LONG-TIME BEHAVIOR FOR CRYSTAL DISLOCATION DYNAMICS

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Abstract. We describe the asymptotic states for the solutions of a nonlocal equation of evolutionary type, which have the physical meaning of the atom dislocation function in a periodic crystal.

More precisely, we can describe accurately the "smoothing effect" on the dislocation function occurring slightly after a "particle collision" (roughly speaking, two opposite transitions layers average out) and, in this way, we can trap the atom dislocation function between a superposition of transition layers which, as time flows, approaches either a constant function or a single heteroclinic (depending on the algebraic properties of the orientations of the initial transition layers).

The results are endowed of explicit and quantitative estimates and, as a byproduct, we show that the ODE systems of particles that governs the evolution of the transition layers does not admit stationary solutions (i.e., roughly speaking, transition layers always move).

1. Introduction

In the scientific literature, several models have been considered in order to describe the motion of the atom dislocations in a crystal. Roughly speaking, a crystal is a structure in which the atoms have the strong tendency to occupy some given site of a lattice; nevertheless, some atom may occupy a different position that the one at rest, and an important question is the accurate description of the evolution of this dislocation function and of its asymptotic and stationary behaviors.

Since different scales come into play in such description, different models have been adopted, in order to deal with phenomena at the atomic, microscopic, mesoscopic and macroscopic scale. Goal of this paper is to consider a microscopic model, inspired by (and, in fact, even more general than) the classical one by Peierls and Nabarro, see e.g. [10] for a detailed description and also Section 2 in [5] for a simple introduction.

In this setting, after a suitable section of a three-dimensional crystal with a transverse plane, the edge dislocation of the atoms along a slip plane is described by a function $v_\varepsilon = v_\varepsilon(t,x)$, where $t \geqslant 0$ is the time variable, $x \in \mathbb{R}$ is the space variable and $\varepsilon > 0$ is the characteristic length of the crystal (say, roughly speaking, the distance between the minimal rest positions of the crystal atoms).

The function v_ε satisfies a nonlocal equation since the evolution along the slip plane is influenced by the whole structure of the crystal, which favors the rest position of the atoms in a lattice, that, in our case, will be taken to be $\mathbb{Z}$.

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More precisely, the influence of the elastic energy of the whole crystal along the slip plane produces a fractional operator, which we denote by $\mathcal{I}_s$ and which is balanced by a force coming from a periodic multi-well potential W produced by the periodic structure of the crystal in the large.

The presence of an external stress σ can also be taken into account (of course, if one aims at "general" results, one has to assume that this stress is sufficiently small to allow a long-time behavior in which the structure of the crystal is dominant with respect to the external forces).

In further detail, we consider here the initial value problem

$$\begin{cases} \partial_t v_\varepsilon = \dfrac{1}{\varepsilon}\left(\mathcal{I}_s v_\varepsilon - \dfrac{1}{\varepsilon^{2s}} W'(v_\varepsilon) + \sigma(t,x)\right) & \text{in } (0,+\infty)\times\mathbb{R} \\ v_\varepsilon(0,\cdot) = v_\varepsilon^0 & \text{on } \mathbb{R} \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small scale parameter, W is a periodic potential and $\mathcal{I}_s$ is the so-called fractional Laplacian of any order $2s \in (0,2)$, that we define (up to a multiplicative normalization constant that we neglect) as

$$\mathcal{I}_s[\varphi](x) := \frac{1}{2}\int_{\mathbb{R}} \frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{1+2s}}\,dy.$$

When $s = \frac{1}{2}$ and $W(v) := 1 - \cos(2\pi v)$, stationary solutions of (1.1) correspond to equilibria in the classical model for dislocation dynamics of Peierls and Nabarro [10] (and indeed the results that we present are new even for such model case). See also [15] or [6] for a basic introduction to the fractional Laplace operator.

We assume that W is a multi-well potential with nondegenerate minima at integer points. More precisely, we suppose that

$$\begin{cases} W \in C^{3,\alpha}(\mathbb{R}) & \text{for some } 0<\alpha<1 \\ W(v+1) = W(v) & \text{for any } v \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R}\setminus\mathbb{Z} \\ W''(0) > 0. \end{cases} \tag{1.2}$$

The function σ represents the external stress and we assume on it the following regularity conditions:

$$\begin{cases} \sigma \in BUC([0,+\infty)\times\mathbb{R}) \quad \text{and for some } M>0 \text{ and } \alpha \in (s,1) \\ \|\sigma_x\|_{L^\infty([0,+\infty)\times\mathbb{R})} + \|\sigma_t\|_{L^\infty([0,+\infty)\times\mathbb{R})} \leqslant M \\ |\sigma_x(t,x+h) - \sigma_x(t,x)| \leqslant M|h|^\alpha, \quad \text{for every } x,h \in \mathbb{R} \text{ and } t \in [0,+\infty), \end{cases} \tag{1.3}$$

where $BUC([0,+\infty)\times\mathbb{R})$ denotes the set of bounded uniformly continuous functions over $[0,+\infty)\times\mathbb{R}$.

It is interesting to observe that the model in (1.1) is one-dimensional, but it refers to a three-dimensional crystal. Roughly speaking, the crystal deformation is supposed to occur along a slip plane, and then the crystal is sectioned by a plane transversal to the slip plane. This reduces the problem to a two-dimensional problem, which, by symmetry, can be set in a half-plane. This two-dimensional problem is of local nature, but it can be equivalently reduced to a nonlocal one-dimensional problem along the boundary of this half-plane.

In this construction, the one-dimensional space in which problem (1.1) is set corresponds to the intersection of two two-dimensional planes in a three-dimensional space (namely, the slip plane and the normal section plane). The dislocation lines in the space are assumed to be straight, or at least transversal to the section plane (and this is also a simplification of the general model). We refer, e.g., to Section 2 in [5] for further discussions and motivations.

In order to detect the long-time evolution of the system in (1.1), we consider initial values that come from a "finite (but arbitrarily large) number" of single atom dislocations.

To make this assumption more explicit, we introduce the so-called basic layer solution u associated to $\mathcal{I}_s$ (see [11, 1, 3]), that is the solution of the stationary equation

$$\begin{cases} \mathcal{I}_s(u) = W'(u) & \text{in } \mathbb{R} \\ u' > 0 & \text{in } \mathbb{R} \\ \lim_{x\to-\infty} u(x) = 0, \quad \lim_{x\to+\infty} u(x) = 1, \quad u(0) = \frac{1}{2}. \end{cases} \tag{1.4}$$

We recall that the standard transition layer is a local minimizer for the (formal) energy functional

$$\frac{1}{4}\iint \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,dx\,dy + \int W(u(x))\,dx.$$

The first term in this functional is an elastic energy and corresponds, from the mathematical point of view, to a Gagliardo seminorm in a fractional Sobolev space. Though the total energy of this system diverges (when $s < 1/2$), it is always possible to consider local minimizers of such energy, by considering compact perturbations (see [11] for details).

Given $x_1^0 < x_2^0 < \cdots < x_N^0$, we say that the function $u\left(\frac{x-x_i^0}{\varepsilon}\right)$ is a transition layer centered at x_i^0 and positively oriented. Similarly, we say that the function $u\left(\frac{x_i^0-x}{\varepsilon}\right) - 1$ is a transition layer centered at x_i^0 and negatively oriented.

We observe that a positive oriented transition layer connects the integer values 0 and 1, with a transition that becomes steeper and steeper as $\varepsilon \to 0$. Viceversa, a negative oriented transition layer connects the integer values 0 and -1.

In this setting, we consider as initial condition in (1.1) the superposition of K positive oriented transition layers with $N-K$ negative oriented transition layers (modified by a small term which takes into account the possible reaction to an external stress), given by the formula

$$v_\varepsilon^0(x) := \frac{\varepsilon^{2s}}{\beta}\sigma(0,x) + \sum_{i=1}^N u\left(\zeta_i \frac{x-x_i^0}{\varepsilon}\right) - (N-K), \tag{1.5}$$

where u is solution of (1.4), $\zeta_1, \dots, \zeta_N \in \{-1, 1\}$, $\sum_{i=1}^N (\zeta_i)^+ = K$, $0 \leqslant K \leqslant N$ and

$$\beta := W''(0) > 0. \tag{1.6}$$

We observe that when $\zeta_i = 1$, the ith transition layer in (1.5) is positively oriented, while when $\zeta_i = -1$, it is negatively oriented. We also point out that, if $\sigma \equiv 0$, then

$$\begin{aligned} &\lim_{x\to-\infty} v_\varepsilon^0(x) = \sum_{\substack{1\leqslant i\leqslant N\\ \zeta_i=-1}} 1-(N-K)=0 \\ \text{and } &\lim_{x\to+\infty} v_\varepsilon^0(x) = \sum_{\substack{1\leqslant i\leqslant N\\ \zeta_i=1}} 1-(N-K)=2K-N. \end{aligned} \tag{1.7}$$

The fact that the initial conditions considered in this framework are superpositions of transition layers is, from the mathematical point of view, a simplification, in the spirit of the theory of "well prepared initial data". From a modeling viewpoint, it takes into account the special (but rather natural) case in which the initial dislocation is caused by a "standard transition", which is a stable solution, and an energy minimizer, for the stationary case.

It would be very desirable to develop general theories describing the evolution of any initial datum, or at least to investigate the stability (in some suitable sense) of the results obtained for well prepared initial data with respect to small perturbations of the initial data itself. It is possible that "many" initial data, after some "chaotic" transient, approach a situation close to the superposition of single transitions, but, as far as we know, a complete investigation of the general case is still not available in the literature.

It has been shown in [7] (when $s=\frac{1}{2}$), in [5] (when $s\in\left(\frac{1}{2},1\right)$) and in [4] (when $s\in\left(0,\frac{1}{2}\right)$) that the evolution of v_ε with the initial condition in (1.5) resembles, as $\varepsilon\to 0$, a step functions with integer values, whose N points of discontinuity, say $(x_1(t),\dots,x_N(t))$, move according to a dynamical system. More precisely, as proved in [12], the potential that drives this dynamical system is either repulsive (when the associated transition layers have the same orientations) or attractive (when they have opposite orientations). In case of attractive potentials, these discontinuity points (sometimes referred in a suggestive but perhaps a bit improper way with the name of "particles") collide in a finite time T_c, see again [12] for a detailed description of this phenomenon.

The explicit system of ordinary differential equations which govern the motion of these jump points $(x_1(t),\dots,x_N(t))$ is given by

$$\begin{cases} \dot{x}_i = \gamma\left(\displaystyle\sum_{j\neq i}\zeta_i\zeta_j\frac{x_i-x_j}{2s|x_i-x_j|^{1+2s}} - \zeta_i\sigma(t,x_i)\right) & \text{in } (0,T_c)\\ x_i(0)=x_i^0, \end{cases} \tag{1.8}$$

for $i=1,\dots,N$, where

$$\gamma := \left(\int_{\mathbb{R}} (u'(x))^2\,dx\right)^{-1}, \tag{1.9}$$

and $0<T_c\leqslant+\infty$ is the collision time of system (1.8).

More explicitly, a collision time T_c is characterized by the fact that

$$x_{i+1}(t) > x_i(t) \quad \text{for any } t\in[0,T_c) \text{ and } i=1,\dots,N-1$$

and there exists i_0 such that

$$x_{i_0+1}(T_c) = x_{i_0}(T_c).$$

If a collision occurs, after the collision time T_c, the dynamical system in (1.8) (as given in [7, 5, 4, 12]) ceases to be well-defined, since at least one of the denominators vanishes, hence the mesoscopic description in the limit as $\varepsilon \to 0$ ceases to be available. Nevertheless, for a fixed $\varepsilon > 0$, the solution v_ε of the evolution equation (1.1) continues to exist and to describe the dislocation dynamics.

In [14], we gave a first explicit description of what happens to the solution v_ε after the collision time when *only two or three* layer solutions are taken into account. Goal of this paper is to further extend this study, by taking into account the superposition of *any number of transition layers*, by describing qualitatively the *asymptotic states* and by providing *quantitative estimates* on the relaxation times needed to approach the limits.

To this goal, we consider several cases, such as:

- the situation in which the first K transition layers are positively oriented and the remaining last $N - K$ negatively oriented (we call this situation the "segregate orientation" case),
- the situation in which there are as many positively oriented as negatively oriented transition layers (we call this situation the "balanced orientation" case),
- the situation in which there are more positively oriented than negatively oriented transition layers (we call this situation the "unbalanced orientation" case; of course the opposite situation in which there are more negatively oriented than positively oriented transition layers can be reduced to this case, up to a spacial reflection).

The results that we obtain are naturally different according to the different cases. In the segregate orientation case we will show that, roughly speaking, the last "positively oriented particle" in the dynamical system (1.8) will collide with the first "negatively oriented particle" at some time T_c; then, slightly after T_c, two transition layers of the solution v_ε will merge the one into the other and annihilate each other (as a consequence, after this, the solution v_ε somehow decreases its oscillations).

We remark that the segregate orientation case is not only interesting in itself, but it also provides a natural comparison for the general case (i.e. it provides the necessary barriers for the other cases, thus reducing each time the picture to the "worst possible scenario").

The balanced orientation case presents the special feature of having $K = N - K$, that is $N = 2K$, which says that the dislocation function goes to zero both at $-\infty$ and at $+\infty$ (recall (1.7)). These conditions at infinity influence the asymptotic behavior in time of v_ε, since we will show that, after a transient time in which collisions occur, the solution v_ε relaxes to zero exponentially fast.

The unbalanced orientation case is somehow more complex. In this case, we have $K > N - K$, so we set $l := 2K - N = K - (N - K) > 0$ (notice that l is the difference between positively oriented and negatively oriented initial transitions). In this situation, the initial dislocation approaches zero at $-\infty$ and l as $x \to +\infty$ (recall again (1.7)).

The asymptotics in time of the dislocation function v_ε is again influenced by these conditions at infinity, since, roughly speaking, the limit behavior as $t \to +\infty$ will try to make an average between the two values at infinity. On the other hand, this "exact" average procedure is not (always) possible for the system and indeed it is not (always) true that v_ε approaches the constant value $\frac{l}{2}$ as $t \to +\infty$.

The heuristic reason for this fact is that the constant $\frac{l}{2}$ is not necessarily a solution of the stationary equation, and even when it is a solution (as in the model case given by the choice of the potential $W(v) := 1 - \cos(2\pi v)$) such solution is unstable from the variational point of view.

In fact, we will show that the constant value $\frac{l}{2}$ is only reached as $t \to +\infty$ "in average" in a possibly dynamical way and in a way which is compatible with the stable solutions of the stationary equation. Namely, if $\frac{l}{2} \in \mathbb{N}$ (i.e. l is even) then[1] indeed $v_\varepsilon \to \frac{l}{2}$ as $t \to +\infty$; but if instead $\frac{l}{2} \notin \mathbb{N}$ (i.e. l is odd) then, for large times, the dislocation function v_ε will approach a transition layer which joins the integer $(l-1)/2$ at $-\infty$ with the integer $(l+1)/2$ at $+\infty$ (that is, a vertical translation of the standard heteroclinic from 0 to 1). Thus, when l is odd, the constant value $\frac{l}{2}$ is not attained in the limit $t \to +\infty$, but instead the system attains a dynamic connection between the values $\frac{l}{2} - \frac{1}{2}$ and $\frac{l}{2} + \frac{1}{2}$.

All these statements will be proved in a quantitative way, by using appropriate comparison functions. We now give a precise mathematical statements of the results that we have just described in words.

