and (4.11). □

Proposition 4.3. *Suppose that Assumptions 1-4, 5* **P-**$\frac{2}{N}$*, 6, and 8 hold. Let* (u, θ) *be a* C^∞ *solution of* (1.4)-(1.5). *Then, for* $i = 1, \ldots, d$ *and any* $x \in \mathbb{T}^N$*, we have*

$$\theta^i(x) \geq \left(1 - \max_{\substack{x\in\mathbb{T}^N \\ j=1,\ldots,d}} H^j(0,x)\right)^{\frac{1}{\alpha}}. \tag{4.12}$$

Moreover, there exists $C > 0$ *that does not depend on the particular solution nor on* ϵ*, such that*

$$\|\theta^i\|_{W^{1,2}(\mathbb{T}^N)} \leq C \tag{4.13}$$

and

$$\|u^i\|_{W^{2,2}(\mathbb{T}^N)} \leq C. \tag{4.14}$$

Proof. We begin the proof by establishing a lower bound on u. Let $i \in \{1, \ldots, d\}$ and $x_0 \in \mathbb{T}^N$ be such that

$$u^i(x_0) = \min_{\substack{j=1,\ldots,d \\ x\in\mathbb{T}^N}} u^j(x).$$

Then, we have

$$Du^i(x_0) = 0,\ D^2u^i(x_0) \geq 0, \tag{4.15}$$

and

$$u^i(x_0) \leq u^j(x_0) \quad \text{for any } j = 1, \ldots, d.$$

In particular, the last inequality implies

$$\beta_\epsilon(u^i(x_0) - u^j(x_0) - \psi^{ij}(x_0)) = \beta'_\epsilon(u^i(x_0) - u^j(x_0) - \psi^{ij}(x_0)) = 0 \quad \text{for any } j = 1, \ldots, d. \tag{4.16}$$

From (1.4), (4.15), and (4.16), we infer that

$$H^i(0, x_0) + u^i(x_0) = g(\theta^i(x_0)). \tag{4.17}$$

We can substitute

$$\theta^i = g^{-1}\left(H^i(Du^i, x) + u^i + \sum_{j\neq i} \beta_\epsilon(u^i - u^j - \psi^{ij})\right)$$

in (1.5) to get

$$\begin{aligned}
&-\theta^i H^i_{p_k p_j} u^i_{x_j x_k} - \theta^i H^i_{p_k x_k} - \frac{1}{g'(\theta^i)}\big[H^i_{p_k} H^i_{p_j} u^i_{x_j x_k} + H^i_{x_k} H^i_{p_k} + H^i_{p_k} u^i_{x_k} \\
&+ \sum_{j\neq i} \beta'_\epsilon(u^i - u^j - \psi^{ij})(u^i - u^j - \psi^{ij})_{x_k} H^i_{p_k}\big] + \theta^i + \sum_{j\neq i} \beta'_\epsilon(u^i - u^j - \psi^{ij})\theta^i \\
&= \sum_{j\neq i} \beta'_\epsilon(u^j - u^i - \psi^{ji})\theta^j + 1.
\end{aligned}$$

Evaluating at $x = x_0$ and using (2.4), (4.15) and (4.16), we obtain

$$-\theta^i H^i_{p_k x_k}(0, x_0) - \frac{1}{g'(\theta^i(x_0))} H^i_{x_k}(0, x_0) H^i_{p_k}(0, x_0) + \theta^i(x_0) \geq 1.$$

Since H^i satisfies (2.9), the preceding inequality can be rewritten as

$$\theta^i(x_0) \geq 1.$$

Then, (4.17) and the last estimate imply

$$u^i(x_0) \geq g^{-1}\left(1 - H^i(0, x_0)\right). \tag{4.18}$$

Now, from (1.4), (4.18), (2.9), (2.10) and the positivity of H^j and β_ϵ, we infer that, for any $x \in \mathbb{T}^N$ and $j = 1, \ldots, d$,

$$g(\theta^j(x)) \geq H^j(Du^j, x) + u^j(x) \geq u^i(x_0) \geq 1 - H^i(0, x_0).$$

Thus, (4.12) follows from the preceding inequality.

Estimate (4.13) follows by combining (3.1), (3.12), (4.12) and the Poincaré inequality. Finally, (4.14) is a consequence of (3.3), (3.6), (3.11), (4.12), and the Poincaré inequality. □

5. Lipschitz bounds

In this section, we prove the Lipschitz continuity of u for any solution (u, θ) of (1.4)-(1.5). These bounds are used to establish the existence of solutions by the continuation method. In contrast to the results in the preceding sections, the estimates here depend on ϵ and are not valid for (1.1)-(1.2).

Lemma 5.1. *Suppose that Assumptions 1-5 and 8 hold, and that either*

- *Assumption 5* **L** *or*
- *Assumptions 5* **P-**$\frac{2}{N}$*, 6 and 7*

hold. Let (u, θ) be a C^∞ solution of (1.4)-(1.5). Then, there exists $C_\epsilon > 0$ that does not depend on the particular solution, such that, for any $i = 1, \ldots, d$,

$$\|\theta^i\|_{L^\infty(\mathbb{T}^N)} \leq C_\epsilon \tag{5.1}$$

and

$$\|u^i\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C_\epsilon. \tag{5.2}$$

Proof. First, note that (5.2) is an immediate consequence of (5.1). Indeed, by combining (5.1) with (2.6), by the positivity of β_ϵ, by the boundedness of u^i (c.f. Proposition 4.1 and Proposition 4.2), and by (1.4), we get

$$C|Du^i|^2 \leq H^i(Du^i) + C \leq g(\theta^i) - u^i + C \leq C.$$

Consequently, we only need to prove (5.1).

If Assumption 5 **L** holds or if Assumptions 5 **P-**$\frac{2}{N}$ and 6 hold by, respectively, Propositions 4.1 and 4.3, then there exists $\theta_0 > 0$, such that, for any $i = 1, \ldots, d$ and for any $x \in \mathbb{T}^N$,

$$\theta^i(x) \geq \theta_0 > 0. \tag{5.3}$$

For any fixed ϵ, by (4.2) and (4.8), there exists a constant, C, depending on ϵ, such that, for any $i, j = 1, \ldots, d$,

$$\beta'_\epsilon(u^i - u^j - \psi^{ij}) \leq C. \tag{5.4}$$

To prove (5.1), we use a technique introduced in [16] and used in [22] to study a mean-field-game obstacle problem. For $p > 0$, we multiply the equation (1.5) by

$\operatorname{div}((\theta^i)^p D_p H^i(Du^i,x))$ and integrate by parts. Accordingly, we get

(5.5)

$$\begin{aligned}
&\int_{\mathbb{T}^N} [\theta^i + \sum_{j\neq i} \beta'_\epsilon(u^i-u^j-\psi^{ij})\theta^i - \sum_{j\neq i}\beta'_\epsilon(u^j-u^i-\psi^{ji})\theta^j]\operatorname{div}((\theta^i)^p D_p H^i) dx\\
&= \int_{\mathbb{T}^N} (\theta^i H^i_{p_k})_{x_k}((\theta^i)^p H^i_{p_j})_{x_j} dx\\
&= \int_{\mathbb{T}^N} (\theta^i H^i_{p_k})_{x_j}((\theta^i)^p H^i_{p_j})_{x_k} dx\\
&= \int_{\mathbb{T}^N} (\theta^i (H^i_{p_k})_{x_j} + \theta^i_{x_j} H^i_{p_k})((\theta^i)^p (H^i_{p_j})_{x_k} + p(\theta^i)^{p-1}\theta^i_{x_k} H^i_{p_j}) dx\\
&= \int_{\mathbb{T}^N} (\theta^i)^{p+1} (H^i_{p_k})_{x_j}(H^i_{p_j})_{x_k} + p(\theta^i)^{p-1} H^i_{p_k}\theta^i_{x_k} H^i_{p_j}\theta^i_{x_j}\\
&+ (p+1)(\theta^i)^p \theta^i_{x_k} H^i_{p_j}(H^i_{p_k})_{x_j} dx\\
&=: \int_{\mathbb{T}^N} I_1 + I_2 + I_3 dx.
\end{aligned}$$