1.1. **The segregate orientation case.** We first consider the particular case in which the first K transition layers in (1.5) are positively oriented and the remaining last $N-K$ negatively oriented, i.e., we assume

$$\zeta_i = \begin{cases} 1 & \text{for } i = 1, \dots, K \\ -1 & \text{for } i = K+1, \dots, N. \end{cases} \tag{1.10}$$

Under this[2] assumption, we show that if the collision time T_c is finite, then the collision occurs between particles x_K and x_{K+1}, and after a time T_ε, which is slightly larger than T_c, the function v_ε is dominated by the superposition of $N-2$ transition layers, the first $K-1$ of them positively oriented and the last $N-K-1$ negatively oriented.

The precise mathematical statement goes as follows:

Theorem 1.1. *Assume that* (1.2), (1.3), (1.10) *hold, that* $0 < K < N$ *and that* $T_c < +\infty$. *Let* v_ε *be the solution of* (1.1)-(1.5) *and* $(x_1(t), \dots, x_N(t))$ *the solution of* (1.8). *Then there exist* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *there exist* $x_1^\varepsilon, \dots, x_{K-1}^\varepsilon, x_{K+2}^\varepsilon, \dots, x_N^\varepsilon \in \mathbb{R}$ *and* T_ε, $\varrho_\varepsilon > 0$, *such that for* $i \in \{1, \dots, K-1, K+2, \dots N\}$,

$$x_i^\varepsilon = x_i(T_c) + o(1) \quad \text{as } \varepsilon \to 0, \tag{1.11}$$

$$x_{i+1}^\varepsilon - x_i^\varepsilon \geqslant c, \tag{1.12}$$

$$T_\varepsilon = T_c + o(1) \quad \text{as } \varepsilon \to 0,$$

$$\varrho_\varepsilon = o(1), \quad \frac{\varepsilon^{2s}}{\varrho_\varepsilon} = o(1), \quad \frac{\varrho_\varepsilon}{\varepsilon^s} = o(1) \quad \text{as } \varepsilon \to 0 \tag{1.13}$$

[1] It is worth to point out that, as expected, the unbalanced orientation case boils down to the balanced orientation case when $l = 0$ (in any case, the quantitative estimates that we obtain in the balanced case are more explicit and precise than the ones for the unbalanced case).

[2] As a matter of fact, we will show in Lemma 3.1, that if $1 - 2s\vartheta_0^{2s}\|\sigma\|_\infty > 0$ and (1.10) holds true, then a collision always occurs in a finite time, i.e., $T_c < +\infty$.

and for any $x \in \mathbb{R}$,

$$v_\varepsilon(T_\varepsilon, x) \leqslant \frac{\varepsilon^{2s}}{\beta}\sigma(T_\varepsilon, x) + \sum_{i=1}^{K-1} u\left(\frac{x - x_i^\varepsilon}{\varepsilon}\right) + \sum_{i=K+2}^{N} u\left(\frac{x_i^\varepsilon - x}{\varepsilon}\right) - (N-K-1) + \varrho_\varepsilon, \tag{1.14}$$

where u *is the solution of* (1.4) *and* β *is given by* (1.6).

The evolution of the dislocation function v_ε from $t < T_c$ to $t > T_c$ is described in Figure 1 (roughly speaking, right after the collision of the Kth particle with the $(K+1)$th particle, the dislocation averages out one oscillation).

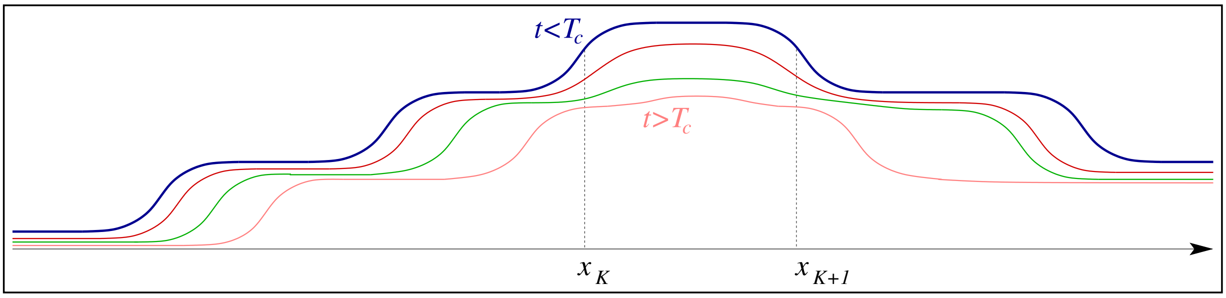

Figure 1 (segregate orientation case): Evolution of the dislocation function as described in Theorem 1.1.

In addition, we can better quantify Theorem 1.1. Indeed, the error term ϱ_ε in (1.14) becomes smaller than ε^{2s} after an additional small time τ_ε as shown in the next theorem, as stated below.

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, if* $N > 2$*, then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *there there exist* $\widetilde{x}_1^\varepsilon, \dots, \widetilde{x}_{K-1}^\varepsilon, \widetilde{x}_{K+2}^\varepsilon, \dots, \widetilde{x}_N^\varepsilon \in \mathbb{R}$*, and* $\widetilde{\varrho}_\varepsilon$*,* $\tau_\varepsilon > 0$*, such that*

$$\tau_\varepsilon = o(1), \quad \widetilde{\varrho}_\varepsilon = o(1)\varepsilon^{2s} \quad \text{as } \varepsilon \to 0, \tag{1.15}$$

for $i \in \{1, \dots, K-1, K+2, \dots N\}$,

$$|\widetilde{x}_i^\varepsilon - x_i^\varepsilon| = o(1) \quad \text{as } \varepsilon \to 0, \tag{1.16}$$

and

(1.17)

$$v_\varepsilon(T_\varepsilon + \tau_\varepsilon, x) \leqslant \frac{\varepsilon^{2s}}{\beta}\sigma(T_\varepsilon + \tau_\varepsilon, x) + \sum_{i=1}^{K-1} u\left(\frac{x - \widetilde{x}_i^\varepsilon}{\varepsilon}\right) + \sum_{i=K+2}^{N} u\left(\frac{\widetilde{x}_i^\varepsilon - x}{\varepsilon}\right) - (N-K-1) + \widetilde{\varrho}_\varepsilon,$$

where T_ε *and the* x_i^ε*'s are given in Theorem* 1.1*,* u *is the solution of* (1.4) *and* β *is given by* (1.6).

1.2. **The balanced orientation case.** Now we consider the case in which $K = N - K$, i.e. the initial configuration presents as many positively oriented layers as negatively oriented ones. In this case, we will use Theorem 1.2 to construct a barrier for the evolution of v_ε. Namely, by an appropriate iteration of Theorem 1.2, we show that, given any initial configuration of an equal number of positive and negative initial dislocations, the system relaxes to the trivial equilibrium (and the relaxation times are exponential). The precise results are stated as follows:

Theorem 1.3. *Assume that* (1.2), (1.3), *hold and that*

$$N = 2K.$$

Let v_ε *be the solution of* (1.1)-(1.5). *Then there exist* $\overline{\sigma} > 0$ *and* $\varepsilon_0 > 0$, *such that if*

$$\|\sigma\|_\infty \leqslant \bar{\sigma}, \tag{1.18}$$

then for any $\varepsilon < \varepsilon_0$ *and any* $(\zeta_1, \dots, \zeta_N) \in \{-1, 1\}^N$ *such that* $\sum_{i=1}^N \zeta_i = 0$, *there exist* $\mathcal{T}_\varepsilon^K, \Lambda_\varepsilon^K > 0$ *such that*

$$|v_\varepsilon(\mathcal{T}_\varepsilon^K, x)| \leqslant \Lambda_\varepsilon^K, \quad \text{for any } x \in \mathbb{R}, \tag{1.19}$$

and

$$\Lambda_\varepsilon^K = o(1) \quad \text{as } \varepsilon \to 0. \tag{1.20}$$

Theorem 1.4. *Under the assumptions of Theorem 1.3, if in addition* $\sigma \equiv 0$, *then there exist* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *we have*

$$|v_\varepsilon(t, x)| \leqslant \Lambda_\varepsilon^K e^{c\frac{\mathcal{T}_\varepsilon^K - t}{\varepsilon^{2s+1}}}, \quad \text{for any } x \in \mathbb{R} \text{ and } t \geqslant \mathcal{T}_\varepsilon^K, \tag{1.21}$$

where $\mathcal{T}_\varepsilon^K$ *and* Λ_ε^K *are given in Theorem 1.3.*

We observe that the exponential decay (for large t) given in (1.21) becomes stronger and stronger for small values of the positive parameter ε (i.e. a small scale of the crystal favors the relaxation of the system).

The situation analytically described in Theorems 1.3 and 1.4 is depicted in Figure 2.

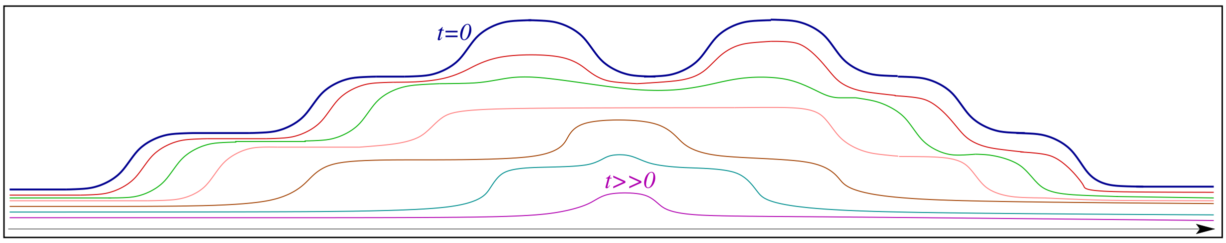

Figure 2 (balanced orientation case): Evolution of the dislocation function as described in Theorems 1.3 and 1.4.

It is worth to point out that the threshold $\bar{\sigma}$ in (1.18) is obtained here by the method of continuity from the case $\sigma \equiv 0$; of course, also in view of concrete applications, we think that it is an interesting problem to obtain explicit quantitative bounds on $\bar{\sigma}$.

1.3. **The unbalanced orientation case.** Now we turn to the general case in which the number of positive initial orientations is not necessarily the same as the number of negative ones. In this case, the limit configuration is either a constant or a single transition, according to the parity of the difference between positive and negative initial orientations. The precise statements go as follows:

Theorem 1.5. *Assume that* (1.2), (1.3), (1.5) *hold and that*

$$N = 2K - l, \quad l \in \mathbb{N}.$$

Let v_ε be the solution of (1.1)-(1.5). *Then there exist $\overline{\sigma} > 0$ and $\varepsilon_0 > 0$ such that if*

$$\|\sigma\|_\infty \leqslant \bar{\sigma}, \tag{1.22}$$

for any $\varepsilon < \varepsilon_0$ and any $(\zeta_1, \ldots, \zeta_N) \in \{-1, 1\}^N$ such that $\sum_{i=1}^N \zeta_i = l$, there exist $\mathcal{T}_\varepsilon^{K-l}, \Lambda_\varepsilon^{K-l} > 0$, $\overline{x}_1^\varepsilon, \ldots, \overline{x}_l^\varepsilon$, $\underline{x}_1^\varepsilon, \ldots, \underline{x}_l^\varepsilon \in \mathbb{R}$, bounded with respect to ε, with $\underline{x}_i^\varepsilon \leqslant \overline{x}_i^\varepsilon$, such that for any $x \in \mathbb{R}$

$$v_\varepsilon(\mathcal{T}_\varepsilon^{K-l}, x) \leqslant \frac{\varepsilon^{2s}}{\beta}\sigma(\mathcal{T}_\varepsilon^{K-l}, x) + \sum_{i=1}^{l} u\left(\frac{x - \underline{x}_i^\varepsilon}{\varepsilon}\right) + \Lambda_\varepsilon^{K-l}, \tag{1.23}$$

and

$$v_\varepsilon(\mathcal{T}_\varepsilon^{K-l}, x) \geqslant \frac{\varepsilon^{2s}}{\beta}\sigma(\mathcal{T}_\varepsilon^{K-l}, x) + \sum_{i=1}^{l} u\left(\frac{x - \overline{x}_i^\varepsilon}{\varepsilon}\right) - \Lambda_\varepsilon^{K-l}, \tag{1.24}$$

where

$$\Lambda_\varepsilon^{K-l} = o(\varepsilon^{2s}) \quad \text{as } \varepsilon \to 0, \tag{1.25}$$

u is the solution of (1.4) *and β is given by* (1.6).

Theorem 1.6. *Under the assumptions of Theorem* 1.5, *if in addition $\sigma \equiv 0$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, we have: for any $R > 0$ there exists $T_0 > \mathcal{T}_\varepsilon^{K-l}$ such that for any $|x| \leqslant R$ and $t > T_0$,*

- *if $l = 2m$, $m \in \mathbb{N}$, then*

$$-C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}} \leqslant v_\varepsilon(t, x) - m \leqslant C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}}. \tag{1.26}$$

- *If $l = 2m + 1$, $m \in \mathbb{N}$, then*

$$v_\varepsilon(t, x) \geqslant m + u\left(\frac{x - \overline{x}^\varepsilon - \alpha_\varepsilon[(1+t)^{\frac{1}{1+2s}} - 1]}{\varepsilon}\right) - C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}}, \tag{1.27}$$

and

$$v_\varepsilon(t, x) \leqslant m + u\left(\frac{x - \underline{x}^\varepsilon + \alpha_\varepsilon[(1+t)^{\frac{1}{1+2s}} - 1]}{\varepsilon}\right) + C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}}, \tag{1.28}$$

where u is the solution of (1.4), *$\alpha_\varepsilon = o(1)$ as $\varepsilon \to 0$, $\underline{x}^\varepsilon$, $\overline{x}^\varepsilon \in \mathbb{R}$ are bounded with respect to ε and $\underline{x}^\varepsilon \leqslant \overline{x}^\varepsilon$.*

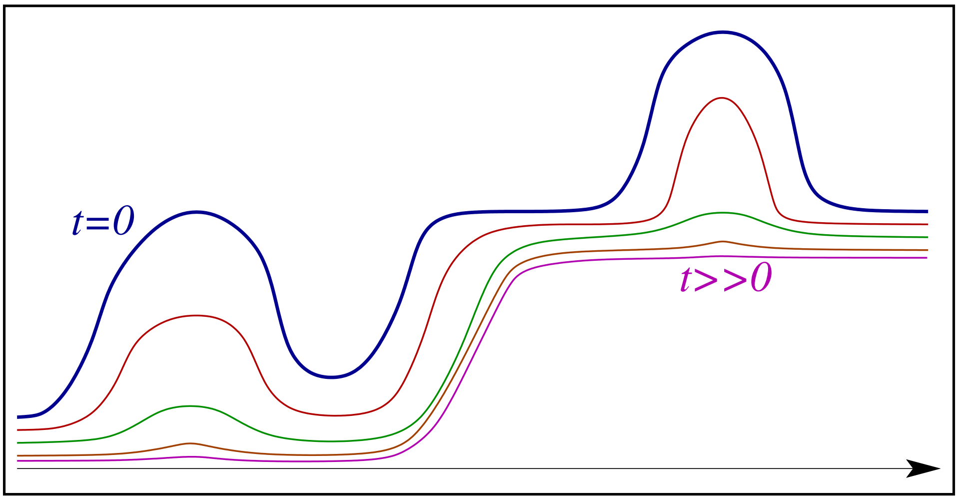

Figure 3 (unbalanced orientation case): Evolution of the dislocation function as described in Theorems 1.5 and 1.6 (l odd, limit case: single transition).

The unbalanced case in which the dislocation function approaches a single heteroclinic is depicted in Figures 3 and 4. We also remark that the index $K-l$ in Theorem 1.5 is related to the number of iterations of Theorems 1.1 and 1.2 needed to perform its proof.

In addition, we point out that there are some quantitative differences between Theorem 1.4 and Theorem 1.6, that is between the balanced and unbalanced orientation cases.

Indeed, when $N = 2K$ (i.e. $m = 0$), the system relaxes to zero exponentially fast, as given by (1.21). Conversely, when $N \neq 2K$, the relaxation times given in (1.26), (1.27) and (1.28) are only polynomial, due to the terms of order $\varepsilon^{2s} t^{-\frac{2s}{2s+1}}$ appearing in these formulas.

The fact is that, in the unbalanced orientation case, the central points of the heteroclinics which provide the barriers move and drifts to infinity: for instance, in case $m = 1$, $N = K = 2$, i.e. when two dislocations with positive orientations are considered, the ODE system can be solved explicitly and one sees that the distance between the dislocations is of the order of $t^{\frac{1}{1+2s}}$ (and this explains the term $t^{\frac{1}{1+2s}}$ in the right hand sides of (1.27) and (1.28)).

This quantitative remark also explains why the decay in time in Theorem 1.6 is polynomial (instead of exponential, as it happens in Theorem 1.4): indeed, the heteroclinics mentioned above, which are centered at distance $O(t^{\frac{1}{1+2s}})$, possess a polynomial tail (with power $-2s$, see e.g. formula (1.6) in [4]): the (rescaled) combination of these two effects produce an error of the form $\left(t^{\frac{1}{1+2s}}/\varepsilon\right)^{-2s}$, and this explains the term of order $\varepsilon^{2s} t^{-\frac{2s}{2s+1}}$ in (1.26), (1.27) and (1.28).

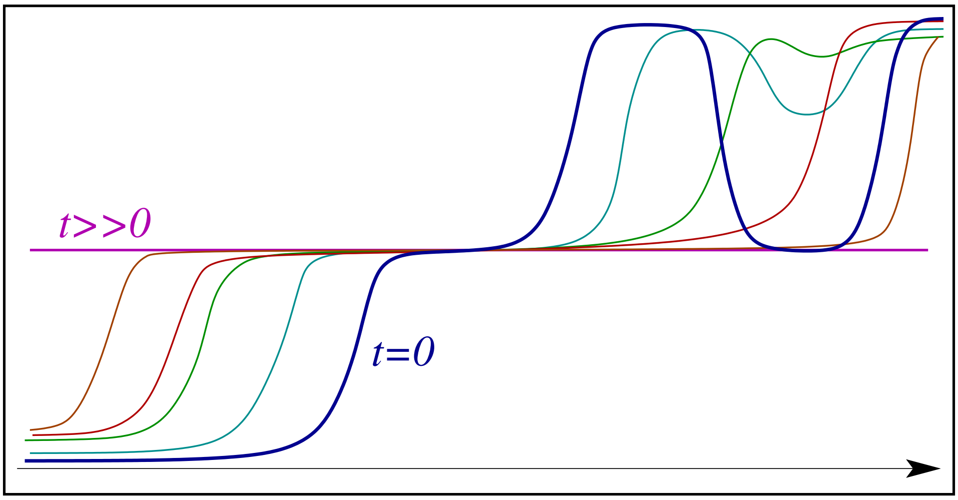

Figure 4 (unbalanced orientation case): Evolution of the dislocation function as described in Theorems 1.5 and 1.6 (l even, limit case: constant).

1.4. **Equilibria of the dynamical system.** An interesting byproduct of our results is that the particles in (1.8) can never remain at rest, namely:

Corollary 1.7. *Assume that* (1.2) *holds true, that* $N \geqslant 2$ *and that* $\sigma \equiv 0$. *Then the ODE system in* (1.8) *does not admit stationary points.*

It is worth to point out that a similar result does not hold for infinitely many particles (an equilibrium being given by alternate particles at the same distance). It is also interesting to observe that our proof of Corollary 1.7 is not based on ODE methods, but on the analysis of the integro-differential equation in (1.1), which provides a further example of link between related, but in principle different, topics, in terms of results, motivations and methods.

The rest of the paper is organized as follows. In Section 2 we collect a series of ancillary results, to be freely exploited in the proofs of the main results.

Then, we prove Theorem 1.1 in Section 3, Theorem 1.2 in Section 4, Theorems 1.3 and 1.5 in Section 5, and Theorems 1.4 and 1.6 in Section 6. Finally, Corollary 1.7 is proved in Section 7.

2. Preliminary observations

2.1. **Toolbox.** In this section we recall some general auxiliary results that will be used in the rest of the paper. In what follows we denote by H the Heaviside function.

Lemma 2.1. *Assume that* (1.2) *holds, then there exists a unique solution* $u \in C^{2,\alpha}(\mathbb{R})$ *of* (1.4). *Moreover, there exist constants* $C, c > 0$ *and* $\kappa > 2s$ *(only depending on* s*) such that*

$$\left| u(x) - H(x) + \frac{1}{2sW''(0)} \frac{x}{|x|^{2s+1}} \right| \leqslant \frac{C}{|x|^{\kappa}}, \quad \text{for } |x| \geqslant 1, \tag{2.1}$$

and

$$\frac{c}{|x|^{1+2s}} \leqslant u'(x) \leqslant \frac{C}{|x|^{1+2s}} \quad \text{for } |x| \geqslant 1. \tag{2.2}$$

Proof. The existence of a unique solution of (1.4) is proven in [11], see also [1]. Estimate (2.1) is proven in [7] for $s = \frac{1}{2}$ and in [5], [4] respectively for $s \in \left(\frac{1}{2}, 1\right)$ and $s \in \left(0, \frac{1}{2}\right)$. Finally, estimate (2.2) is shown in [1]. □

Next, we introduce the function ψ to be the solution of

$$\begin{cases} \mathcal{I}_s\psi - W''(u)\psi = u' + \eta(W''(u) - W''(0)) & \text{in } \mathbb{R} \\ \psi(-\infty) = 0 = \psi(+\infty), \end{cases} \tag{2.3}$$

where u is the solution of (1.4) and

$$\eta := \frac{1}{W''(0)} \int_{\mathbb{R}} (u'(x))^2 dx = \frac{1}{\gamma\beta}. \tag{2.4}$$

For a detailed heuristic motivation of equation (2.3), see Section 3.1 of [7]. For later purposes, we recall the following decay estimate on the solution of (2.3):

Lemma 2.2. *Assume that* (1.2) *holds, then there exists a unique solution* ψ *to* (2.3). *Furthermore* $\psi \in C^{1,\alpha}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *for some* $\alpha \in (0,1)$ *and there exists* $C > 0$ *such that for any* $x \in \mathbb{R}$

$$|\psi'(x)| \leqslant \frac{C}{1 + |x|^{1+2s}}. \tag{2.5}$$

Proof. The existence of a unique solution of (2.3) is proven in [7] for $s = \frac{1}{2}$ and in [5], [4] respectively for $s \in \left(\frac{1}{2}, 1\right)$ and $s \in \left(0, \frac{1}{2}\right)$. Estimate (2.5) is shown in [13]. □

3. Proof of Theorem 1.1

Let $(x_1(t), \ldots, x_N(t))$ be the solution of (1.8), where the ζ_i's are given by (1.10). Let us denote, for $i = 1, \ldots, N-1$

$$\vartheta_i(t) := x_{i+1}(t) - x_i(t),$$

and

$$\vartheta_i^0 := x_{i+1}^0 - x_i^0.$$

Let us start by showing that if the assumption (3.1) below is satisfied, then the condition $T_c < +\infty$ holds true and a collision occurs between the particles x_K and x_{K+1}.

Lemma 3.1. *Assume that*

$$1 - 2s(\vartheta_K^0)^{2s}\|\sigma\|_\infty > 0. \tag{3.1}$$

Then $\vartheta_K(t)$ *is decreasing and there exists* T_c *satisfying*

$$T_c \leqslant \frac{s(\vartheta_K^0)^{1+2s}}{(2s+1)\gamma(1 - 2s(\vartheta_K^0)^{2s}\|\sigma\|_\infty)}, \tag{3.2}$$

such that

$$\vartheta_K(T_c) = 0.$$

Proof. From (1.8) and (1.10), we infer that

$$\begin{aligned}
\dot\vartheta_K &= \dot x_{K+1} - \dot x_K\\
&= \gamma\Bigg(\sum_{j\neq K+1}\zeta_{K+1}\zeta_j\frac{x_{K+1}-x_j}{2s|x_{K+1}-x_j|^{1+2s}} - \zeta_{K+1}\sigma(t,x_{K+1})\\
&\quad - \sum_{j\neq K}\zeta_K\zeta_j\frac{x_K-x_j}{2s|x_K-x_j|^{1+2s}} + \zeta_K\sigma(t,x_K)\Bigg)\\
&= \gamma\Bigg(-\sum_{j=1}^{K-1}\frac{1}{2s(x_{K+1}-x_j)^{2s}} - \frac{1}{2s(x_{K+1}-x_K)^{2s}} - \sum_{j=K+2}^{N}\frac{1}{2s(x_j-x_{K+1})^{2s}}\\
&\quad -\sum_{j=1}^{K-1}\frac{1}{2s(x_K-x_j)^{2s}} - \frac{1}{2s(x_{K+1}-x_K)^{2s}} - \sum_{j=K+2}^{N}\frac{1}{2s(x_j-x_K)^{2s}}\\
&\quad +\sigma(t,x_{K+1})+\sigma(t,x_K)\Bigg)\\
&\leqslant \gamma\left(-\frac{1}{s\vartheta_K^{2s}} + 2\|\sigma\|_\infty\right).
\end{aligned}$$

Therefore, ϑ_K is subsolution of

$$\dot\vartheta = -\frac{\gamma}{s\vartheta^{2s}} + 2\gamma\|\sigma\|_\infty, \tag{3.3}$$

with initial condition

$$\vartheta_K(0) = \vartheta_K^0.$$

If $\sigma\equiv 0$, then $\dot\vartheta_K<0$. If $\sigma\not\equiv 0$ then equation (3.3) has the stationary solution $\vartheta_s(t) := \left(\frac{1}{2s\|\sigma\|_\infty}\right)^{\frac{1}{2s}}$. If assumption (3.1) is satisfied, then $\vartheta_K^0<\vartheta_s$ and since ϑ_K cannot touch ϑ_s, its derivative remains negative. Hence

$$\vartheta_K \leqslant \vartheta_K^0 \quad\text{and}\quad \dot\vartheta_K < -\frac{\gamma}{s(\vartheta_K^0)^{2s}} + 2\gamma\|\sigma\|_\infty < 0.$$

As a consequence, there exists a finite time T_c such that $\vartheta_K(T_c)=0$. Since ϑ_K is subsolution of (3.3) and it is decreasing, we have

$$s\vartheta_K^{2s}\dot\vartheta_K \leqslant -\gamma + 2s\gamma\|\sigma\|_\infty\vartheta_K^{2s} \leqslant -\gamma + 2s\gamma\|\sigma\|_\infty(\vartheta_K^0)^{2s}.$$

Integrating in $(0,T_c)$, we get

$$\frac{s}{2s+1}(\vartheta_K^{2s+1}(T_c)-\vartheta_K^{2s+1}(0)) = -\frac{s}{2s+1}(\vartheta_K^0)^{2s+1} \leqslant -\gamma(1-2s\|\sigma\|_\infty(\vartheta_K^0)^{2s})T_c$$

which gives (3.2). □

While the particles x_K and x_{K+1} collide at time T_c, the remaining particles stay at positive distance one from each other, as stated in the lemma below.

Lemma 3.2. *There exists $c>0$ depending on s,N the ϑ_i^0's and T_c, such that, for any $t\in[0,T_c]$ and $i\neq K$, we have*

$$\vartheta_i(t)\geqslant c. \tag{3.4}$$

Proof. Let us prove (3.4) for $i = 1, \dots, K-1$. Similarly one can show (3.4) for $i = K+1, \dots, N-1$. For $1 \leqslant i < j \leqslant K$, let us denote

$$\vartheta_{j,i}(t) := x_j(t) - x_i(t).$$

We first show that

$$\vartheta_{K,1}(t) \geqslant (x_K^0 - x_1^0)e^{-\gamma\|\sigma_x\|_\infty t}. \tag{3.5}$$

Indeed, from (1.8) and (1.10), we have

$$\begin{aligned}
\dot{\vartheta}_{K,1} &= \gamma\left(\sum_{l=1}^{K-1}\frac{1}{2s(x_K - x_l)^{2s}} + \sum_{l=K+1}^{N}\frac{1}{2s(x_l - x_K)^{2s}} + \sum_{l=2}^{K}\frac{1}{2s(x_l - x_1)^{2s}}\right. \\
&\quad \left. - \sum_{l=K+1}^{N}\frac{1}{2s(x_l - x_1)^{2s}} - \sigma(t, x_K) + \sigma(t, x_1)\right) \\
&\geqslant \gamma\left(\sum_{l=1}^{K-1}\frac{1}{2s(x_K - x_l)^{2s}} + \sum_{l=2}^{K}\frac{1}{2s(x_l - x_1)^{2s}} - \sigma(t, x_K) + \sigma(t, x_1)\right) \\
&\geqslant -\gamma\|\sigma_x\|_\infty \vartheta_{K,1},
\end{aligned}$$

which implies (3.5).

Now, suppose by contradiction that there exist $1 \leqslant i < j \leqslant K$ and a first time $T > 0$ such that

$$\vartheta_{j,i}(T) = 0. \tag{3.6}$$

From (3.5), either $i > 1$ or $j < K$. Suppose for instance $i > 1$. Choose i and j to be respectively the minimum and the maximum index such that (3.6) holds, i.e., $x_i(T) - x_{i-1}(T) > 0$ and either $x_{j+1}(T) - x_j(T) > 0$ or $j = K$. Then, there exists $C_0 > 0$ such that for any $t \in [0, T]$,

$$-\frac{1}{2s(x_i(t) - x_{i-1}(t))^{2s}} \geqslant -C_0, \tag{3.7}$$

and, if $j < K$,

$$-\frac{1}{2s(x_{j+1}(t) - x_j(t))^{2s}} \geqslant -C_0. \tag{3.8}$$

Then, using (1.8), (1.10), (3.7) and (3.8), we get

$$\begin{aligned}
\dot\vartheta_{j,i} &= \gamma\left(\sum_{l=1}^{j-1}\frac{1}{2s(x_j-x_l)^{2s}} - \sum_{l=j+1}^{K}\frac{1}{2s(x_l-x_j)^{2s}} + \sum_{l=K+1}^{N} + \frac{1}{2s(x_l-x_j)^{2s}}\right.\\
&\quad -\sum_{l=1}^{i-1}\frac{1}{2s(x_i-x_l)^{2s}} + \sum_{l=i+1}^{K}\frac{1}{2s(x_l-x_i)^{2s}} - \sum_{l=K+1}^{N} + \frac{1}{2s(x_l-x_i)^{2s}}\\
&\quad \left. -\sigma(t,x_j)+\sigma(t,x_i)\right)\\
&\geqslant \gamma\left(\sum_{l=1}^{j-1}\frac{1}{2s(x_j-x_l)^{2s}} - \sum_{l=j+1}^{K}\frac{1}{2s(x_l-x_j)^{2s}}\right.\\
&\quad -\sum_{l=1}^{i-1}\frac{1}{2s(x_i-x_l)^{2s}} + \sum_{l=i+1}^{K}\frac{1}{2s(x_l-x_i)^{2s}}\\
&\quad \left. -\sigma(t,x_j)+\sigma(t,x_i)\right)\\
&\geqslant \gamma\left(\frac{1}{s\vartheta_{j,i}^{2s}} - \sum_{l=j+1}^{K}\frac{1}{2s(x_l-x_j)^{2s}} - \sum_{l=1}^{i-1}\frac{1}{2s(x_i-x_l)^{2s}} - \sigma(t,x_j)+\sigma(t,x_i)\right)\\
&\geqslant \gamma\left(\frac{1}{s\vartheta_{j,i}^{2s}} - C - \|\sigma_x\|_\infty\vartheta_{j,i}\right),
\end{aligned}$$

where $C=(K-j+i-1)C_0$. Now, (3.6) implies that for any $\delta>0$ there exists $t_\delta>0$ such that $0<\vartheta_{j,i}(t)\leqslant\delta$ for any $t\in(T-t_\delta,T)$. Choosing δ small enough so that

$$\frac{1}{s\delta^{2s}} - C - \|\sigma_x\|_\infty\delta > 0,$$

from the computation above we see that $\vartheta_{j,i}$ is increasing in $(T-t_\delta,T)$ and this contradicts (3.6). Estimate (3.4) for $i<K$ is then proven. A similar argument gives (3.4) when $i>K$. □

Now, as firstly seen in [7, 5, 4, 14], we consider an auxiliary small parameter $\delta>0$ and define $(\overline{x}_1(t),\dots,\overline{x}_N(t))$ to be the solution to the following system: for $i=1,\dots,N$

$$\begin{cases}\dot{\overline{x}}_i = \gamma\left(\sum_{j\neq i}\zeta_i\zeta_j\frac{\overline{x}_i-\overline{x}_j}{2s|\overline{x}_i-\overline{x}_j|^{1+2s}} - \zeta_i\sigma(t,\overline{x}_i) - \zeta_i\delta\right) & \text{in } (0,T_c^\delta)\\ \overline{x}_i(0) = x_i^0 - \zeta_i\delta, \end{cases} \tag{3.9}$$

where the ζ_i's are given by (1.10) and T_c^δ is the collision time of the perturbed system (3.9). Let us denote for $i=1,\dots,N-1$

$$\overline{\vartheta}_i(t) := \overline{x}_{i+1}(t) - \overline{x}_i(t). \tag{3.10}$$

The following results have been proven in [14] in the case $N=3$. Since the proofs do not change in the case $N>3$, we skip them and we refer to the analogous results in [14].