Using (2.5) in Assumption 4, we get

$$\begin{aligned}
I_1 &= (\theta^i)^{p+1}(H^i_{p_k p_l} u_{x_l x_j} + H^i_{p_k x_j})(H^i_{p_j p_m} u_{x_m x_k} + H^i_{p_j x_k})\\
&\geq (\theta^i)^{p+1}[\gamma^2|D^2u^i|^2 - C(1+|Du^i|)|D^2u^i| - C(1+|Du^i|^2)]\\
&\geq (\theta^i)^{p+1}\tilde\gamma^2|D^2u^i|^2 - C(\theta^i)^{p+1}(1+|Du^i|^2)
\end{aligned}$$

for some $\tilde\gamma > 0$. Clearly,

$$I_2 = p(\theta^i)^{p-1}|D_pH^i\cdot D\theta^i|^2.$$

Next, we estimate I_3 from below. From equation (1.4), we gather that

$$H^i_{p_j} u^i_{x_j x_l} = g'(\theta^i)\theta^i_{x_l} - H^i_{x_l} - u^i_{x_l} - \sum_{j\neq i}\beta'_\epsilon(u^i-u^j-\psi^{ij})(u^i-u^j-\psi^{ij})_{x_l}. \tag{5.6}$$

The estimate (2.8) in Assumption 4 and the lower bound (5.3) on θ^i imply the existence of a positive constant, C_0 (depending on ϵ), such that

$$g'(\theta^i)\theta^i \geq C_0 > 0. \tag{5.7}$$

Then, using (2.4), (2.5), (5.4), (5.6), (5.7), and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
I_3 &= (p+1)(\theta^i)^p\theta^i_{x_k} H^i_{p_j}(H^i_{p_k p_l} u^i_{x_l x_j} + H^i_{p_k x_j})\\
&= (p+1)g'(\theta^i)(\theta^i)^p H^i_{p_k p_l}\theta^i_{x_k}\theta^i_{x_l} + (p+1)(\theta^i)^p\theta^i_{x_k}(H^i_{p_j}H^i_{p_k x_j} - H^i_{p_k p_l}H^i_{x_l})\\
&- (p+1)(\theta^i)^p H^i_{p_k p_l}\theta^i_{x_k}\left(u^i_{x_l} + \sum_{j\neq i}\beta'_\epsilon(u^i-u^j-\psi^{ij})(u^i-u^j-\psi^{ij})_{x_l}\right)\\
&\geq (p+1)\gamma C_0(\theta^i)^{p-1}|D\theta^i|^2 - C(p+1)(\theta^i)^p|D\theta^i|(1+|Du^i|^2)\\
&- C(p+1)(\theta^i)^p|D\theta^i|(1+|Du^i|) - \sum_{j\neq i} C(p+1)(\theta^i)^p|D\theta^i|(1+|Du^j|)\\
&\geq C(p+1)(\theta^i)^{p-1}|D\theta^i|^2 - C(p+1)(\theta^i)^{p+1}\left(1+|Du^i|^4 + \sum_{j\neq i}|Du^j|^2\right).
\end{aligned}$$

Next, we bound the left-hand side of (5.5) from above. Using (2.5), (5.4), and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&(\theta^i+\sum_{j\neq i}\beta_\epsilon'(u^i-u^j-\psi^{ij})\theta^i)\operatorname{div}((\theta^i)^pD_pH^i)\\
&=[\theta^i+\sum_{j\neq i}\beta_\epsilon'(u^i-u^j-\psi^{ij})\theta^i][p(\theta^i)^{p-1}D_pH^i\cdot D\theta^i+(\theta^i)^pH^i_{p_kp_j}u^i_{x_jx_k}+(\theta^i)^pH^i_{p_kx_k}]\\
&\leq C\theta^i[p(\theta^i)^{p-1}|D_pH^i\cdot D\theta^i|+(\theta^i)^p|H^i_{p_kp_j}u^i_{x_jx_k}|+(\theta^i)^p|H^i_{p_kx_k}|]\\
&\leq \frac{p}{2}(\theta^i)^{p-1}|D_pH^i\cdot D\theta^i|^2+\frac{\tilde{\gamma}^2}{2}(\theta^i)^{p+1}|D^2u^i|^2+Cp(\theta^i)^{p+1}(1+|Du^i|).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&-\sum_{j\neq i}\beta_\epsilon'(u^j-u^i-\psi^{ji})\theta^j\operatorname{div}((\theta^i)^pD_pH^i)\\
&\leq \sum_{j\neq i}C\theta^j\left|p(\theta^i)^{p-1}|D_pH^i\cdot D\theta^i|+(\theta^i)^p|H^i_{p_kp_j}u^i_{x_jx_k}|+(\theta^i)^p|H^i_{p_kx_k}|\right|\\
&\leq \sum_{j\neq i}C\theta^j\left[p(\theta^i)^{\frac{p-1}{2}}(\theta^i)^{\frac{p-1}{2}}\left|D_pH^i\cdot D\theta^i\right|+(\theta^i)^{\frac{p-1}{2}}(\theta^i)^{\frac{p+1}{2}}\left|H^i_{p_kp_j}u^i_{x_jx_k}\right|+(\theta^i)^{\frac{p-1}{2}}(\theta^i)^{\frac{p+1}{2}}\left|H^i_{p_kx_k}\right|\right]\\
&\leq \sum_{j\neq i}Cp(\theta^j)^2(\theta^i)^{p-1}+\frac{p}{2}(\theta^i)^{p-1}|D_pH^i\cdot D\theta^i|^2+\frac{\tilde{\gamma}^2}{2}(\theta^i)^{p+1}|D^2u^i|^2+Cp(\theta^i)^{p+1}(1+|Du^i|^2).
\end{aligned}
$$

From the preceding estimates, we conclude that

$$
\begin{aligned}
&C(p+1)\int_{\mathbb{T}^N}(\theta^i)^{p-1}|D\theta^i|^2dx-C(p+1)\int_{\mathbb{T}^N}(\theta^i)^{p+1}(1+|Du^i|^4+\sum_{j\neq i}|Du^j|^2)dx\\
&+p\int_{\mathbb{T}^N}(\theta^i)^{p-1}|D_pH^i\cdot D\theta^i|^2dx\\
&+\tilde{\gamma}^2\int_{\mathbb{T}^N}(\theta^i)^{p+1}|D^2u^i|^2-C\int_{\mathbb{T}^N}(\theta^i)^{p+1}(1+|Du^i|^2)dx\\
&\leq\int_{\mathbb{T}^N}I_1+I_2+I_3dx\\
&=\int_{\mathbb{T}^N}[\theta^i+\sum_{j\neq i}\beta_\epsilon'(u^i-u^j-\psi^{ij})\theta^i-\sum_{j\neq i}\beta_\epsilon'(u^j-u^i-\psi^{ji})\theta^j]\operatorname{div}((\theta^i)^pD_pH^i)dx\\
&\leq p\int_{\mathbb{T}^N}(\theta^i)^{p-1}|D_pH^i\cdot D\theta^i|^2dx+\tilde{\gamma}^2\int_{\mathbb{T}^N}(\theta^i)^{p+1}|D^2u^i|^2dx\\
&+Cp\int_{\mathbb{T}^N}(\theta^i)^{p+1}(1+|Du^i|^2)dx+\sum_{j\neq i}Cp\int_{\mathbb{T}^N}(\theta^j)^2(\theta^i)^{p-1}.
\end{aligned}
$$

Consequently,

$$
\int_{\mathbb{T}^N}(\theta^i)^{p-1}|D\theta^i|^2dx\leq\int_{\mathbb{T}^N}(\theta^i)^{p+1}(1+|Du^i|^4+\sum_{j\neq i}|Du^j|^2)dx+\sum_{j\neq i}C\int_{\mathbb{T}^N}(\theta^j)^2(\theta^i)^{p-1}dx.
$$