Proposition 3.3. *Let $(x_1,\dots,x_N)$ and $(\overline{x}_1,\dots,\overline{x}_N)$ be the solution respectively of system* (1.8) *and* (3.9). *Let $T_c<+\infty$ and T_c^δ be the collision time respectively of* (1.8) *and* (3.9). *Then we have*

$$\lim_{\delta\to 0} T_c^\delta = T_c, \tag{3.11}$$

and for $i = 1, \dots, N$

$$\lim_{\delta\to 0} \overline{x}_i(t) = x_i(t) \quad \textit{for any } t \in [0, T_c). \tag{3.12}$$

Proof. See the proof of Proposition 5.1 in [14]. □

Proposition 3.4. *Let* $(\overline{x}_1, \dots, \overline{x}_N)$ *be the solution to system* (3.9) *and* $(\overline{\vartheta}_1, \dots, \overline{\vartheta}_{N-1})$ *given by* (3.10). *Then, for any* $0 \leqslant \delta \leqslant 1$ *the function* $\min_{i=1,\dots,N} \overline{\vartheta}_i$ *is Hölder continuous in* $[0, T_c^\delta]$ *with Hölder constant uniform in* δ.

Proof. See the proof of Proposition 5.2 in [14]. □

Next, we set

$$\overline{c}_i(t) := \dot{\overline{x}}_i(t), \quad i = 1, \dots, N \tag{3.13}$$

and

$$\overline{\sigma} := \frac{\sigma + \delta}{\beta}, \tag{3.14}$$

where β is given by (1.6). Let u and ψ be respectively the solution of (1.4) and (2.3). We define

(3.15)

$$\overline{v}_\varepsilon(t, x) := \varepsilon^{2s}\overline{\sigma}(t, x) + \sum_{i=1}^N u\left(\zeta_i \frac{x - \overline{x}_i(t)}{\varepsilon}\right) - (N - K) - \sum_{i=1}^N \zeta_i \varepsilon^{2s} \overline{c}_i(t) \psi\left(\zeta_i \frac{x - \overline{x}_i(t)}{\varepsilon}\right).$$

The situation is depicted in Figure 5.

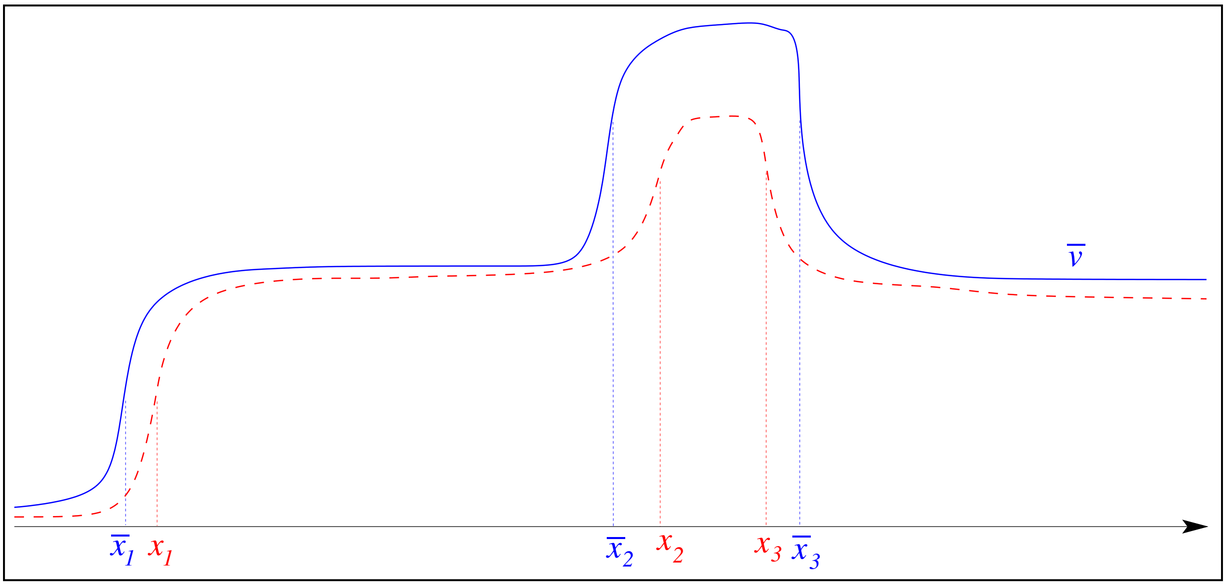

Figure 5: The barrier of Proposition 3.5.

Under the appropriate choice of the parameters, the function $\overline{v}_\varepsilon$ is a supersolution of (1.1)-(1.5), as next results point out:

Proposition 3.5. *There exist $\varepsilon_0 > 0$ and ϑ_ε, $\delta_\varepsilon > 0$ with*

$$\vartheta_\varepsilon,\ \delta_\varepsilon,\ \varepsilon\vartheta_\varepsilon^{-2} = o(1) \quad \text{as } \varepsilon \to 0 \tag{3.16}$$

such that for any $\varepsilon < \varepsilon_0$, if $(\overline{x}_1, \dots, \overline{x}_N)$ is a solution of the ODE system in (3.9) *with $\delta \geqslant \delta_\varepsilon$, then the function $\overline{v}_\varepsilon$ defined in* (3.15) *satisfies*

$$\varepsilon(\overline{v}_\varepsilon)_t - \mathcal{I}_s \overline{v}_\varepsilon + \frac{1}{\varepsilon^{2s}} W'(\overline{v}_\varepsilon) - \sigma \geqslant 0$$

for any $x \in \mathbb{R}$ and any $t \in (0, T_c^\delta)$ such that $\overline{x}_{i+1}(t) - \overline{x}_i(t) \geqslant \vartheta_\varepsilon$ for $i = 1, \dots, N-1$.

Proof. See the proof of Proposition 5.3 in [14]. □

Lemma 3.6. *Let $v_\varepsilon^0(x)$ be defined by* (1.5)*. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and δ_ε given by Proposition* 3.5*, if $(\overline{x}_1, \dots, \overline{x}_N)$ is the solution to system* (3.9) *with $\delta = \delta_\varepsilon$, then the function $\overline{v}_\varepsilon$ defined in* (3.15) *satisfies*

$$v_\varepsilon^0(x) \leqslant \overline{v}_\varepsilon(0, x) \quad \text{for any } x \in \mathbb{R}.$$

Proof. See the proof of Lemma 5.4 in [14]. □

Now we consider the barrier function $\overline{v}_\varepsilon$ defined in (3.15), where $(\overline{x}_1, \dots, \overline{x}_N)$ is the solution to system (3.9) in which we fix $\delta = \delta_\varepsilon$, with δ_ε given by Proposition 3.5. For ε small enough, from (3.11), (3.12) and (3.4), we infer that there exists $T_\varepsilon^1 > 0$ such that

$$\overline{\vartheta}_K(T_\varepsilon^1) = \overline{x}_{K+1}(T_\varepsilon^1) - \overline{x}_K(T_\varepsilon^1) = \vartheta_\varepsilon, \tag{3.17}$$

and

$$\overline{\vartheta}_K(t) = \overline{x}_{K+1}(t) - \overline{x}_K(t) > \vartheta_\varepsilon \quad \text{for any } t < T_\varepsilon^1,$$

and there exists a constant $c_0 > 0$ independent of ε such that

$$\overline{x}_{i+1}(t) - \overline{x}_i(t) \geqslant c_0 \quad \text{for any } t \leqslant T_\varepsilon^1 \text{ and } i \neq K. \tag{3.18}$$

From (3.9), (3.13) and (3.17), we infer that

$$|\overline{c}_K(T_\varepsilon^1)| \leqslant C\vartheta_\varepsilon^{-2s}. \tag{3.19}$$

By Proposition 3.5 and Lemma 3.6, the function $\overline{v}_\varepsilon$ defined in (3.15), is a supersolution of (1.1)-(1.5) in $(0, T_\varepsilon^1) \times \mathbb{R}$, and the comparison principle (see e.g. Proposition 5.10 in [8]) implies

$$v_\varepsilon(t, x) \leqslant \overline{v}_\varepsilon(t, x) \quad \text{for any } (t, x) \in [0, T_\varepsilon^1] \times \mathbb{R}. \tag{3.20}$$

Moreover, since $\vartheta_\varepsilon = o(1)$ as $\varepsilon \to 0$, we have

$$T_\varepsilon^1 = T_c + o(1) \quad \text{as } \varepsilon \to 0. \tag{3.21}$$

Indeed, if up to subsequences, T_ε^1 converges as $\varepsilon \to 0$ to some $T > 0$, since $T_\varepsilon^1 \leqslant T_c^{\delta_\varepsilon}$ then by (3.11) we have $T \leqslant T_c$. Suppose by contradiction that

$$T < T_c. \tag{3.22}$$

Then by Proposition 3.4 and (3.17)

$$|\overline{\vartheta}_K(T_\varepsilon^1) - \overline{\vartheta}_K(T)| = |\vartheta_\varepsilon - \overline{\vartheta}_K(T)| \leqslant C|T_\varepsilon^1 - T|^\alpha,$$

for some $C > 0$ and $\alpha \in (0, 1)$ independent of ε. This and (3.12) imply that $\vartheta_K(T) = 0$ which is in contradiction with (3.22). Thus (3.21) is proven.

Next, to conclude the proof of Theorem 1.1, we are going to show that starting from T^1_ε, after a small time t_ε, the function v_ε satisfies (1.14), for some $\varrho_\varepsilon = o(1)$ and some $x^\varepsilon_i = x_i(T_c) + o(1)$, $i \neq K$, $K+1$, as $\varepsilon \to 0$. For this scope, we denote

$$\overline{x}^\varepsilon_i := \overline{x}_i(T^1_\varepsilon), \quad i = 1, \dots, N.$$

We recall that from (3.17)

$$\overline{x}^\varepsilon_{K+1} - \overline{x}^\varepsilon_K = \vartheta_\varepsilon. \tag{3.23}$$

We show (1.14) for $x \leqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$, similarly one can prove it for $x \geqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$. For this aim let us introduce the following further perturbed system: for $\hat\delta > \delta_\varepsilon$, $L > 1$ such that $\overline{x}^\varepsilon_{K+1} + L\vartheta_\varepsilon < \overline{x}^\varepsilon_{K+2}$, and ζ_i's given by (1.10),

$$\begin{cases} \dot{\hat{x}}_i = \gamma \left(\displaystyle\sum_{j \neq i} \zeta_i \zeta_j \frac{\hat{x}_i - \hat{x}_j}{2s|\hat{x}_i - \hat{x}_j|^{1+2s}} - \zeta_i \sigma(t, \hat{x}_i) - \zeta_i \hat\delta \right) & \text{in } (0, T^{\hat\delta}_c) \\ \hat{x}_{K+1}(0) = \overline{x}^\varepsilon_{K+1} + L\vartheta_\varepsilon & \\ \hat{x}_i(0) = \overline{x}^\varepsilon_i - \zeta_i \vartheta_\varepsilon & i \neq K+1, \end{cases} \tag{3.24}$$

where $T^{\hat\delta}_c$ is the collision time of the system (3.24). We set

$$\hat{c}_i(t) := \dot{\hat{x}}_i(t), \quad i = 1, \dots, N \tag{3.25}$$

and

$$\hat\sigma := \frac{\sigma + \hat\delta}{\beta}, \tag{3.26}$$

where β is given by (1.6).
We define
(3.27)

$$\hat{v}_\varepsilon(t,x) := \varepsilon^{2s} \hat\sigma(t,x) + \sum_{i=1}^N u\left(\zeta_i \frac{x - \hat{x}_i(t)}{\varepsilon}\right) - (N-K) - \sum_{i=1}^N \zeta_i \varepsilon^{2s} \hat{c}_i(t) \psi\left(\zeta_i \frac{x - \hat{x}_i(t)}{\varepsilon}\right),$$

where again u and ψ are respectively the solution of (1.4) and (2.3).