Applying Young's inequality, we gather

$$
(\theta^j)^2(\theta^i)^{p-1}\leq\frac{2}{p+1}(\theta^j)^{p+1}+\frac{p-1}{p+1}(\theta^i)^{p+1}.
$$

Therefore,

(5.8)
$$\int_{\mathbb{T}^N} (\theta^i)^{p-1}|D\theta^i|^2 dx \le \int_{\mathbb{T}^N} (\theta^i)^{p+1}(1+|Du^i|^4 + \sum_{j\neq i} |Du^j|^2)dx + \sum_{j\neq i} C \int_{\mathbb{T}^N} (\theta^j)^{p+1} dx.$$

If $N = 1$, (5.1) is a consequence of Morrey's Theorem. If $N = 2$, then $\theta^i \in L^p$ for all p. Moreover, in this case, the argument that follows can be modified by replacing the Sobolev exponent, 2^*, by any arbitrarily large number, M. Therefore, we assume that $N > 2$. Accordingly, by (3.13), we have $\theta^i \in L^{\frac{2^*(1+\alpha)}{2}}$. In addition, Sobolev's inequality provides the bound

$$\begin{aligned}\left(\int_{\mathbb{T}^N} (\theta^i)^{\frac{p+1}{2}2^*} dx\right)^{\frac{2}{2^*}} &\le C\int_{\mathbb{T}^N} (\theta^i)^{p+1}dx + C\int_{\mathbb{T}^N} |D((\theta^i)^{\frac{p+1}{2}})|^2 dx \\ &= C\int_{\mathbb{T}^N} (\theta^i)^{p+1}dx + C(p+1)^2\int_{\mathbb{T}^N} (\theta^i)^{p-1}|D\theta^i|^2 dx.\end{aligned} \tag{5.9}$$

Let $\beta := \sqrt{\frac{2^*}{2}} = \sqrt{\frac{N}{N-2}} > 1$. Consider first the **P** case. Then, Assumption 7 implies that

$$2\alpha \le (\alpha+1)\beta^2\frac{\beta-1}{\beta}.$$

The previous inequality together with (2.6) implies that

$$|Du^i|^4 \le C(g(\theta^i))^2 + C \le C(\theta^i)^{2\alpha} + C \le C(1+(\theta^i)^{(\alpha+1)\beta^2\frac{\beta-1}{\beta}}).$$

The same inequality holds in the logarithmic case with $\alpha = 0$:

$$|Du^i|^4 \le C(g(\theta^i))^2 + C \le C(\log(\theta^i))^2 + C \le C(1+(\theta^i)^{\beta^2\frac{\beta-1}{\beta}}).$$

Therefore, from Hölder's inequality, we get

$$\begin{aligned}&\int_{\mathbb{T}^N} (\theta^i)^{p+1}(1+|Du^i|^4)dx \le C\int_{\mathbb{T}^N} (\theta^i)^{p+1}(1+(\theta^i)^{(\alpha+1)\beta^2\frac{\beta-1}{\beta}})dx \\ &\le C\int_{\mathbb{T}^N} (\theta^i)^{p+1}dx + C\left(\int_{\mathbb{T}^N} (\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}\left(\int_{\mathbb{T}^N} (\theta^i)^{(\alpha+1)\beta^2}dx\right)^{\frac{\beta-1}{\beta}} \\ &\le C\int_{\mathbb{T}^N} (\theta^i)^{p+1}dx + C\left(\int_{\mathbb{T}^N} (\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}} \\ &\le C\left(\int_{\mathbb{T}^N} (\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_{\mathbb{T}^N} (\theta^i)^{p+1}(1+|Du^j|^2)dx \le C\int_{\mathbb{T}^N} (\theta^i)^{p+1}(1+(\theta^j)^{(\alpha+1)\beta^2\frac{\beta-1}{\beta}})dx \\ &\le C\int_{\mathbb{T}^N} (\theta^i)^{p+1}dx + C\left(\int_{\mathbb{T}^N} (\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}\left(\int_{\mathbb{T}^N} (\theta^j)^{(\alpha+1)\beta^2}dx\right)^{\frac{\beta-1}{\beta}} \\ &\le C\int_{\mathbb{T}^N} (\theta^i)^{p+1}dx + C\left(\int_{\mathbb{T}^N} (\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}} \\ &\le C\left(\int_{\mathbb{T}^N} (\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}.\end{aligned}$$

The last two inequalities, combined with (5.8) and (5.9) give the bound

$$\left(\int_{\mathbb{T}^N}(\theta^i)^{(p+1)\beta^2}dx\right)^{\frac{1}{\beta^2}} \leq Cp^2\left(\int_{\mathbb{T}^N}(\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}} + Cp^2\sum_{j\neq i}\left(\int_{\mathbb{T}^N}(\theta^j)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}$$

for $i = 1,\dots,d$. Summing on i, we finally obtain

$$\sum_{i=1}^{d}\left(\int_{\mathbb{T}^N}(\theta^i)^{(p+1)\beta^2}dx\right)^{\frac{1}{\beta^2}} \leq Cp^2\sum_{i=1}^{d}\left(\int_{\mathbb{T}^N}(\theta^i)^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}. \tag{5.10}$$

Arguing as in [16], we get

$$\sum_{i=1}^{d}\|\theta^i\|_{L^\infty(\mathbb{T}^N)} \leq C$$

and, hence, (5.1). □

Corollary 5.2. *Suppose that Assumptions 1-5 and 8 hold, and either*

- *Assumption 5* **L** *or*
- *Assumptions 5* **P-**$\frac{2}{N}$*, 6 and 7*

hold. Let (u,θ) *be a* C^∞ *solution of* (1.4)-(1.5)*. Then, for any* $k \in \mathbb{N}$*, there exists* $C_{\epsilon,k} > 0$ *that does not depend on the particular solution, such that*

$$\|u^i\|_{W^{k,\infty}(\mathbb{T}^N)} + \|\theta^i\|_{W^{k,\infty}(\mathbb{T}^N)} \leq C_{\epsilon,k}, \tag{5.11}$$

for any $i = 1,\dots,d$.

Proof. If Assumption 5 **L** holds or if Assumptions 5 **P-**$\frac{2}{N}$ and 6 hold by, respectively, Propositions 4.1, and 4.3, then there exists $\theta_0 > 0$, such that, for any $i = 1,\dots,d$ and any $x \in \mathbb{T}^N$,

$$\theta^i(x) \geq \theta_0 > 0. \tag{5.12}$$

Thus, we use

$$\theta^i = g^{-1}\left(H^i(Du^i,x) + u^i + \sum_{j\neq i}\beta_\epsilon(u^i - u^j - \psi^{ij})\right)$$

in (1.5) to get

$$\begin{aligned}
&-\theta^i H^i_{p_kp_j}u^i_{x_jx_k} - \theta^i H^i_{p_kx_k} - \frac{1}{g'(\theta^i)}\big[H^i_{p_k}H^i_{p_j}u^i_{x_jx_k} + H^i_{x_k}H^i_{p_k} + H^i_{p_k}u^i_{x_k}\\
&+\sum_{j\neq i}\beta'_\epsilon(u^i-u^j-\psi^{ij})(u^i-u^j-\psi^{ij})_{x_k}H^i_{p_k}\big] + \theta^i + \sum_{j\neq i}\beta'_\epsilon(u^i-u^j-\psi^{ij})\theta^i\\
&=\sum_{j\neq i}\beta'_\epsilon(u^j-u^i-\psi^{ji})\theta^j + 1.
\end{aligned}$$

From estimates (5.1), (5.2), and (5.12), we have that the previous equation is a uniformly elliptic equation for each i. Therefore, from the elliptic regularity theory, we infer that

$$\|u^i\|_{W^{2,p}(\mathbb{T}^N)} + \|\theta^i\|_{W^{1,p}(\mathbb{T}^N)} \leq C_\epsilon$$

for any $1 < p < \infty$. Repeated differentiation and a bootstrapping argument give (5.11). □

6. Proof of Theorem 1.2

In this section, we show the existence and uniqueness of a classical solution of (1.4)-(1.5). In the proof of existence, we use the continuation method. In the proof of uniqueness, we rely on a monotonicity argument. Here, we work under Assumptions 1-4 and either 5 **L** or 5 **P**-$\frac{2}{N}$ together with Assumptions 6 and 7.