Lemma 3.7. *There exist ε_0, $\hat\delta_\varepsilon > 0$ with $\delta_\varepsilon < \hat\delta_\varepsilon = \delta_\varepsilon + o(1)$ as $\varepsilon \to 0$, where δ_ε is given by Proposition 3.5, such that if $(\hat{x}_1, \dots, \hat{x}_N)$ is the solution to system (3.24) with $\hat\delta = \hat\delta_\varepsilon$, then the function $\hat{v}_\varepsilon$ defined in (3.27) satisfies*

$$\hat{v}_\varepsilon(0,x) \geqslant \overline{v}_\varepsilon(T^1_\varepsilon, x) \quad \text{for any } x \in \mathbb{R}.$$

Proof. See the proof of Lemma 5.6 in [14]. □

Lemma 3.8. *Let*

$$t_\varepsilon := \frac{4s(L+2)^{2s} \vartheta_\varepsilon^{2s+1}}{\gamma[1 - 2s(L+2)^{2s} \vartheta_\varepsilon^{2s} (\|\sigma\|_\infty + \hat\delta)]}. \tag{3.28}$$

Then there exists $L > 1$, $c_1 > 0$, and $\varepsilon_0, \hat\delta_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $\hat\delta < \hat\delta_0$,

$$\overline{x}^\varepsilon_{K+1} + L\vartheta_\varepsilon < \overline{x}^\varepsilon_{K+2}, \tag{3.29}$$

and the solution $(\hat{x}_1, \dots, \hat{x}_N)$ to system (3.24) satisfies

$$\hat{x}_K(t_\varepsilon) \geqslant \overline{x}^\varepsilon_{K+1}, \tag{3.30}$$

for any $t \in [0, t_\varepsilon]$*,* $\hat{x}_{K+1}(t) - \hat{x}_K(t)$ *is decreasing and*

$$\hat{x}_{K+1}(t) - \hat{x}_K(t) \geqslant \hat{x}_{K+1}(t_\varepsilon) - \hat{x}_K(t_\varepsilon) \geqslant \vartheta_\varepsilon, \tag{3.31}$$

and for any $t \in [0, t_\varepsilon]$ *and* $i \neq K$

$$\hat{x}_{i+1}(t) - \hat{x}_i(t) \geqslant c_1. \tag{3.32}$$

Proof. Let us denote

$$\hat{\vartheta}_K(t) := \hat{x}_{K+1}(t) - \hat{x}_K(t).$$

Then, from Lemma 3.1, for ε and $\hat{\delta}$ small enough, such that

$$\hat{\vartheta}_K(0) = (L+2)\vartheta_\varepsilon < \left[\frac{1}{2s(\|\sigma\|_\infty + \hat{\delta})}\right]^{\frac{1}{2s}},$$

$\hat{\vartheta}_K$ is decreasing, therefore for $t > 0$,

$$\hat{\vartheta}_K(t) < (L+2)\vartheta_\varepsilon. \tag{3.33}$$

Moreover, there exists $\tau > 0$ satisfying

$$\tau < \frac{s(L+2)^{1+2s}\vartheta_\varepsilon^{1+2s}}{(2s+1)\gamma(1 - 2s(\vartheta_\varepsilon)^{2s}(L+2)^{2s}(\|\sigma\|_\infty + \hat{\delta}))}, \tag{3.34}$$

such that $\hat{\vartheta}_K(\tau) = \vartheta_\varepsilon$. Remark that $\tau = o(1)$ as $\varepsilon \to 0$, then from (3.18) we infer that, for ε and $\hat{\delta}$ small enough, there exists a constant c_1 independent of ε and $\hat{\delta}$ such that

$$\hat{\vartheta}_i(t) \geqslant c_1 \quad \text{for any } t \in [0, \tau] \text{ and } i \neq K. \tag{3.35}$$

From (3.35) (where the ζ_i's are given by (1.10)) and (3.24), we infer that

$$\begin{aligned}
\dot{\hat{\vartheta}}_K = \gamma \Bigg(& -\sum_{j=1}^{K-1} \frac{1}{2s(\hat{x}_{K+1} - \hat{x}_j)^{2s}} - \frac{1}{2s(\hat{x}_{K+1} - \hat{x}_K)^{2s}} - \sum_{j=K+2}^{N} \frac{1}{2s(\hat{x}_j - \hat{x}_{K+1})^{2s}} \\
& - \sum_{j=1}^{K-1} \frac{1}{2s(\hat{x}_K - \hat{x}_j)^{2s}} - \frac{1}{2s(\hat{x}_{K+1} - \hat{x}_K)^{2s}} - \sum_{j=K+2}^{N} \frac{1}{2s(\hat{x}_j - \hat{x}_K)^{2s}} \\
& + \sigma(t, \hat{x}_{K+1}) + \sigma(t, \hat{x}_K) + 2\hat{\delta} \Bigg) \\
\geqslant \gamma \Bigg(& -\frac{1}{s\hat{\vartheta}_K^{2s}} - \sum_{j=1}^{K-1} \frac{1}{2s(\hat{\vartheta}_K + \ldots + \hat{\vartheta}_j)^{2s}} - \sum_{j=K+2}^{N} \frac{1}{2s(\hat{\vartheta}_{K+1} + \ldots + \hat{\vartheta}_{j-1})^{2s}} \\
& - \sum_{j=1}^{K-1} \frac{1}{2s(\hat{\vartheta}_{K-1} + \ldots + \hat{\vartheta}_j)^{2s}} - \sum_{j=K+2}^{N} \frac{1}{2s(\hat{\vartheta}_K + \ldots + \hat{\vartheta}_{j-1})^{2s}} - 2\|\sigma\|_{L^\infty} \Bigg) \\
\geqslant \gamma \Bigg(& -\frac{1}{s\hat{\vartheta}_K^{2s}} - C - 2\|\sigma\|_{L^\infty} \Bigg),
\end{aligned}$$

for some $C > 0$ independent of ε and $\hat{\delta}$.

Combining the previous estimate with (3.33), we get, for any $t \in (0, \tau)$,

$$\dot{\hat{\vartheta}}_K \geqslant \gamma \left(\frac{-1 - (2\|\sigma\|_\infty + C)s\hat{\vartheta}_K^{2s}}{s\hat{\vartheta}_K^{2s}} \right) \geqslant \gamma \left(\frac{-1 - (2\|\sigma\|_\infty + C)s(L+2)^{2s}\vartheta_\varepsilon^{2s}}{s\hat{\vartheta}_K^{2s}} \right),$$

i.e.,

$$\hat{\vartheta}_K^{2s}\dot{\hat{\vartheta}}_K \geqslant \frac{\gamma}{s}\left(-1-(2s\|\sigma\|_\infty + C)(L+2)^{2s}\vartheta_\varepsilon^{2s}\right).$$

Integrating the previous inequality in $(0,\tau)$, we get

$$\begin{aligned}\frac{1}{2s+1}(\hat{\vartheta}_K^{2s+1}(\tau)-\hat{\vartheta}_K^{2s+1}(0)) &= \frac{1}{2s+1}\vartheta_\varepsilon^{2s+1}(1-(L+2)^{2s+1})\\ &\geqslant \frac{\gamma}{s}\left(-1-(2s\|\sigma\|_\infty + C)(L+2)^{2s}\vartheta_\varepsilon^{2s}\right)\tau,\end{aligned}$$

from which

$$\tau \geqslant \frac{s\vartheta_\varepsilon^{2s+1}[(L+2)^{2s+1}-1]}{\gamma(2s+1)(1+(2s\|\sigma\|_\infty + C)(L+2)^{2s}\vartheta_\varepsilon^{2s})}. \tag{3.36}$$

Next, (3.24) and (3.33) imply

(3.37)

$$\begin{aligned}\dot{\hat{x}}_K =&\gamma\left(\sum_{j=1}^{K-1}\frac{1}{2s(\hat{x}_K-\hat{x}_j)^{2s}}+\frac{1}{2s(\hat{x}_{K+1}-\hat{x}_K)^{2s}}+\sum_{j=K+2}^{N}\frac{1}{2s(\hat{x}_j-\hat{x}_K)^{2s}}-\sigma(t,\hat{x}_K)-\hat{\delta}\right)\\ &\geqslant \gamma\left(\frac{1}{2s\hat{\vartheta}_K^{2s}}-\sigma(t,\hat{x}_K)-\hat{\delta}\right)\\ &\geqslant \gamma\left(\frac{1}{2s(L+2)^{2s}(\vartheta_\varepsilon)^{2s}}-\|\sigma\|_\infty-\hat{\delta}\right)\\ &> 0.\end{aligned}$$

Let t be the time such that $\hat{x}_K(t)=\overline{x}^\varepsilon_{K+1}=\hat{x}_K(0)+2\vartheta_\varepsilon$, then integrating (3.37) in $(0,t)$ we get

$$\hat{x}_K(t)-\hat{x}_K(0)=2\vartheta_\varepsilon \geqslant \gamma\left(\frac{1}{2s(L+2)^{2s}\vartheta_\varepsilon^{2s}}-\|\sigma\|_\infty-\hat{\delta}\right)t,$$

from which

$$t\leqslant t_\varepsilon \tag{3.38}$$

where t_ε is defined by (3.28).

Comparing τ with t_ε, from (3.28) and (3.36), we see that it is possible to choose L big enough so that

$$\tau > t_\varepsilon \geqslant t.$$

Estimate (3.35) and $\tau > t_\varepsilon$ imply (3.32). Moreover, from (3.18), for any fixed L, we can choose ε small enough so that (3.29) holds. For such a choice of L, the decreasing monotonicity of $\hat{\vartheta}_K$ implies (3.31). Finally, (3.38) and the increasing monotonicity of $\hat{x}_K$ give

$$\hat{x}_K(t_\varepsilon)\geqslant \hat{x}_K(t)=\overline{x}^\varepsilon_{K+1},$$

which proves (3.30). This concludes the proof of the lemma. □

We consider now as barrier the function $\hat{v}_\varepsilon$ defined in (3.27), where we fix $\hat{\delta}=\hat{\delta}_\varepsilon$ in system (3.24), with $\hat{\delta}_\varepsilon$ given by Lemma 3.7, and L given by Lemma 3.8. For ε small enough, from (3.31), (3.32) and Proposition 3.5, the function $\hat{v}_\varepsilon$ satisfies

$$\varepsilon(\hat{v}_\varepsilon)_t-\mathcal{I}_s\hat{v}_\varepsilon+\frac{1}{\varepsilon^{2s}}W'(\hat{v}_\varepsilon)-\sigma(t,x)\geqslant 0 \quad \text{in } (0,t_\varepsilon)\times\mathbb{R}$$

where t_ε is given by (3.28). Moreover from (3.20) and Lemma 3.7

$$v_\varepsilon(T^1_\varepsilon, x) \leqslant \hat v_\varepsilon(0,x) \quad \text{for any } x\in\mathbb{R}.$$

The comparison principle then implies

$$v_\varepsilon(T^1_\varepsilon + t, x) \leqslant \hat v_\varepsilon(t,x) \quad \text{for any } (t,x)\in[0,t_\varepsilon]\times\mathbb{R}. \tag{3.39}$$

Now, for $x \leqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$, from (3.17), (3.30) and (3.31) we know that

$$x - \hat x_K(t_\varepsilon) \leqslant -\frac{\vartheta_\varepsilon}{2} \quad \text{and} \quad \hat x_{K+1}(t_\varepsilon) - x \geqslant \frac{3\vartheta_\varepsilon}{2}.$$

Therefore, from estimate (2.1) we have

$$u\left(\frac{x-\hat x_K(t_\varepsilon)}{\varepsilon}\right) + u\left(\frac{\hat x_{K+1}(t_\varepsilon)-x}{\varepsilon}\right) - 1 \leqslant C\varepsilon^{2s}\vartheta_\varepsilon^{-2s}. \tag{3.40}$$

Moreover (3.31), (3.32), (3.24) and (3.25) imply that for $i = K, K+1$

$$|\hat c_i(t_\varepsilon)| \leqslant C\vartheta_\varepsilon^{-2s}. \tag{3.41}$$

From the (3.40), (3.41) and (3.39), we infer that, for $x \leqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$, we have

$$\begin{aligned} v_\varepsilon(T^1_\varepsilon + t_\varepsilon, x) &\leqslant \varepsilon^{2s}\hat\sigma(t_\varepsilon,x) + \sum_{\substack{i=1\\ i\neq K,K+1}}^N u\left(\zeta_i\frac{x-\hat x_i(t_\varepsilon)}{\varepsilon}\right) - (N-K-1)\\ &\quad - \sum_{\substack{i=1\\ i\neq K,K+1}}^N \zeta_i\varepsilon^{2s}\hat c_i(t)\psi\left(\zeta_i\frac{x-\hat x_i(t_\varepsilon)}{\varepsilon}\right) + C\varepsilon^{2s}\vartheta_\varepsilon^{-2s}\\ &\leqslant \frac{\varepsilon^{2s}}{\beta}\sigma(t_\varepsilon,x) + \sum_{\substack{i=1\\ i\neq K,K+1}}^N u\left(\zeta_i\frac{x-\hat x_i(t_\varepsilon)}{\varepsilon}\right) - (N-K-1) + \varrho_\varepsilon, \end{aligned} \tag{3.42}$$

where

$$\varrho_\varepsilon = O(\varepsilon^{2s}\vartheta_\varepsilon^{-2s}).$$

From (3.16), we see that ϱ_ε satisfies (1.13).

Similarly, one can prove that

$$v_\varepsilon(T^1_\varepsilon + t, x) \leqslant \hat w_\varepsilon(t,x) \quad \text{for any } (t,x)\in[0,t_\varepsilon]\times\mathbb{R}. \tag{3.43}$$

where $\hat w_\varepsilon$ is defined by

$$\hat w_\varepsilon(t,x) := \varepsilon^{2s}\hat\sigma(t,x) + \sum_{i=1}^N u\left(\zeta_i\frac{x-\hat y_i(t)}{\varepsilon}\right) - (N-K) - \sum_{i=1}^N \zeta_i\varepsilon^{2s}\hat d_i(t)\psi\left(\zeta_i\frac{x-\hat y_i(t)}{\varepsilon}\right),$$

where $(\hat y_1,\dots,\hat y_N)$ is the solution of the system (3.24) with initial condition

$$\begin{aligned} \hat y_i(0) &= \overline{x}^\varepsilon_i - \zeta_i\vartheta_\varepsilon, \quad i\neq K,\\ \hat y_K(0) &= \overline{x}^\varepsilon_K - L\vartheta_\varepsilon, \end{aligned} \tag{3.44}$$

for L large enough, small ε and $\hat\delta = \hat\delta_\varepsilon$, and

$$\hat d_i := \dot{\hat y}_i(t), \quad i = 1,\dots,N.$$

As before, from (3.43), we get that, for $x \geqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$,

$$v_\varepsilon(T^1_\varepsilon + t_\varepsilon, x) \leqslant \frac{\varepsilon^{2s}}{\beta}\sigma(t_\varepsilon, x) + \sum_{\substack{i=1\\ i\neq K,K+1}}^{N} u\left(\zeta_i \frac{x-\hat{y}_i(t_\varepsilon)}{\varepsilon}\right) - (N-K-1) + \varrho_\varepsilon. \tag{3.45}$$

Now, from (3.18), (3.24) and (3.44), we see that

$$|\hat{x}_i(t_\varepsilon) - \hat{y}_i(t_\varepsilon)| = o(1) \quad \text{as } \varepsilon \to 0, \text{ for } i \neq K, K+1. \tag{3.46}$$

Estimates (3.46) combined with (3.18), imply that there exists a constant $c > 0$ independent of ε such that, for $i \neq K, K+1$,

$$\begin{aligned}&\max(\hat{x}_{i-1}(t_\varepsilon), \hat{y}_{i-1}(t_\varepsilon)) + c \leqslant \min(\hat{x}_i(t_\varepsilon), \hat{y}_i(t_\varepsilon))\\ &\leqslant \max(\hat{x}_i(t_\varepsilon), \hat{y}_i(t_\varepsilon)) \leqslant \min(\hat{x}_{i+1}(t_\varepsilon), \hat{y}_{i+1}(t_\varepsilon)) - c.\end{aligned}$$

Therefore, if we define

$$x^\varepsilon_i := \begin{cases}\min(\hat{x}_i(t_\varepsilon), \hat{y}_i(t_\varepsilon)) & \text{for } i = 1, \dots, K-1\\ \max(\hat{x}_i(t_\varepsilon), \hat{y}_i(t_\varepsilon)) & \text{for } i = K+1, \dots, N,\end{cases}$$

we see that the x^ε_i's satisfy (1.12). Moreover, for $x \leqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$, from (3.42), (1.10) and the monotonicity of u we get

$$\begin{aligned}v_\varepsilon(T^1_\varepsilon + t_\varepsilon, x) &\leqslant \frac{\varepsilon^{2s}}{\beta}\sigma(t_\varepsilon, x) + \sum_{\substack{i=1\\ i\neq K,K+1}}^{N} u\left(\zeta_i \frac{x-\hat{x}_i(t_\varepsilon)}{\varepsilon}\right) - (N-K-1) + \varrho_\varepsilon\\ &= \frac{\varepsilon^{2s}}{\beta}\sigma(t_\varepsilon, x) + \sum_{i=1}^{K-1} u\left(\frac{x-\hat{x}_i(t_\varepsilon)}{\varepsilon}\right) + \sum_{i=K+2}^{N} u\left(\frac{\hat{x}_i(t_\varepsilon)-x}{\varepsilon}\right)\\ &\quad - (N-K-1) + \varrho_\varepsilon\\ &\leqslant \varepsilon^{2s}\hat{\sigma}(t_\varepsilon, x) + \sum_{i=1}^{K-1} u\left(\frac{x-x^\varepsilon_i}{\varepsilon}\right) + \sum_{i=K+2}^{N} u\left(\frac{x^\varepsilon_i-x}{\varepsilon}\right)\\ &\quad - (N-K-1) + \varrho_\varepsilon,\end{aligned}$$

which gives (1.14) for $x \leqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$. Similarly, from (3.45) and the monotonicity of u we get (1.14) for $x \geqslant \overline{x}^\varepsilon_K + \frac{\vartheta_\varepsilon}{2}$. Estimates (1.13) follow from (3.16). This concludes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Let us consider the function

$$h(t,x) := \frac{\varepsilon^{2s}}{\beta}\sigma(t,x) + \sum_{\substack{i=1\\ i\neq K,K+1}}^{N} u\left(\zeta_i \frac{x-x_i(t)}{\varepsilon}\right) - (N-K-1) + \varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}} \tag{4.1}$$

where

$$x_i(t) := x^\varepsilon_i + \zeta_i K_\varepsilon \varrho_\varepsilon (e^{-\frac{\mu t}{\varepsilon^{2s+1}}} - 1), \tag{4.2}$$

the x_i^ε's and ϱ_ε are given by Theorem 1.1 and the ζ_i's satisfy (1.10) We show that, choosing conveniently K_ε and μ, h is a supersolution of the equation (1.1) for small times, as next result states:

Lemma 4.1. *There exist $\varepsilon_0 > 0$ and $\mu > 0$, such that for any $\varepsilon < \varepsilon_0$, there exist K_ε, $\tau_\varepsilon > 0$ such that*

$$\varrho_\varepsilon K_\varepsilon = o(1), \quad \tau_\varepsilon = o(1), \quad \varrho_\varepsilon e^{-\frac{\mu\tau_\varepsilon}{\varepsilon^{2s+1}}} = \varepsilon^{2s} o(1) \quad \text{as } \varepsilon \to 0, \tag{4.3}$$

and the function h defined in (4.1)-(4.2) satisfies

$$\varepsilon h_t - \mathcal{I}_s h + \frac{1}{\varepsilon^{2s}} W'(h) - \sigma(t,x) \geqslant 0 \quad \text{for any } (t,x) \in (0,\tau_\varepsilon) \times \mathbb{R}.$$

Proof. Let α, $\gamma \in (0,1)$ be such that

$$\frac{s}{2s+1} < \alpha < \frac{1}{2}, \tag{4.4}$$

and

$$0 < \gamma < \min\{4s(1-\alpha) - 2s, \alpha(2s+1) - s\}. \tag{4.5}$$

Let τ_ε be such that

$$\varrho_\varepsilon e^{-\frac{\mu\tau_\varepsilon}{\varepsilon^{2s+1}}} = \varepsilon^{2s+\gamma}, \tag{4.6}$$

i.e.,

$$\tau_\varepsilon = \frac{\varepsilon^{2s+1}}{\mu} \log\left(\varrho_\varepsilon \varepsilon^{-(2s+\gamma)}\right).$$

Remark that from (1.13),

$$\tau_\varepsilon = o(1) \quad \text{as } \varepsilon \to 0.$$

Using (4.2), we compute

$$\begin{aligned} \varepsilon h_t &= \frac{\varepsilon^{2s+1}}{\beta}\sigma_t - \sum_{\substack{i=1\\ i\neq K,K+1}}^{N} \zeta_i \dot{x}_i u'\left(\zeta_i \frac{x-x_i(t)}{\varepsilon}\right) - \varepsilon^{-2s}\varrho_\varepsilon \mu e^{-\frac{\mu t}{\varepsilon^{2s+1}}} \\ &= \varepsilon^{-2s-1} K_\varepsilon \varrho_\varepsilon \mu e^{-\frac{\mu t}{\varepsilon^{2s+1}}} \sum_{\substack{i=1\\ i\neq K,K+1}}^{N} u'\left(\zeta_i \frac{x-x_i(t)}{\varepsilon}\right) - \varepsilon^{-2s}\varrho_\varepsilon \mu e^{-\frac{\mu t}{\varepsilon^{2s+1}}} + O(\varepsilon^{2s+1}), \end{aligned}$$

and

$$\mathcal{I}_s h = \frac{\varepsilon^{2s}}{\beta}\mathcal{I}_s\sigma + \varepsilon^{-2s} \sum_{\substack{i=1\\ i\neq K,K+1}}^{N} \mathcal{I}_s u\left(\zeta_i \frac{x-x_i(t)}{\varepsilon}\right) = \sum_{\substack{i=1\\ i\neq K,K+1}}^{N} \varepsilon^{-2s} W'\left(u\left(\zeta_i \frac{x-x_i(t)}{\varepsilon}\right)\right) + O(\varepsilon^{2s}).$$

Then, using the periodicity of W', we get

(4.7)

$$\begin{aligned}
&\varepsilon h_t - \mathcal{I}_s h + \frac{1}{\varepsilon^{2s}} W'(h) - \sigma \\
&= \varepsilon^{-2s-1} K_\varepsilon \varrho_\varepsilon \mu e^{-\frac{\mu t}{\varepsilon^{2s+1}}} \sum_{\substack{i=1 \\ i\neq K,K+1}}^{N} u'\left(\zeta_i \frac{x - x_i(t)}{\varepsilon}\right) - \varepsilon^{-2s} \varrho_\varepsilon \mu e^{-\frac{\mu t}{\varepsilon^{2s+1}}} \\
&+ \varepsilon^{-2s} W'\left(\frac{\varepsilon^{2s}}{\beta}\sigma + \sum_{\substack{i=1 \\ i\neq K,K+1}}^{N} u\left(\zeta_i \frac{x - x_i(t)}{\varepsilon}\right) + \varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\right) - \sum_{\substack{i=1 \\ i\neq K,K+1}}^{N} \varepsilon^{-2s} W'\left(u\left(\zeta_i \frac{x - x_i(t)}{\varepsilon}\right)\right) \\
&- \sigma + O(\varepsilon^{2s}).
\end{aligned}$$

Case 1. Suppose that there exits i_0 such that x is close to $x_{i_0}(t)$ more than ε^α:

$$|x - x_{i_0}(t)| \leqslant \varepsilon^\alpha$$

for fixed α satisfying (4.4). Then estimate (2.2) implies

$$u'\left(\zeta_{i_0} \frac{x - x_{i_0}(t)}{\varepsilon}\right) \geqslant c\varepsilon^{(1-\alpha)(1+2s)}. \tag{4.8}$$

For $i \neq i_0$, we simply have

$$u'\left(\zeta_i \frac{x - x_i(t)}{\varepsilon}\right) \geqslant 0. \tag{4.9}$$

From (4.2) and the fact that the $x_i(t)$'s are well separated at time $t = 0$ by (1.12), we infer that for K_ε such that $K_\varepsilon \varrho_\varepsilon = o(1)$ as $\varepsilon \to 0$, the $x_i(t)$'s stay well separated for any $t \in (0, \tau_\varepsilon)$. Therefore, if x is close $x_{i_0}(t)$, then there exists $c > 0$ independent of ε, such that for any $i \neq i_0$,

$$|x - x_i(t)| \geqslant c.$$

This combined with (2.1) yields, for $i \neq i_0$,

$$\left|\widetilde{u}\left(\zeta_i \frac{x - x_i(t)}{\varepsilon}\right)\right| \leqslant C\varepsilon^{2s}, \tag{4.10}$$

where here and in what follows, we denote by C several constants independent of ε and by

$$\widetilde{u}(x) := u(x) - H(x),$$

where H is the Heaviside function. Hence, from the Lipschitz regularity and the periodicity of W', we get

$$
\begin{aligned}
&\varepsilon^{-2s}W'\left(\frac{\varepsilon^{2s}}{\beta}\sigma+\sum_{\substack{i=1\\ i\neq K,K+1}}^{N}u\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)+\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\right)-\varepsilon^{-2s}W'\left(u\left(\zeta_{i_0}\frac{x-x_{i_0}(t)}{\varepsilon}\right)\right)\\
&=\varepsilon^{-2s}W'\left(\frac{\varepsilon^{2s}}{\beta}\sigma+u\left(\zeta_{i_0}\frac{x-x_{i_0}(t)}{\varepsilon}\right)+\sum_{i\neq i_0}\widetilde{u}\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)+\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\right)\\
&-\varepsilon^{-2s}W'\left(u\left(\zeta_{i_0}\frac{x-x_{i_0}(t)}{\varepsilon}\right)\right)\\
&\geqslant -C\varepsilon^{-2s}\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}-C.
\end{aligned}
$$

Moreover, from (4.10), the Lipschitz regularity of W' and $W'(0)=0$, we infer that

$$
\sum_{i\neq i_0}\varepsilon^{-2s}\left|W'\left(u\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)\right)\right|\leqslant C.
$$

Therefore, from (4.7), using the previous estimates, (4.6), (4.8) and (4.9), we get, for any $(t,x)\in(0,\tau_\varepsilon)\times\mathbb{R}$,

$$
\begin{aligned}
\varepsilon h_t-\mathcal{I}_s h+\frac{1}{\varepsilon^{2s}}W'(h)-\sigma &\geqslant \frac{K_\varepsilon\varrho_\varepsilon\mu}{\varepsilon^{2s+1}}e^{-\frac{\mu t}{\varepsilon^{2s+1}}}c\varepsilon^{(1-\alpha)(1+2s)}-\frac{\varrho_\varepsilon\mu}{\varepsilon^{2s}}e^{-\frac{\mu t}{\varepsilon^{2s+1}}}-\frac{C\varrho_\varepsilon}{\varepsilon^{2s}}e^{-\frac{\mu t}{\varepsilon^{2s+1}}}-C\\
&=\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\left(cK_\varepsilon\mu\varepsilon^{-\alpha(1+2s)}-\mu\varepsilon^{-2s}-C\varepsilon^{-2s}-C\varrho_\varepsilon^{-1}e^{\frac{\mu t}{\varepsilon^{2s+1}}}\right)\\
&\geqslant \varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\left(cK_\varepsilon\mu\varepsilon^{-\alpha(1+2s)}-C\varepsilon^{-2s-\gamma}\right)\\
&=0
\end{aligned}
$$

if

$$
K_\varepsilon\mu=\frac{C}{c}\varepsilon^{\alpha(2s+1)-2s-\gamma}. \tag{4.11}
$$

Remark that since $\frac{\varrho_\varepsilon}{\varepsilon^s}$ =o(1) as $\varepsilon\to0$ by (1.13), for fixed μ independent of ε, we have

$$
\varrho_\varepsilon K_\varepsilon=o(1)\varepsilon^{\alpha(2s+1)-s-\gamma}=o(1)\quad\text{as }\varepsilon\to0,
$$

for γ satisfying (4.5).

Case 2. Suppose that, for any $i=1,\dots,N-2$,

$$
|x-x_i(t)|\geqslant\varepsilon^\alpha.
$$

Then, estimate (2.1) implies

$$
\left|\widetilde{u}\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)\right|\leqslant C\varepsilon^{2s(1-\alpha)}. \tag{4.12}
$$

Making a Taylor expansion of W' around 0, using that $W'(0)=0$, $W''(0)=\beta>0$ and (4.12), we get

$$\begin{aligned}
&\varepsilon^{-2s}W'\left(\frac{\varepsilon^{2s}}{\beta}\sigma+\sum_{\substack{i=1\\ i\neq K,K+1}}^{N}u\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)+\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\right)\\
&=\varepsilon^{-2s}W'\left(\frac{\varepsilon^{2s}}{\beta}\sigma+\sum_{\substack{i=1\\ i\neq K,K+1}}^{N}\widetilde{u}\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)+\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\right)\\
&=\beta\varepsilon^{-2s}\left(\frac{\varepsilon^{2s}}{\beta}\sigma+\sum_{\substack{i=1\\ i\neq K,K+1}}^{N}\widetilde{u}\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)+\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\right)+\varepsilon^{-2s}O(\varepsilon^{2s(1-\alpha)})^2\\
&+\varepsilon^{-2s}O\left(\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\right)^2\\
&\geqslant\sigma+\beta\varepsilon^{-2s}\sum_{\substack{i=1\\ i\neq K,K+1}}^{N}\widetilde{u}\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)+\frac{\beta}{2}\varepsilon^{-2s}\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}+O(\varepsilon^{4s(1-\alpha)-2s}),
\end{aligned}$$

for ε small enough. Similarly, we have

$$\sum_{\substack{i=1\\ i\neq K,K+1}}^{N}\varepsilon^{-2s}W'\left(u\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)\right)=\beta\varepsilon^{-2s}\sum_{\substack{i=1\\ i\neq K,K+1}}^{N}\widetilde{u}\left(\zeta_i\frac{x-x_i(t)}{\varepsilon}\right)+O(\varepsilon^{4s(1-\alpha)-2s}).$$

Combining the previous estimates with (4.7) and using that $u'>0$, (4.5) and (4.6), yields, for any $(t,x)\in\mathbb{R}\times(0,\tau_\varepsilon)$,

$$\begin{aligned}
\varepsilon h_t-\mathcal{I}_s h+\frac{1}{\varepsilon^{2s}}W'(h)-\sigma&\geqslant\frac{\beta}{2}\varepsilon^{-2s}\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}-\varepsilon^{-2s}\varrho_\varepsilon\mu e^{-\frac{\mu t}{\varepsilon^{2s+1}}}+O(\varepsilon^{4s(1-\alpha)-2s})\\
&=\varepsilon^{-2s}\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\left(\frac{\beta}{2}-\mu+O(\varepsilon^{4s(1-\alpha)})\varrho_\varepsilon^{-1}e^{\frac{\mu t}{\varepsilon^{2s+1}}}\right)\\
&\geqslant\varepsilon^{-2s}\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\left(\frac{\beta}{2}-\mu-C\varepsilon^{4s(1-\alpha)-2s-\gamma}\right)\\
&=\varepsilon^{-2s}\varrho_\varepsilon e^{-\frac{\mu t}{\varepsilon^{2s+1}}}\left(\frac{\beta}{2}-\mu-o(1)\right)\\
&\geqslant 0,
\end{aligned}$$

if we fix μ independent of ε such that

$$\mu\leqslant\frac{\beta}{4},\tag{4.13}$$

and ε is small enough. The lemma is then proven choosing τ_ε, K_ε and μ satisfying respectively (4.6), (4.11) and (4.13), with α and γ satisfying respectively (4.4) and (4.5). □

Let us now conclude the proof of Theorem 1.2. From Theorem 1.1 we have

$$v_\varepsilon(T_\varepsilon^1,x)\leqslant h(0,x)\quad\text{for any } x\in\mathbb{R}.$$

Moreover, for μ, K_ε and τ_ε given by Lemma 4.1 and ε small enough, the function $h(t,x)$ is a supersolution of the equation (1.1). The comparison principle then implies

$$v_\varepsilon(T^1_\varepsilon + t, x) \leqslant h(t,x) \quad \text{for any } (t,x) \in (0,\tau_\varepsilon] \times \mathbb{R}.$$

Choosing $t = \tau_\varepsilon$ above, we get (1.17) with

$$\widetilde{\varrho}_\varepsilon := \varrho_\varepsilon e^{-\frac{\mu\tau_\varepsilon}{\varepsilon^{2s+1}}}$$

satisfying (1.15).

Finally, (1.16) is a consequence of (4.2) and (4.3).