6.1. **Existence.** To prove the existence of a classical solution of (1.4)-(1.5) using the continuation method, we define

$$H^i_\lambda(p,x) := \lambda H^i(p,x) + (1-\lambda)\frac{|p|^2}{2}$$

for $0 \le \lambda \le 1$, $(p,x) \in \mathbb{R}^N \times \mathbb{T}^N$, and $i = 1, \dots, d$. We introduce the mean-field game

$$H^i_\lambda(Du^i_\lambda, x) + u^i_\lambda + \sum_{j\neq i} \beta_\epsilon(u^i_\lambda - u^j_\lambda - \psi^{ij}) = g(\theta^i_\lambda), \tag{6.1}$$

$$-\mathrm{div}(D_pH^i_\lambda(Du^i_\lambda, x)\theta^i_\lambda) + \theta^i_\lambda + \sum_{j\neq i} \beta'_\epsilon(u^i_\lambda - u^j_\lambda - \psi^{ij})\theta^i_\lambda - \beta'_\epsilon(u^j_\lambda - u^i_\lambda - \psi^{ji})\theta^j_\lambda = 1 \tag{6.2}$$

for $x \in \mathbb{T}^N$ and $i = 1, \dots, d$.

Next, for $k \in \mathbb{N}$, we set $E^k := (H^k(\mathbb{T}^N))^d$, $E^0 := (L^2(\mathbb{T}^N))^d$. If $k > \frac{N}{2}$, E^k is an algebra. Moreover, $E^k \subset \left(C^\gamma(\mathbb{T}^N)\right)^d$ for any $0 < \gamma < 2 - \frac{N}{k}$. Given $\theta_0 > 0$ and $k > \frac{N}{2}$, we set

$$E^k_{\theta_0} := \{\theta \in E^k \,|\, \theta^i \ge \theta_0,\ i = 1, \dots, d\}.$$

Finally, for $k > \frac{N}{2}$, we define $F : [0,1] \times E^{k+2} \times E^{k+1}_{\theta_0} \to E^k \times E^{k+1}$ by

$$F(\lambda, u, \theta) :=$$

$$\begin{pmatrix} \mathrm{div}(D_pH^i_\lambda(Du^i_\lambda, x)\theta^i_\lambda) - \theta^i_\lambda - \sum_{j\neq i}(\beta'_{\epsilon,\lambda}(u^i_\lambda - u^j_\lambda - \psi^{ij})\theta^i_\lambda - \beta'_{\epsilon,\lambda}(u^j_\lambda - u^i_\lambda - \psi^{ji})\theta^j_\lambda) + 1 \\ H^i_\lambda(Du^i_\lambda, x) + u^i_\lambda + \sum_{j\neq i} \beta_\epsilon(u^i_\lambda - u^j_\lambda - \psi^{ij}) - g(\theta^i_\lambda) \end{pmatrix}.$$

Then, (6.1)-(6.2) is equivalent to

$$F(\lambda, u_\lambda, \theta_\lambda) = 0.$$

Let

$$\Lambda := \{\lambda \in [0,1] \,|\, \text{(6.1)-(6.2) has a classical solution } (u_\lambda, \theta_\lambda)\}.$$

Next, we show that

$$\Lambda = [0,1].$$

We divide the proof of this identity into the three following claims.

Claim 1: $0 \in \Lambda$. Indeed, for $\lambda = 0$, we have the explicit solution:

$$u^i_0 = g(1), \quad \theta^i_0 = 1, \quad i = 1, \dots, d.$$

Claim 2: Λ *is closed.* To prove this claim, we show that, for any sequence, $(\lambda_k)_k \subset \Lambda$, such that $\lambda_k \to \lambda_0$ as $k \to +\infty$, we have that $\lambda_0 \in \Lambda$. Accordingly, let $(u_{\lambda_k}, \theta_{\lambda_k})$ be a classical solution of (6.1)-(6.2) for $\lambda = \lambda_k$. Recall that H_λ satisfies Assumptions 1-4 and Assumption 6 uniformly in $0 \le \lambda \le 1$. Therefore, by (5.11), we can bound the derivatives of any order of $(u_{\lambda_k}, \theta_{\lambda_k})$ by a constant that is independent of k. Consequently, we can extract a subsequence of smooth solutions converging to a limit function (u,θ) that solves (6.1)-(6.2) with $\lambda=\lambda_0$. Thus, $\lambda_0\in\Lambda$.

Claim 3: Λ is open. To prove this last claim, we need to check that, for any $\lambda_0\in\Lambda$, there exists a neighborhood of λ_0 contained in Λ. To do so, we use the implicit function theorem. To simplify the notation, for $h=\beta,\beta',\beta''$, we set

$$h_{\epsilon,\lambda_0}(i,j):=h_\epsilon(u^i_{\lambda_0}-u^j_{\lambda_0}-\psi^{ij}).$$

For $\lambda_0\in\Lambda$, we consider the Fréchet derivative, $\mathcal{L}_{\lambda_0}:E^{k+2}\times E^{k+1}\to E^k\times E^{k+1}$, of $(u,\theta)\to F(\lambda_0,u,\theta)$ at $(u_{\lambda_0},\theta_{\lambda_0})$. We have

$$\mathcal{L}_{\lambda_0}(v,f) = \begin{pmatrix} (H^i_{\lambda_0,p_kp_j}(Du^i_{\lambda_0},x)v^i_{x_j}\theta^i_{\lambda_0}+H^i_{\lambda_0,p_k}(Du^i_{\lambda_0},x)f^i)_{x_k}-f^i \\ -\sum_{j\neq i}[(\beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0}+\beta''_{\epsilon,\lambda_0}(j,i)\theta^j_{\lambda_0})(v^i-v^j)+\beta'_{\epsilon,\lambda_0}(i,j)f^i-\beta'_{\epsilon,\lambda_0}(j,i)f^j] \\ \\ D_pH^i_{\lambda_0}(Du^i_{\lambda_0},x)\cdot Dv^i+v^i+\sum_{j\neq i}(\beta'_{\epsilon,\lambda_0}(i,j)(v^i-v^j))-g'(\theta^i_{\lambda_0})f^i \end{pmatrix}. \tag{6.3}$$

Because of the a priori bounds for smooth solutions (5.11) and either estimate (4.3), in the **L** case, or (4.12), in the **P-**$\frac{2}{N}$ case, the operator, $\mathcal{L}_{\lambda_0}$, is well defined in $E^{k+2}\times E^{k+1}$ for any $k\geq 0$. Next, we prove that $\mathcal{L}_{\lambda_0}$ is an isomorphism from $E^{k+2}\times E^{k+1}$ to $E^k\times E^{k+1}$ for any $k\geq 0$.

Let $w=(v,f)\in E^1\times E^0$. Define the bilinear form, $B_{\lambda_0}[w_1,w_2]:E^1\times E^0\to\mathbb{R}$, by

$$B_{\lambda_0}[w_1,w_2]:=\sum_{i=1}^d B^i_{\lambda_0}[w_1,w_2], \tag{6.4}$$

where

$$\begin{aligned} &B^i_{\lambda_0}[w_1,w_2]:= \\ &\int_{\mathbb{T}^N}-H^i_{\lambda_0,p_kp_j}(Du^i_{\lambda_0},x)v^i_{1,x_j}v^i_{2,x_k}\theta^i_{\lambda_0}-D_pH^i_{\lambda_0}(Du^i_{\lambda_0},x)\cdot Dv^i_2f^i_1-f^i_1v^i_2dx \\ &-\int_{\mathbb{T}^N}\Big[\sum_{j\neq i}(\beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0}+\beta''_{\epsilon,\lambda_0}(j,i)\theta^j_{\lambda_0})(v^i_1-v^j_1)+\beta'_{\epsilon,\lambda_0}(i,j)f^i_1-\beta'_{\epsilon,\lambda_0}(j,i)f^j_1\Big]v^i_2dx \\ &+\int_{\mathbb{T}^N}[D_pH^i_{\lambda_0}(Du^i_{\lambda_0},x)\cdot Dv^i_1+v^i_1+\sum_{j\neq i}(\beta'_{\epsilon,\lambda_0}(i,j)(v^i_1-v^j_1))-g'(\theta^i_{\lambda_0})f^i_1]f^i_2dx. \end{aligned}$$

If $w_1\in E^{k+2}\times E^{k+1}$ with $k\geq 0$, then

$$B_{\lambda_0}[w_1,w_2]=\int_{\mathbb{T}^N}\mathcal{L}_{\lambda_0}(w_1)\cdot w_2dx.$$

The following lemma is a straightforward consequence of estimate (5.11) combined with either (4.3) or (4.12).