5. Proof of Theorems 1.3 and 1.5

We perform a unique proof of Theorems 1.3 and 1.5. Let $N = 2K - l$, with either $l = 0$ (Theorem 1.3) or $0 \neq l \in \mathbb{N}$ (Theorem 1.5). First of all, notice that, given $x^0_1, \dots, x^0_N$, for any $(\zeta_1, \dots, \zeta_N) \in \{-1,1\}^N$ such that $\sum_{i=1}^N \zeta_i = l$, the initial datum v^0_ε, defined in (1.5), is below the function w^0_ε in which the positive particles are the first K and the negative ones the remaining last $K - l$, i.e., for any $x \in \mathbb{R}$,

$$v^0_\varepsilon(x) \leqslant w^0_\varepsilon(x) := \frac{\varepsilon^{2s}}{\beta}\sigma(0,x) + \sum_{i=1}^{K} u\left(\frac{x - x^0_i}{\varepsilon}\right) + \sum_{i=K+1}^{N} u\left(\frac{x^0_i - x}{\varepsilon}\right) - (N - K).$$

The comparison principle then implies,

$$v_\varepsilon(t,x) \leqslant w_\varepsilon(t,x) \quad \text{for any } (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{5.1}$$

where w_ε is the solution of (1.1) with initial datum w^0_ε. Therefore, when $l = 0$, to show that there exist $\mathcal{T}^K_\varepsilon$ and $\Lambda^K_\varepsilon = o(1)$ as $\varepsilon \to 0$ such that

$$v_\varepsilon(\mathcal{T}^K_\varepsilon, x) \leqslant \Lambda^K_\varepsilon \quad \text{for any } x \in \mathbb{R}, \tag{5.2}$$

it suffices to prove (5.2) for $w_\varepsilon(t,x)$. When $l \in \mathbb{N}$ it suffices to show (1.23) for $w_\varepsilon(t,x)$.

Hence, let us consider the solution $(x_1(t), \dots, x_N(t))$ of the ODE's system (1.8) with

$$\zeta_i = \begin{cases} 1 & \text{for } i = 1, \dots, K \\ -1 & \text{for } i = K+1, \dots, N. \end{cases}$$

As usual, let us denote, for $i = 1, \dots, N-1$,

$$\vartheta_i(t) := x_{i+1}(t) - x_i(t)$$

and

$$\vartheta^0_i := x^0_{i+1} - x^0_i.$$

Let us first assume $\sigma \equiv 0$. From Lemma 3.1, for any initial configuration of the particles, a collision between the particles x_K and x_{K+1} of system (1.8) occurs at a finite time, that we denote by T^1_c, satisfying

$$T^1_c \leqslant \frac{s(\vartheta^0_K)^{1+2s}}{(2s+1)\gamma}.$$

Then by Theorems 1.1 and 1.2, there exist $T^1_\varepsilon, o^1_\varepsilon > 0$ and $\widetilde{x}^{1,\varepsilon}_1, \dots, \widetilde{x}^{1,\varepsilon}_{K-1}, \widetilde{x}^{1,\varepsilon}_{K+2} \dots, \widetilde{x}^{1,\varepsilon}_N$, such that, for $i \in \{1, \dots, K-1, K+2, \dots, N\}$,

$$\widetilde{x}^{1,\varepsilon}_i = x_i(T_c) + o(1) \quad \text{as } \varepsilon \to 0$$

$$T^1_\varepsilon = T^1_c + o(1), \quad 0 < o^1_\varepsilon := \beta\frac{\widetilde{\varrho}_\varepsilon}{\varepsilon^{2s}} = o(1) \quad \text{as } \varepsilon \to 0,$$

and

$$w_\varepsilon(T_\varepsilon^1, x) \leqslant \frac{\varepsilon^{2s}}{\beta} o_\varepsilon^1 + \sum_{i=1}^{K-1} u\left(\frac{x - \widetilde{x}_i^{1,\varepsilon}}{\varepsilon}\right) + \sum_{i=K+2}^{N} u\left(\frac{\widetilde{x}_i^{1,\varepsilon} - x}{\varepsilon}\right) - (N - K - 1). \tag{5.3}$$

Now, let us denote by $w_\varepsilon^1(t,x)$ the solution of system (1.1), with $\sigma = o_\varepsilon^1$ and initial datum the right-hand side of (5.3). Then, from the comparison principle, we have, for any $(t,x) \in (0,+\infty) \times \mathbb{R}$,

$$w_\varepsilon(T_\varepsilon^1 + t, x) \leqslant w_\varepsilon^1(t,x). \tag{5.4}$$

From Lemma 3.1, for ε small enough, the collision time, that we denote by T_c^2, of the following ODE's system: for $i \in \{1, \dots, K-1, K+2, \dots, N\}$,

$$\begin{cases} \dot{x}_i = \gamma \left(\sum_{j \neq i} \zeta_i \zeta_j \frac{x_i - x_j}{2s|x_i - x_j|^{1+2s}} + o_\varepsilon^1 \right) & \text{in } (0, T_c^2) \\ x_i(0) = \widetilde{x}_i^{1,\varepsilon}, \end{cases} \tag{5.5}$$

where

$$\zeta_i = \begin{cases} 1 & \text{for } i = 1, \dots, K-1 \\ -1 & \text{for } i = K+2, \dots, N, \end{cases}$$

is finite. Therefore, by Theorems 1.1 and 1.2 and (5.4) , there exist $T_\varepsilon^2, o_\varepsilon^2 > 0$ and $\widetilde{x}_1^{2,\varepsilon}, \dots, \widetilde{x}_{K-2}^{2,\varepsilon}, \widetilde{x}_{K+3}^{2,\varepsilon} \dots, \widetilde{x}_N^{2,\varepsilon}$, such that, for $i \in \{1, \dots, K-2, K+3, \dots, N\}$, we have

$$\widetilde{x}_i^{2,\varepsilon} = x_i(T_c^2) + o(1) \quad \text{as } \varepsilon \to 0$$

where $(x_1, \dots, x_{K-2}, x_{K+3}, \dots, x_N)$ is the solution of (5.5),

$$T_\varepsilon^2 = T_c^2 + o(1), \quad o_\varepsilon^2 = o(1) \quad \text{as } \varepsilon \to 0,$$

and

$$w_\varepsilon(T_\varepsilon^1 + T_\varepsilon^2, x) \leqslant \frac{\varepsilon^{2s}}{\beta}(o_\varepsilon^1 + o_\varepsilon^2) + \sum_{i=1}^{K-2} u\left(\frac{x - \widetilde{x}_i^{2,\varepsilon}}{\varepsilon}\right) + \sum_{i=K+3}^{N} u\left(\frac{\widetilde{x}_i^{2,\varepsilon} - x}{\varepsilon}\right) - (N - K - 2).$$

Let us first assume $l = 0$. Then, repeating the argument, we see that, after K collisions, if we denote

$$\mathcal{T}_\varepsilon^K := T_\varepsilon^1 + \ldots + T_\varepsilon^K$$

and

$$\Lambda_\varepsilon^K := \frac{\varepsilon^{2s}}{\beta}(o_\varepsilon^1 + \ldots + o_\varepsilon^{K-1}) + \varrho_\varepsilon^K,$$

then, for any $x \in \mathbb{R}$,

$$w_\varepsilon(\mathcal{T}_\varepsilon^K, x) \leqslant \Lambda_\varepsilon^K.$$

The last estimate and (5.1) imply (5.2). Remark that Theorem 1.2 cannot be applied after the last collision, since there are only two remaining particles before the last collision occurs, therefore the hypothesis $N > 2$ of the theorem is not satisfied.

Similarly, when $l \in \mathbb{N}$, after $K - l$ collisions, if we denote

$$\mathcal{T}_\varepsilon^{K-l} := T_\varepsilon^1 + \ldots + T_\varepsilon^{K-l}$$

and

$$\Lambda_\varepsilon^{K-l} := \frac{\varepsilon^{2s}}{\beta}(o_\varepsilon^1 + \ldots + o_\varepsilon^{K-l}),$$

we get that $w_\varepsilon(t,x)$, and therefore by (5.1) $v_\varepsilon(t,x)$, satisfies inequality (1.23), with $\Lambda_\varepsilon^{K-l}$ satisfying (1.25). Differently from the previous case, when $l\in\mathbb{N}$, Theorem 1.2 can be applied after the last collision, since there are more than two remaining particles before the last collision occurs. To show (1.24) when $l\in\mathbb{N}$ and

$$v_\varepsilon(\mathcal{T}_\varepsilon^K,x)\geqslant -\Lambda_\varepsilon^K \quad \text{for any } x\in\mathbb{R}, \tag{5.6}$$

when $l=0$, we consider the function z_ε to be the solution of (1.1) with initial datum z_ε^0 in which the negative particles are now the first $K-l$ and the positive ones the remaining last K, i.e.,

$$z_\varepsilon^0(x):=\frac{\varepsilon^{2s}}{\beta}\sigma(0,x)+\sum_{i=1}^{K-l}u\left(\frac{x_i^0-x}{\varepsilon}\right)+\sum_{i=K-l+1}^{N}u\left(\frac{x-x_i^0}{\varepsilon}\right)-(N-K).$$

The comparison principle then implies

$$v_\varepsilon(t,x)\geqslant z_\varepsilon(t,x)\quad\text{for any } (t,x)\in(0,+\infty)\times\mathbb{R}.$$

A similar argument as before, then gives (1.24) when $l\in\mathbb{N}$ and (5.6) when $l=0$. This concludes the proof of Theorems 1.3 and 1.5 in the case $\sigma\equiv 0$.

The result for $\sigma\not\equiv 0$ such that $\|\sigma\|_\infty\leqslant\overline{\sigma}$ with $\overline{\sigma}$ small enough, follows from the case $\sigma\equiv 0$ and the continuity up to the collision time, of the solution of the ODE's system

$$\begin{cases}\dot{x}_i=\gamma\left(\displaystyle\sum_{j\neq i}\zeta_i\zeta_j\frac{x_i-x_j}{2s|x_i-x_j|^{1+2s}}-\zeta_i\delta\right) & \text{in } (0,T_c)\\ x_i(0)=\widetilde{x}_i^\varepsilon,\end{cases}$$

with respect to the parameter δ (Proposition 3.3).

6. Proof of Theorems 1.4 and 1.6

6.1. **Proof of Theorem 1.4.** The proof of Theorem 1.4 follows the same steps as in the proof of Theorem 1.2 in [14] and we only sketch it. Consider the function $h(\tau,\xi)$ which is solution of

$$\begin{cases}h_\tau+W'(h)=0, & \forall\tau\in(0,+\infty)\\ h(0,\xi)=\xi.\end{cases} \tag{6.1}$$

Then assumptions (1.2) and (1.20) imply that there exists $\varepsilon_0>0$ such that for any $\varepsilon<\varepsilon_0$, h satisfies: $h(\tau,0)\equiv 0$; if $\xi\in(0,\Lambda_\varepsilon^K]$, then

$$0<h(\tau,\xi)\leqslant\xi e^{-\frac{\beta}{2}\tau};$$

if $\xi\in[-\Lambda_\varepsilon^K,0)$, then

$$-\xi e^{-\frac{\beta}{2}\tau}\leqslant h(\tau,\xi)<0,$$

where $\beta=W''(0)>0$. Now, the function $\tilde{h}(t,x):=h(\frac{t-\mathcal{T}_\varepsilon^K}{\varepsilon^{2s+1}},\Lambda_\varepsilon^K)$, where $\mathcal{T}_\varepsilon^K$ is given by Theorem 1.3, is solution of the equation (1.1) for $\sigma\equiv 0$ and $t>\mathcal{T}_\varepsilon^K$, with $\tilde{h}(\mathcal{T}_\varepsilon^K,x)=\Lambda_\varepsilon^K$. Then, the comparison principle and estimate (1.19) imply

$$v_\varepsilon(t,x)\leqslant\tilde{h}(t,x)\leqslant\Lambda_\varepsilon^K e^{-\frac{\beta}{2}\frac{t-\mathcal{T}_\varepsilon^K}{\varepsilon^{2s+1}}}\quad\text{for any } x\in\mathbb{R},\ t>\mathcal{T}_\varepsilon^K.$$

Similarly, one can prove that

$$v_\varepsilon(t,x)\geqslant-\Lambda_\varepsilon^K e^{-\frac{\beta}{2}\frac{t-\mathcal{T}_\varepsilon^K}{\varepsilon^{2s+1}}}\quad\text{for any } x\in\mathbb{R},\ t>\mathcal{T}_\varepsilon^K,$$

and this proves (1.21).

6.2. **Proof of Theorem 1.6.** We start by proving a general result for the solution of the following system of ODE's:

$$\begin{cases} \dot{x}_i = \gamma \sum_{j\neq i} \frac{x_i - x_j}{2s|x_i - x_j|^{1+2s}} - \delta'(t) & \text{in } (0, T_c) \\ x_i(0) = x_i^0, \end{cases} \tag{6.2}$$

$i = 1, \dots, N$, where δ is a differentiable function.

Lemma 6.1. *Let $\delta : [0, +\infty) \to \mathbb{R}$ be differentiable in $(0, +\infty)$. Let $(x_1(t), \dots, x_N(t))$ be the solution of (6.2) with $x_{i+1}^0 - x_i^0 = \vartheta_0 > 0$, for any $i = 1, \dots, N-1$. Then there exists a constant k depending on N, γ, s and ϑ_0, such that for any $i = 1, \dots, N-1$, we have*

$$x_{i+1}(t) - x_i(t) \geqslant k(1+t)^{\frac{1}{1+2s}} \quad \textit{for any } t > 0. \tag{6.3}$$

Moreover, if $N = 2m$, $m \in \mathbb{N}$, then

$$x_{m+1}(t) + x_m(t) = x_{m+1}^0 + x_m^0 + 2\delta(0) - 2\delta(t) \quad \textit{for any } t > 0, \tag{6.4}$$

if instead, $N = 2m+1$, $m \in \mathbb{N}$, then

$$x_{m+1}(t) = x_{m+1}^0 + \delta(0) - \delta(t) \quad \textit{for any } t > 0. \tag{6.5}$$

In particular $T_c = +\infty$.

Proof. We perform the proof of the lemma in the case $N = 2m$, being the case $N = 2m+1$ similar. Let us first consider the case $\delta \equiv 0$. Since the system of ODE's in (6.2) is invariant under translations of particles, that is, $(x_1(t) + a, \dots, x_N(t) + a)$ is solution of the ODE's in (6.2), for any $a \in \mathbb{R}$, without loss of generality we may assume that the initial configuration of the particles is symmetric with respect to the origin. Therefore, suppose that, for $i = 1, \dots, m$,

$$x_{m+i}^0 = -x_{m-i+1}^0.$$

Then, the solution of (6.2) satisfies, for $i = 1, \dots, m$,

$$x_{m+i}(t) = -x(t)_{m-i+1}, \quad \text{for any } t > 0. \tag{6.6}$$

Indeed, let $(y_{m+1}(t), \dots, y_{2m}(t))$ be the solution of the following system: for $i = 1, \dots, m$

$$\begin{cases} \dot{y}_{m+i} = \gamma \left(\sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{y_{m+i} - y_{m+j}}{2s|y_{m+i} - y_{m+j}|^{1+2s}} + \sum_{j=1}^{m} \frac{y_{m+i} + y_{m+j}}{2s|y_{m+i} + y_{m+j}|^{1+2s}} \right) & \text{in } (0, T_c) \\ y_{m+i}(0) = x_{m+i}^0. \end{cases}$$

Set, for $i = 1, \dots, m$ and $t \geqslant 0$,

$$y_{m-i+1}(t) := -y_{m+i}(t).$$

Then $(y_1(t), \dots, y_N(t))$ is solution of (6.2) and by uniqueness it coincides with $(x_1(t), \dots, x_N(t))$. This implies that $(x_1(t), \dots, x_N(t))$ satisfies property (6.6). In particular (6.4) holds true. Next, denote

$$\vartheta_{j,i}(t) := x_j(t) - x_i(t).$$

In order to prove (6.3), we show that for $j = 1, \dots, m$, there exists $k_j > 0$ such that

$$\vartheta_{2m-j+1,j}(t) \geqslant k_j(1+t)^{\frac{1}{1+2s}}. \tag{6.7}$$

We prove (6.7) by induction. Let $j = 1$. From (6.2), we see that $\vartheta_{2m,1}(t)$ solves:

$$\dot{\vartheta}_{2m,1} = \gamma\left(\sum_{j=1}^{2m-1}\frac{1}{2s(x_{2m}-x_j)^{2s}} + \sum_{j=2}^{2m}\frac{1}{2s(x_j-x_1)^{2s}}\right) \geqslant \frac{\gamma}{s\vartheta_{2m,1}^{2s}}.$$

A solution of equation $\dot{\vartheta} = \frac{\gamma}{s\vartheta^{2s}}$ is given by $\vartheta(t) = \left((N-1)^{1+2s}\vartheta_0^{1+2s} + \frac{(2s+1)\gamma}{s}t\right)^{\frac{1}{2s+1}}$. Since in addition, $\vartheta(0) = (N-1)\vartheta_0 = \vartheta_{2m,1}(0)$, by comparison $\vartheta_{2m,1}(t) \geqslant \vartheta(t)$ for any $t > 0$. This implies (6.7) for $j = 1$, with $k_1 = \min\left\{(N-1)\vartheta_0, \left(\frac{(2s+1)\gamma}{s}\right)^{\frac{1}{2s+1}}\right\}$.