Lemma 6.1. *Let B be the bilinear form given by* (6.4). *Then, there exists $C>0$ such that*

$$|B_{\lambda_0}[w_1,w_2]|\leq C\|w_1\|_{E^1\times E^0}\|w_2\|_{E^1\times E^0}$$

for any $w_1,w_2\in E^1\times E^0$.

Thus, in view of Riesz's representation theorem for Hilbert spaces, there exists a continuous linear mapping, $A : E^1 \times E^0 \to E^1 \times E^0$, such that

$$B_{\lambda_0}[w_1, w_2] = (Aw_1, w_2)_{E^1 \times E^0}. \tag{6.5}$$

Lemma 6.2. *The operator, A, defined in* (6.5) *is injective.*

Proof. Let $w = (v, f)$. Then, we have

$$B^i_{\lambda_0}[w, w] = \int_{\mathbb{T}^N} -H^i_{\lambda_0, p_k p_j}(Du^i_{\lambda_0}, x) v^i_{x_j} v^i_{x_k} \theta^i_{\lambda_0}$$
$$- \int_{\mathbb{T}^N} \sum_{j \neq i} (\beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0} + \beta''_{\epsilon,\lambda_0}(j,i)\theta^j_{\lambda_0})(v^i - v^j)v^i + g'(\theta^i_{\lambda_0})(f^i)^2 dx.$$

Summing the preceding expression on i, using the identity

$$\sum_i \sum_{j \neq i} (\beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0} + \beta''_{\epsilon,\lambda_0}(j,i)\theta^j_{\lambda_0})(v^i - v^j)v^i$$
$$= \sum_i \sum_{j \neq i} \beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0}(v^i - v^j)^2,$$

the convexity property (2.4) from Assumption 4 and either (4.3), in the **L** case, or (4.12), in the **P-**$\frac{2}{N}$ case, we get

$$\begin{aligned} B_{\lambda_0}[w, w] &= \sum_{i=1}^d B^i_{\lambda_0}[w, w] \\ &\le -\sum_{i=1}^d \int_{\mathbb{T}^N} H^i_{\lambda_0, p_k p_j}(Du^i_{\lambda_0}, x) v^i_{x_j} v^i_{x_k} \theta^i_{\lambda_0} + g'(\theta^i_{\lambda_0})(f^i)^2 dx \\ &- \sum_i \sum_{j \neq i} \int_{\mathbb{T}^N} \beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0}(v^i - v^j)^2 \\ &\le -C_{\lambda_0} \int_{\mathbb{T}^N} \|Dv\|^2 + \|f\|^2 dx \\ &- \theta_0 \sum_i \sum_{j \neq i} \int_{\mathbb{T}^N} \beta''_{\epsilon,\lambda_0}(i,j)(v^i - v^j)^2 \, dx. \end{aligned} \tag{6.6}$$

According to the previous inequality, if $Aw = 0$, we have $w = (\mu, 0)$ for some $\mu \in \mathbb{R}^d$. Next, by computing

$$0 = (Aw, (0, \mu)) = B[(\mu, 0), (0, \mu)] = \sum_{i=1}^d \int_{\mathbb{T}^N} v^i_1 f^i_2 dx = |\mu|^2,$$

we conclude that $\mu = 0$. □

Lemma 6.3. *The operator, A, given by* (6.5) *is surjective.*

Proof. First, we prove that the range of A is closed in $E^1 \times E^0$. For that, take a Cauchy sequence, $(z_n)_n$, in the range of A; that is, $z_n = Aw_n$ for some sequence $(w_n)_n$ in $E^1 \times E^0$. We claim that $(w_n)_n$ is a Cauchy sequence. Let $w_n = (v_n, f_n)$.

Then, according to (6.6), we have

$$\begin{aligned}(z_n - z_m, w_n - w_m)_{E^1\times E^0} &= (A(w_n - w_m), w_n - w_m)_{E^1\times E^0}\\ &= B[w_n - w_m, w_n - w_m]\\ &\leq -C(\|D(v_n - v_m)\|_{E^0}^2 + \|f_n - f_m\|_{E^0}^2)\\ &- \sum_i \sum_{j\neq i} \int_{\mathbb{T}^N} \beta''_{\epsilon,\lambda_0}(i,j)((v_n^i - v_m^i) - (v_n^j - v_m^j))^2\,dx.\end{aligned}$$

Moreover, we have

$$\begin{aligned}|(z_n - z_m, &w_n - w_m)_{E^1\times E^0}| \leq \|z_n - z_m\|_{E^1\times E^0}\|w_n - w_m\|_{E^1\times E^0}\\ &= \|z_n - z_m\|_{E^1\times E^0}(\|v_n - v_m\|_{E^0} + \|Dv_n - Dv_m\|_{E^0} + \|f_n - f_m\|_{E^0})\\ &\leq \delta(\|Dv_n - Dv_m\|_{E^0}^2 + \|f_n - f_m\|_{E^0}^2) + C_\delta\|z_n - z_m\|_{E^1\times E^0}^2\\ &+ \|z_n - z_m\|_{E^1\times E^0}\|v_n - v_m\|_{E^0}.\end{aligned}$$

Let μ be a positive constant to be chosen later. By selecting a suitably small δ and combining the inequalities above, we get

$$\begin{aligned}&\|Dv_n - Dv_m\|_{E^0}^2 + \|f_n - f_m\|_{E^0}^2\\ &+ \sum_i \sum_{j\neq i} \int_{\mathbb{T}^N\cap\{u^i - u^j - \psi^{ij} > 0\}} \beta''_{\epsilon,\lambda_0}(i,j)((v_n^i - v_m^i) - (v_n^j - v_m^j))^2\,dx\\ &\quad\leq C\|z_n - z_m\|_{E^1\times E^0}^2 + \|z_n - z_m\|_{E^1\times E^0}\|v_n - v_m\|_{E^0}\\ &\quad\leq C_\mu\|z_n - z_m\|_{E^1\times E^0}^2 + \mu\|v_n - v_m\|_{E^0}^2.\end{aligned}\tag{6.7}$$

Next, we have

$$\begin{aligned}B[w_n - w_m, (0, v_n - v_m)] =&\\ &\int_{\mathbb{T}^N}\sum_{i=1}^d \left[D_pH^i_{\lambda_0}(Du^i_{\lambda_0}, x)\cdot\left(D(v_n^i - v_m^i)\right)(v_n^i - v_m^i) + (v_n^i - v_m^i)^2\right]dx\\ &+ \int_{\mathbb{T}^N}\sum_{i=1}^d\sum_{i\neq j}\beta'_{\epsilon,\lambda_0}(i,j)[(v_n^i - v_m^i) - (v_n^j - v_m^j)](v_n^i - v_m^i)\,dx\\ &- \int_{\mathbb{T}^N}\sum_{i=1}^d g'(\theta^i_{\lambda_0})(f_n^i - f_n^i)(v_n^i - v_m^i)\,dx.\end{aligned}$$

Set

$$M = \left[\sum_{i=1}^d \int_{\mathbb{T}^N}\left(\sum_{i\neq j}\beta'_{\epsilon,\lambda_0}(i,j)[(v_n^i - v_m^i) - (v_n^j - v_m^j)]\right)^2\right]^{1/2}.$$

Then, using (5.1), (5.2), Hölder's inequality, and Young's inequality, we get

$$\begin{aligned}&B[w_n - w_m, (0, v_n - v_m)]\\ &\geq \|v_n - v_m\|_{E^0}^2\\ &- C(\|Dv_n - Dv_m\|_{E^0}^2 + \|v_n - v_m\|_{E^0}\|f_n - f_m\|_{E^0} + M\|v_n - v_m\|_{E^0})\\ &\geq C\|v_n - v_m\|_{E^0}^2 - C(\|Dv_n - Dv_m\|_{E^0}^2 + \|f_n - f_m\|_{E^0}^2 + M^2).\end{aligned}\tag{6.8}$$

On the other hand, (6.5) implies that

$$B[w_n - w_m, (0, v_n - v_m)] \le C\|z_n - z_m\|_{E^1\times E^0}\|v_n - v_m\|_{E^0}.$$

Therefore,

$$\begin{aligned}
&C\|v_n - v_m\|_{E^0}^2 - C(\|Dv_n - Dv_m\|_{E^0}^2 + \|f_n - f_m\|_{E^0}^2 + M^2)\\
&\le C\|z_n - z_m\|_{E^1\times E^0}\|v_n - v_m\|_{E^0}\\
&\le C\|z_n - z_m\|_{E^1\times E^0}^2 + \mu_2\|v_n - v_m\|_{E^0}^2
\end{aligned}$$

for some μ_2 to be selected later. Next, taking into account that

$$M^2 \le C\sum_i\sum_{j\neq i}\int_{\mathbb{T}^N}\beta''_{\epsilon,\lambda_0}(i,j)[(v_n^i - v_m^i) - (v_n^j - v_m^j)]^2\,dx$$

and using (6.7) and (6.8), we obtain

$$\|v_n - v_m\|_{E^0}^2 \le C\|z_n - z_m\|_{E^1\times E^0}^2.$$

Consequently, v_n is a Cauchy sequence in L^2. Finally, from (6.7) we infer that w_n is a Cauchy sequence in $E^1\times E^0$ because of the bound

$$\|w_n - w_m\|_{E^1\times E^0}^2 \le C\|z_n - z_m\|_{E^1\times E^0}^2.$$

The last inequality and the continuity of A imply that $R(A)$ is closed.

Finally, we prove that $R(A) = E^1\times E^0$. Suppose that $R(A)\neq E^1\times E^0$. Since $R(A)$ is closed, there exists $z\in R(A)^\perp$ with $z\neq 0$, such that $B_{\lambda_0}[z,z] = 0$. The argument in the proof of Lemma 6.2 implies that $z = 0$, which is a contradiction. □

Lemma 6.4. *The operator, $\mathcal{L}_{\lambda_0} : E^{k+2}\times E^{k+1}\to E^k\times E^{k+1}$, given by* (6.3) *is an isomorphism for all $k\in\mathbb{N}$.*

Proof. By Lemma 6.2, $\mathcal{L}_{\lambda_0}$ is injective. Therefore, it suffices to prove that $\mathcal{L}_{\lambda_0}$ is surjective. To do so, we fix $w_0\in E^0\times E^1$ with $w_0 = (v_0, f_0)$. We claim that there exists a solution, $w_1\in E^2\times E^1$, to $\mathcal{L}_{\lambda_0}w_1 = w_0$.

Consider the bounded linear functional, $w\to(w_0,w)_{(E^0)^2}$, in $E^1\times E^0$. According to the Riesz representation theorem, there exists $\widetilde{w}\in E^1\times E^0$, such that $(w_0,w)_{(E^0)^2} = (\widetilde{w},w)_{E^1\times E^0}$ for any $w\in E^1\times E^0$. In light of Lemmas 6.2 and 6.3, the operator A is invertible in $E^1\times E^0$. We define $w_1 := A^{-1}\widetilde{w}$. Then, for any $w\in E^1\times E^0$,

$$(Aw_1, w)_{E^1\times E^0} = (w_0, w)_{E^0\times E^0};$$

that is, $w_1 = (v, f)$ is a weak solution of

(6.9)
$$\begin{cases}(H^i_{\lambda_0,p_kp_j}(Du^i_{\lambda_0},x)v^i_{x_j}\theta^i_{\lambda_0} + H^i_{\lambda_0,p_k}(Du^i_{\lambda_0},x)f^i)_{x_k} - f^i\\ -\sum_{j\neq i}[(\beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0} + \beta''_{\epsilon,\lambda_0}(j,i)\theta^j_{\lambda_0})(v^i - v^j) + \beta'_{\epsilon,\lambda_0}(i,j)f^i - \beta'_{\epsilon,\lambda_0}(j,i)f^j] = v^i_0\\ \\ D_pH^i_{\lambda_0}(Du^i_{\lambda_0},x)\cdot Dv^i + v^i + \sum_{j\neq i}(\beta'_{\epsilon,\lambda_0}(i,j)(v^i - v^j)) - g'(\theta^i_{\lambda_0})f^i = f^i_0\end{cases}$$

for $x\in\mathbb{T}^N$ and $i = 1,\dots,d$. From the second equation in (6.9), we obtain

$$f^i = (g'(\theta^i_{\lambda_0}))^{-1}(D_pH^i_{\lambda_0}(Du^i_{\lambda_0},x)\cdot Dv^i + v^i + \sum_{j\neq i}(\beta'_{\epsilon,\lambda_0}(i,j)(v^i - v^j)) - f^i_0). \tag{6.10}$$

Using (6.10) in the first equation of (6.9), we see that v^i is a weak solution to

$$
\begin{aligned}
&[H^i_{\lambda_0,p_kp_j}(Du^i_{\lambda_0},x)v^i_{x_j}\theta^i_{\lambda_0}+H^i_{\lambda_0,p_k}(Du^i_{\lambda_0},x)H^i_{\lambda_0,p_j}(Du^i_{\lambda_0},x)v^i_{x_j}(g'(\theta^i_{\lambda_0}))^{-1}]_{x_k}\\
&=-[(g'(\theta^i_{\lambda_0}))^{-1}(v^i+\sum_{j\neq i}(\beta'_{\epsilon,\lambda_0}(i,j)v^i-\beta'_{\epsilon,\lambda_0}(i,j)v^j)-f^i_0)]_{x_k}+f^i\\
&+\sum_{j\neq i}[(\beta''_{\epsilon,\lambda_0}(i,j)\theta^i_{\lambda_0}+\beta''_{\epsilon,\lambda_0}(j,i)\theta^j_{\lambda_0})(v^i-v^j)+\beta'_{\epsilon,\lambda_0}(i,j)f^i-\beta'_{\epsilon,\lambda_0}(j,i)f^j]+v^i_0.
\end{aligned}
$$

For any $i=1,\dots,d$, the right-hand side of the previous equation is in $L^2(\mathbb{T}^N)$. Thus, the elliptic regularity theory implies that $v\in E^2$. Consequently, (6.10) gives $f\in E^1$.

By induction, if $w_0=(v_0,f_0)\in E^k\times E^{k+1}$, then $w_1=(v,f)\in E^{k+2}\times E^{k+1}$. This concludes the proof of the lemma. □

Claim 3 is now a straightforward consequence of Lemma 6.4 combined with the implicit function theorem in Banach spaces, see, for instance, [15].

Finally, we gather the previous results to prove the existence claim in Theorem 1.2.

Proof of Theorem 1.2 - existence. Claims 1-3 imply that $\Lambda=[0,1]$. In particular $1\in\Lambda$, i.e., there exists a C^∞ solution of (1.4)-(1.5). □

6.2. **Uniqueness.** Here, we complete the proof of Theorem 1.2 by proving the uniqueness of solutions for (1.4)-(1.5) using the monotonicity method.

Proof of Theorem 1.2 - uniqueness. Let (u_1,θ_1) and (u_2,θ_2) be classical solutions of (1.4)-(1.5). First, we take (1.4) with $(u,\theta)=(u_1,\theta_1)$ and $(u,\theta)=(u_2,\theta_2)$ and subtract the corresponding equations. Next, we multiply by $\theta^i_1-\theta^i_2$ and integrate. Accordingly, we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}^N}[H^i(Du^i_1,x)-H^i(Du^i_2,x)](\theta^i_1-\theta^i_2)+(u^i_1-u^i_2)(\theta^i_1-\theta^i_2)dx\\
&+\sum_{j\neq i}\int_{\mathbb{T}^N}(\beta_\epsilon(u^i_1-u^j_1-\psi^{ij})-\beta_\epsilon(u^i_2-u^j_2-\psi^{ij}))(\theta^i_1-\theta^i_2)dx\\
&=\int_{\mathbb{T}^N}(g(\theta^i_1)-g(\theta^i_2))(\theta^i_1-\theta^i_2)dx.
\end{aligned}
$$

Now, we take (1.5) with $(u,\theta)=(u_1,\theta_1)$ and $(u,\theta)=(u_2,\theta_2)$. Next, we subtract the corresponding equations, multiply by $u^i_1-u^i_2$, and integrate by parts. Accordingly, we get

$$
\begin{aligned}
0=&\int_{\mathbb{T}^N}(D_pH^i(Du^i_1,x)\theta^i_1-D_pH^i(Du^i_2,x)\theta^i_2)D(u^i_1-u^i_2)+(u^i_1-u^i_2)(\theta^i_1-\theta^i_2)dx\\
&+\sum_{j\neq i}\int_{\mathbb{T}^N}\Big[\beta'_\epsilon(u^i_1-u^j_1-\psi^{ij})\theta^i_1-\beta'_\epsilon(u^j_1-u^i_1-\psi^{ji})\theta^j_1\\
&-\Big(\beta'_\epsilon(u^i_2-u^j_2-\psi^{ij})\theta^i_2-\beta'_\epsilon(u^j_2-u^i_2-\psi^{ji})\theta^j_2\Big)\Big](u^i_1-u^i_2)dx.
\end{aligned}
$$

Finally, we subtract the two previous identities, sum on i, and use the monotonicity of g to conclude that

$$\begin{aligned}
0 &\leq \sum_{i=1}^{d} \int_{\mathbb{T}^N} (g(\theta_1^i) - g(\theta_2^i))(\theta_1^i - \theta_2^i) dx \\
&= \sum_{i=1}^{d} \int_{\mathbb{T}^N} [(H^i(Du_1^i, x) - H^i(Du_2^i, x))(\theta_1^i - \theta_2^i) \\
&- (D_p H^i(Du_1^i, x)\theta_1 - D_p H^i(Du_2^i, x)\theta_2) D(u_1^i - u_2^i)] dx \\
&+ \sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} (\beta_\epsilon(u_1^i - u_1^j - \psi^{ij}) - \beta_\epsilon(u_2^i - u_2^j - \psi^{ij}))(\theta_1^i - \theta_2^i) dx \\
&- \sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} \Big[\beta_\epsilon'(u_1^i - u_1^j - \psi^{ij})\theta_1^i - \beta_\epsilon'(u_1^j - u_1^i - \psi^{ji})\theta_1^j \\
&- \Big(\beta_\epsilon'(u_2^i - u_2^j - \psi^{ij})\theta_2^i - \beta_\epsilon'(u_2^j - u_2^i - \psi^{ji})\theta_2^j\Big)\Big] (u_1^i - u_2^i) dx.
\end{aligned}$$

Now, from the convexity of β_ϵ, we infer that

$$\begin{aligned}
&\sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} (\beta_\epsilon(u_1^i - u_1^j - \psi^{ij}) - \beta_\epsilon(u_2^i - u_2^j - \psi^{ij}))(\theta_1^i - \theta_2^i) dx \\
&- \sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} [\beta_\epsilon'(u_1^i - u_1^j - \psi^{ij})\theta_1^i - \beta_\epsilon'(u_1^j - u_1^i - \psi^{ji})\theta_1^j \\
&- (\beta_\epsilon'(u_2^i - u_2^j - \psi^{ij})\theta_2^i - \beta_\epsilon'(u_2^j - u_2^i - \psi^{ji})\theta_2^j)](u_1^i - u_2^i) dx \\
&= \sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} (\beta_\epsilon(u_1^i - u_1^j - \psi^{ij}) - \beta_\epsilon(u_2^i - u_2^j - \psi^{ij}))(\theta_1^i - \theta_2^i) dx \\
&- \sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} [\beta_\epsilon'(u_1^i - u_1^j - \psi^{ij})\theta_1^i - \beta_\epsilon'(u_2^i - u_2^j - \psi^{ij})\theta_2^i](u_1^i - u_2^i - u_1^j + u_2^j) dx \\
&= -\sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} [\beta_\epsilon(u_1^i - u_1^j - \psi^{ij}) - \beta_\epsilon(u_2^i - u_2^j - \psi^{ij}) \\
&- \beta_\epsilon'(u_2^i - u_2^j - \psi^{ij})(u_1^i - u_1^j - u_2^i + u_2^j)]\theta_2^i dx \\
&- \sum_{i=1}^{d} \sum_{j\neq i} \int_{\mathbb{T}^N} [\beta_\epsilon(u_2^i - u_2^j - \psi^{ij}) - \beta_\epsilon(u_1^i - u_1^j - \psi^{ij}) \\
&- \beta_\epsilon'(u_1^i - u_1^j - \psi^{ij})(u_2^i - u_2^j - u_1^i + u_1^j)]\theta_1^i dx \\
&\leq 0.
\end{aligned}$$

Moreover, using (2.4) of Assumption 4, we get

$$\begin{aligned}
&\sum_{i=1}^{d}\int_{\mathbb{T}^N}[(H^i(Du_1^i,x)-H^i(Du_2^i,x))(\theta_1^i-\theta_2^i)\\
&-(D_pH^i(Du_1^i,x)\theta_1-D_pH^i(Du_2^i,x)\theta_2)D(u_1^i-u_2^i)]dx\\
&=-\sum_{i=1}^{d}\int_{\mathbb{T}^N}[H^i(Du_1^i,x)-H^i(Du_2^i,x)-D_pH^i(Du_2^i,x)D(u_1^i-u_2^i)]\theta_2^i dx\\
&-\sum_{i=1}^{d}\int_{\mathbb{T}^N}[H^i(Du_2^i,x)-H^i(Du_1^i,x)-D_pH^i(Du_1^i,x)D(u_2^i-u_1^i)]\theta_1^i dx\\
&\leq -C\sum_{i=1}^{d}\int_{\mathbb{T}^N}|D(u_1^i-u_2^i)|^2dx.
\end{aligned}$$

By combining the last two inequalities, we conclude that

$$0\leq -C\sum_{i=1}^{d}\int_{\mathbb{T}^N}|D(u_1^i-u_2^i)|^2dx.$$

Thus, we infer that $u_1^i=u_2^i$ for any $i=1,\dots,d$. Consequently,

$$\begin{aligned}
\theta_1^i&=g^{-1}\left(H^i(Du_1^i,x)+u_1^i+\sum_{j\neq i}\beta_\epsilon(u_1^i-u_2^j-\psi^{ij})\right)\\
&=g^{-1}\left(H^i(Du_2^i,x)+u_2^i+\sum_{j\neq i}\beta_\epsilon(u_2^i-u_2^j-\psi^{ij})\right)\\
&=\theta_2^i.
\end{aligned}$$

This concludes the proof of the uniqueness of the solution of (1.4)-(1.5). □

7. Proof of Theorem 1.1

Now, we use Theorem 1.2 and a limiting procedure to prove Theorem 1.1.

Proof of Theorem 1.1. Let $(u_\epsilon,\theta_\epsilon)$ be the classical solution of (1.4)-(1.5), whose existence is guaranteed by Theorem 1.2. By estimates (4.5), (4.2), and (4.4) if Assumption 5 **L** holds, or by estimates (4.8), (4.14) and (4.13) if Assumptions 5 **P-**$\frac{2}{N}$ and 6 hold, there exist $\gamma\in(0,1)$, $u\in(W^{2,2}(\mathbb{T}^N))^d\cap(C^\gamma(\mathbb{T}^N))^d$, and $\theta\in(W^{1,2}(\mathbb{T}^N))^d$, such that, up to extracting a subsequence, as $\epsilon\to 0$,

$$u_\epsilon\to u\quad\text{in }(L^\infty(\mathbb{T}^N))^d,$$

$$Du_\epsilon\to Du,\quad\theta_\epsilon\to\theta\quad\text{in }(L^2(\mathbb{T}^N))^d,$$

$$D^2u_\epsilon\rightharpoonup D^2u,\quad\text{in }(L^2(\mathbb{T}^N))^d,$$

and, for any $i=1,\dots,d$ and $j\neq i$,

$$u^i-u^j-\psi^{ij}\leq 0\quad\text{in }\mathbb{T}^N.$$

The uniform convergence of u^i_ϵ to u^i implies that, if $x \in \mathbb{T}^N$ is such that $u^i(x) - u^j(x) - \psi^{ij}(x) < 0$, then, for a small enough ϵ, we have $u^i_\epsilon(x) - u^j_\epsilon(x) - \psi^{ij}(x) < 0$. Consequently,

$$\beta_\epsilon(u^i_\epsilon(x) - u^j_\epsilon(x) - \psi^{ij}(x)) = \beta'_\epsilon(u^i_\epsilon(x) - u^j_\epsilon(x) - \psi^{ij}(x)) = 0.$$

We deduce that the limit, (u, θ), is a weak solution of

$$H^i(Du^i, x) + u^i \le g(\theta^i) \quad \text{in } \mathbb{T}^N,$$

$$H^i(Du^i, x) + u^i = g(\theta^i) \quad \text{in } \cap_{j \neq i} \{u^i - u^j - \psi^{ij} < 0\}.$$

Next, for $j \neq i$, we introduce the measures

$$\nu^{ij}_\epsilon = \beta'_\epsilon(u^i_\epsilon(x) - u^j_\epsilon(x) - \psi^{ij}(x))\theta^j.$$

By (3.5), we have that $\int_{\mathbb{T}^N} \nu^{ij}_\epsilon \, dx \le C$ for some constant, C, independent of ϵ. Thus, there exist non-negative measures, ν^{ij}, such that

$$-\text{div}(D_p H^i(Du^i, x)\theta^i) + \theta^i + \sum_{j \neq i} \left(\nu^{ij} - \nu^{ji}\right) = 1.$$

Moreover, ν^{ij} is supported in the set $u^i - u^j - \psi^{ij} = 0$. □

8. Uniqueness of the limit

In this last section, we discuss the uniqueness of the limit, (u, θ), in the proof of Theorem 1.1.

Proposition 8.1. *Suppose that Assumptions 1-4 and 8 hold, and that either*

- *Assumption 5* **L** *or*
- *Assumptions 5* **P-**$\frac{2}{N}$*, 6 and 7*

hold. For $\epsilon > 0$, let $(u_\epsilon, \theta_\epsilon)$ be the solution of (1.4)-(1.5)*. Then, the limit, (u, θ), as $\epsilon \to 0$ of the family $(u_\epsilon, \theta_\epsilon)$ exists; that is, (u, θ) is independent of the choice of subsequence.*

Proof. For $k = 0, 1$, consider a sequence, ϵ^k_n, converging to 0. Let $(u_k, \theta_k) = \lim_{n\to\infty}(u_{\epsilon^k_n}, \theta_{\epsilon^k_n})$. By (1.1), we have

$$H^i(Du^i_k, x) + u^i_k - g(\theta^i_k) \le 0$$

and

$$u^i_k - u^j_k - \psi^{ij} \le 0.$$

For $k = 0, 1$, let $\tilde{k} = 1 - k$. Taking into account the preceding inequalities and the uniform convexity of H^i, we have

$$\begin{aligned}
0 \ge & H^i(Du^i_k, x) - g(\theta^i_k) + u^i_k + \sum_j \beta_{\epsilon^{\tilde{k}}_n}(u^i_k - u^j_k - \psi^{ij}) \\
\ge & H^i(Du^i_{\epsilon^{\tilde{k}}_n}, x) - g(\theta^i_{\epsilon^{\tilde{k}}_n}) + u^i_{\epsilon^{\tilde{k}}_n} + \sum_j \beta_{\epsilon^{\tilde{k}}_n}(u^i_{\epsilon^{\tilde{k}}_n} - u^j_{\epsilon^{\tilde{k}}_n} - \psi^{ij}) \\
& + u^i_k - u^i_{\epsilon^{\tilde{k}}_n} + D_p H^i(Du^i_{\epsilon^{\tilde{k}}_n}, x) \cdot D(u^i_k - u^i_{\epsilon^{\tilde{k}}_n}) \\
& + \sum_{j \neq i} \beta'_{\epsilon^{\tilde{k}}_n}(u^i_{\epsilon^{\tilde{k}}_n} - u^j_{\epsilon^{\tilde{k}}_n} - \psi^{ij})((u^i_k - u^i_{\epsilon^{\tilde{k}}_n}) - (u^j_k - u^j_{\epsilon^{\tilde{k}}_n})) \\
& + c\gamma |D(u^i_k - u^i_{\epsilon^{\tilde{k}}_n})|^2 + g(\theta^i_{\epsilon^{\tilde{k}}_n}) - g(\theta^i_k)
\end{aligned}$$

for some $c > 0$. Integrating with respect to $\theta^i_{\epsilon^{\tilde{k}}_n}$ and adding over i and k give

$$\sum_{i,k}\int_{\mathbb{T}^N}\left[\gamma|D(u^i_k-u^i_{\epsilon^{\tilde{k}}_n})|^2\theta^i_{\epsilon^{\tilde{k}}_n}+(g(\theta^i_{\epsilon^{\tilde{k}}_n})-g(\theta^i_k))\theta^i_{\epsilon^{\tilde{k}}_n}+u^i_k-u^i_{\epsilon^{\tilde{k}}_n}\right]dx\leq 0.$$

Because $\theta^i_{\epsilon^{\tilde{k}}_n}$ is bounded in $(W^{1,2}(\mathbb{T}^N))^d$ and because $u_{\epsilon^{\tilde{k}}_n}$ is bounded in $(W^{2,2}(\mathbb{T}^N))^d$, by extracting a further subsequence, if necessary, we have the following almost everywhere convergences:

$$|D(u^i_k-u^i_{\epsilon^{\tilde{k}}_n})|^2\theta^i_{\epsilon^{\tilde{k}}_n}\to|D(u^i_k-u^i_{\tilde{k}})|^2\theta^i_{\tilde{k}}$$

and

$$g(\theta^i_{\epsilon^{\tilde{k}}_n})\theta^i_{\epsilon^{\tilde{k}}_n}\to g(\theta^i_{\tilde{k}})\theta^i_{\tilde{k}}.$$

Furthermore,

$$\int_{\mathbb{T}^N}g(\theta^i_k)\theta^i_{\epsilon^{\tilde{k}}_n}dx\to\int_{\mathbb{T}^N}g(\theta^i_k)\theta^i_{\tilde{k}}dx.$$

Consequently, taking into account that

$$\lim_{n\to\infty}\sum_k\int_{\mathbb{T}^N}u^i_k-u^i_{\epsilon^{\tilde{k}}_n}dx=0$$

and that $z\mapsto zg(z)$ is bounded by below, and using Fatou's Lemma, we obtain

$$\sum_{i,k}\int_{\mathbb{T}^N}\gamma\ |D(u^i_k-u^i_{\tilde{k}})|^2\theta^i_{\tilde{k}}+(g(\theta^i_{\tilde{k}})-g(\theta^i_k))\theta^i_{\tilde{k}}\leq 0.$$

Because g is monotone increasing, the preceding inequality implies that $\theta_1=\theta_2$ and that $Du_1=Du_2$. Thus, using (1.1), we have $u_1=u_2$. □

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