Now assume that (6.7) holds true for $j = 1, \dots, m-1$ and let us prove it for $j = m$. Remark that, from (6.6), we have, for $j = 1, \dots, m$,

$$\begin{aligned}\vartheta_{2m-j+1,j} &= x_{2m-j+1} - x_j = x_{2m-j+1} - x_{m+1} + \vartheta_{m+1,m} + x_m - x_j\\ &= 2(x_{2m-j+1} - x_{m+1}) + \vartheta_{m+1,m} = 2(x_m - x_j) + \vartheta_{m+1,m}.\end{aligned} \tag{6.8}$$

Therefore, from (6.2), we see that $\vartheta_{m+1,m}(t)$ solves:

$$\begin{aligned}\dot{\vartheta}_{m+1,m} = \frac{\gamma}{2s}&\left(\sum_{j=1}^{m}\frac{1}{(x_{m+1}-x_j)^{2s}} - \sum_{j=m+2}^{2m}\frac{1}{(x_j-x_{m+1})^{2s}}\right.\\ &\left.-\sum_{j=1}^{m-1}\frac{1}{(x_m-x_j)^{2s}} + \sum_{j=m+1}^{2m}\frac{1}{(x_j-x_m)^{2s}}\right)\\ \geqslant \frac{\gamma}{2s}&\left(\frac{2}{\vartheta_{m+1,m}^{2s}} - \sum_{j=1}^{m-1}\frac{1}{(x_{2m-j+1}-x_{m+1})^{2s}} - \sum_{j=1}^{m-1}\frac{1}{(x_m-x_j)^{2s}}\right).\end{aligned}$$

Now, since from (6.8) we have that

$$x_m - x_j = x_{2m-j+1} - x_{m+1} = \frac{\vartheta_{2m-j+1,j} - \vartheta_{m+1,m}}{2},$$

the last inequality can be rewritten as

$$\dot{\vartheta}_{m+1,m} \geqslant \frac{\gamma}{s}\left(\frac{1}{\vartheta_{m+1,m}^{2s}} - \sum_{j=1}^{m-1}\frac{2^{2s}}{(\vartheta_{2m-j+1,j} - \vartheta_{m+1,m})^{2s}}\right).$$

Then, using (6.7) for $j = 1, \dots, m-1$, from the previous inequalities we get

$$\dot{\vartheta}_{m+1,m} \geqslant \frac{\gamma}{s}\left(\frac{1}{\vartheta_{m+1,m}^{2s}} - \sum_{j=1}^{m-1}\frac{2^{2s}}{(k_j(1+t)^{\frac{1}{1+2s}} - \vartheta_{m+1,m}^{2s})^{2s}}\right).$$

Now, we consider the function $g(t) = k(1+t)^{\frac{1}{1+2s}}$ for some $0 < k < k_j$ to be determined. We have

$$\dot g - \frac{\gamma}{s}\left(\frac{1}{g^{2s}} - \sum_{j=1}^{m-1}\frac{2^{2s}}{(k_j(1+t)^{\frac{1}{1+2s}} - g)^{2s}}\right)$$
$$= (1+t)^{-\frac{2s}{1+2s}}\left(\frac{k}{1+2s} - \frac{\gamma}{s}\left(k^{-2s} - \sum_{j=1}^{m-1} 2^{2s}(k_j - k)^{-2s}\right)\right) \leqslant 0,$$

for $k > 0$ small enough. Therefore, there exists $k > 0$ such that g is subsolution of the equation

$$\dot\vartheta = \frac{\gamma}{s}\left(\frac{1}{\vartheta^{2s}} - \sum_{j=1}^{m-1}\frac{2^{2s}}{(k_j(1+t)^{\frac{1}{1+2s}} - \vartheta)^{2s}}\right).$$

Since in addition, for $k \leqslant \vartheta_0$, we have that $g(0) \leqslant \vartheta_{m+1,m}(0)$, by comparison we get $g(t) \leqslant \vartheta_{m+1,m}(t)$ for any $t > 0$, i.e., (6.7) for $j = m$, with $k_j = k$. This concludes the proof of (6.7). We are now ready to prove (6.3). From (6.6) it suffices to show (6.3) for $i = m, \dots, N-1$. We proceed by induction. Inequality (6.3) for $i = m$ is given by (6.7) for $j = m$. Assume now that (6.3) holds true for $i = m, \dots, N-2$. Then, from (6.2), we see that $\vartheta_{N,N-1}(t) = x_N(t) - x_{N-1}(t)$ solves:

$$\dot\vartheta_{N,N-1} = \frac{\gamma}{2s}\left(\frac{2}{\vartheta^{2s}_{N,N-1}} + \sum_{j=1}^{N-2}\frac{1}{(x_N - x_j)^{2s}} - \sum_{j=1}^{N-2}\frac{1}{(x_{N-1} - x_j)^{2s}}\right)$$
$$\geqslant \frac{\gamma}{s}\left(\frac{1}{\vartheta^{2s}_{N,N-1}} - \frac{C}{(1+t)^{\frac{2s}{2s+1}}}\right),$$

for some $C > 0$. Arguing as before, we get (6.3) for $i = N-1$ and this concludes the proof of the lemma when $\delta \equiv 0$. Now, let us consider the general case, when the assumption $\delta \equiv 0$ does not hold. Define $z_i(t) := x_i(t) + \delta(t)$, for $i = 1, \dots, N$. Then, $(z_1(t), \dots, z_N(t))$ is solution of the initial value problem (6.2) with $\delta \equiv 0$ and initial conditions $x_i^0 + \delta(0)$. Therefore, the results just proven in the case $\delta \equiv 0$ and applied to $(z_1(t), \dots, z_N(t))$, yield (6.3), (6.4) and (6.5) for $(x_1(t), \dots, x_N(t))$. This concludes the proof of the lemma for $N = 2m$.

When $N = 2m+1$, and $\delta \equiv 0$, again, without loss of generality, we may assume that the initial configuration of the particles is symmetric with respect to the origin, that is for $i = 1, \dots, m$,

$$x^0_{m+1+i} = -x^0_{m+1-i}.$$

Then, the solution of (6.2) satisfies, for $i = 1, \dots, m$,

$$x_{m+1+i}(t) = -x(t)_{m+1-i}, \quad \text{for any } t > 0,$$

and

$$x_{m+1}(t) = x^0_{m+1}, \quad \text{for any } t > 0.$$

Then, the proof proceeds like the case $N = 2m$. One proves by induction that for $j = 1, \dots, m$, there exists $k_j > 0$ such that

$$\vartheta_{2m+2-j,j}(t) \geqslant k_j(1+t)^{\frac{1}{1+2s}}.$$

Using these inequalities to estimate from below the derivatives of $\vartheta_{i+1,i}$ and proceeding again by induction, (6.3) follows. When $\delta \not\equiv 0$, then one applies the results just proven to the functions $z_i(t) := x_i(t) + \delta(t)$ to obtain (6.3) and (6.5) for $\delta \not\equiv 0$. $\square$

Let us now prove Theorem 1.6. In order to do it, we consider appropriate barriers for the solution v_ε of (1.1)-(1.5) with $\sigma \equiv 0$. Set

$$\vartheta_m := \min_{i=1,\dots,l-1} \underline{x}^\varepsilon_{i+1} - \underline{x}^\varepsilon_i$$

and

$$0 \leqslant \sigma_\varepsilon := \frac{\Lambda^{K-l}_\varepsilon}{\varepsilon^{2s}} = o(1) \quad \text{as } \varepsilon \to 0,$$

where $\underline{x}^\varepsilon_1, \dots, \underline{x}^\varepsilon_l$ and $\Lambda^{K-l}_\varepsilon$ are given by Theorem 1.5. Let $w_\varepsilon(t,x)$ be the solution of (1.1) with $\sigma \equiv 0$ and with the following initial condition

$$w_\varepsilon(0,x) = \sum_{i=1}^{l} u\left(\frac{x - \underline{y}^\varepsilon_i}{\varepsilon}\right) + \varepsilon^{2s}\sigma_\varepsilon,$$

where u is the solution of (1.4), and $\underline{y}^\varepsilon_1, \dots, \underline{y}^\varepsilon_l$ are defined as follows

$$\underline{y}^\varepsilon_1 := \underline{x}^\varepsilon_1, \qquad \underline{y}^\varepsilon_i := \underline{x}^\varepsilon_i + \vartheta_m, \text{ for } i = 2, \dots, l.$$

From (1.23) and the monotonicity of u, we have that $v_\varepsilon(\mathcal{T}^{K-l}_\varepsilon, x) \leqslant w_\varepsilon(0,x)$ for any $x \in \mathbb{R}$. Then by the comparison principle

$$v_\varepsilon(\mathcal{T}^{K-l}_\varepsilon + t, x) \leqslant w_\varepsilon(t,x) \quad \text{for any } (t,x) \in (0,+\infty) \times \mathbb{R}. \tag{6.9}$$

Now, we argue as in Section 3. Consider the function

$$\overline{w}_\varepsilon(t,x) := \varepsilon^{2s}\overline{\sigma}_\varepsilon + \sum_{i=1}^{l} u\left(\frac{x - x_i(t)}{\varepsilon}\right) - \sum_{i=1}^{l} \varepsilon^{2s} c_i(t)\psi\left(\frac{x - x_i(t)}{\varepsilon}\right) \tag{6.10}$$

where u and ψ are respectively the solution of (1.4) and (2.3), $(x_1(t), \dots, x_l(t))$ is the solution of (6.2) with

$$N = l, \quad \delta(t) = (1+2s)(\sigma_\varepsilon + \delta_\varepsilon)(1+t)^{\frac{1}{1+2s}} \quad \text{and } x^0_i = \underline{y}^\varepsilon_i - \delta_\varepsilon, \tag{6.11}$$

and where

$$\begin{aligned} &c_i(t) = \dot{x}_i(t), \\ &\overline{\sigma}_\varepsilon(t) = \frac{\delta'(t)}{W''(0)} = \frac{(\sigma_\varepsilon + \delta_\varepsilon)(1+t)^{-\frac{2s}{1+2s}}}{W''(0)}, \end{aligned} \tag{6.12}$$

and $\delta_\varepsilon = o(1)$ as $\varepsilon \to 0$ to be determined. We want to show that there exists δ_ε such that the function $\overline{w}_\varepsilon(t,x)$ is an upper barrier for $w_\varepsilon(t,x)$. By Lemma 3.6, we have that

$$w_\varepsilon(0,x) \leqslant \overline{w}_\varepsilon(0,x) \quad \text{for any } x \in \mathbb{R}. \tag{6.13}$$

Moreover, $\overline{w}_\varepsilon(t,x)$ is a supersolution of (1.1), as stated in the following proposition.

Proposition 6.2. *There exist $\varepsilon_0 > 0$ and $0 < \delta_\varepsilon = o(1)$ as $\varepsilon \to 0$, such that such that for any $\varepsilon < \varepsilon_0$, if $(x_1, \dots, x_l)$ is a solution of the ODE system in (6.2) where N and $\delta(t)$, are given by (6.11), then the function $\overline{w}_\varepsilon$ defined by (6.10) satisfies*

$$\varepsilon(\overline{w}_\varepsilon)_t - \mathcal{I}_s\overline{w}_\varepsilon + \frac{1}{\varepsilon^{2s}}W'(\overline{w}_\varepsilon) \geqslant 0 \tag{6.14}$$

for any $(t,x) \in (0,+\infty) \times \mathbb{R}$.

Proposition 6.2 generalizes Proposition 3.5 in the case in which the particles x_i's have all the same orientation. Indeed, thanks to Lemma 6.1, in the former proposition the error term δ', appearing in system (6.2), is allowed to go to 0 as $t \to +\infty$. The proof of Proposition 6.2 is a technical modification of the proof of Proposition 3.5 given in [14]. Therefore, we postpone it to the Appendix.

Now, let us choose δ_ε such that (6.13) and (6.14) hold. Then the comparison principle implies

$$w_\varepsilon(t,x) \leqslant \overline{w}_\varepsilon(t,x) \quad \text{for any } (t,x) \in (0,+\infty)\times\mathbb{R}. \tag{6.15}$$

Let us first consider the case $l = 2m$. By Lemma 6.1 applied with δ defined as in (6.11), we have that

$$\begin{aligned} x_{m+1}(t) &= \frac{x_{m+1}(t)}{2} + \frac{x_{m+1}(t)}{2} = \frac{x^0_{m+1}+x^0_m}{2} + \delta(0) - \delta(t) + \frac{1}{2}(x_{m+1}(t) - x_m(t)) \\ &\geqslant \frac{x^0_{m+1}+x^0_m}{2} + \left(\frac{k}{2} - (1+2s)(\sigma_\varepsilon + \delta_\varepsilon)\right)(1+t)^{\frac{1}{1+2s}} \\ &\geqslant \frac{x^0_{m+1}+x^0_m}{2} + \frac{k}{4}(1+t)^{\frac{1}{1+2s}}, \end{aligned}$$

for ε small enough. Similarly,

$$x_m(t) \leqslant \frac{x^0_{m+1}+x^0_m}{2} - \frac{k}{4}(t+1)^{\frac{1}{1+2s}},$$

for ε small enough. From the previous estimates and (6.3), we infer that, for any $R>0$ there exists $t_0>0$ such that if $|x| \leqslant R$, we have, for any $t>t_0$,

$$x_m(t) < x < x_{m+1}(t), \quad \text{and } |x - x_i(t)| \geqslant C(1+t)^{\frac{1}{2s+1}}, \quad \text{for any } i=1,\dots,l.$$

Therefore, from (2.1), we have

$$u\left(\frac{x-x_i(t)}{\varepsilon}\right) \leqslant \begin{cases} 1 + C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}} & \text{if } i=1,\dots,m \\ C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}} & \text{if } i=m+1,\dots,l. \end{cases}$$

Next, from (6.2) and (6.3), we see that

$$|c_i(t)| \leqslant C(1+t)^{-\frac{2s}{2s+1}}.$$

From the previous estimates, (6.9), (6.10) and (6.15), we conclude that, for $t>t_0$,

$$v_\varepsilon(\mathcal{T}^{K-l}_\varepsilon + t, x) \leqslant m + C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}}. \tag{6.16}$$

Similarly, choosing as lower barrier the function $z_\varepsilon(t,x)$ solution of (1.1) with $\sigma\equiv 0$ and initial condition

$$z_\varepsilon(0,x) = \sum_{i=1}^{l} u\left(\frac{x-\overline{y}^\varepsilon_i}{\varepsilon}\right) - \varepsilon^{2s}\sigma_\varepsilon,$$

where

$$\overline{y}^\varepsilon_1 := \overline{x}^\varepsilon_1, \qquad \overline{y}^\varepsilon_i := \overline{x}^\varepsilon_i + \vartheta_M, \text{ for } i=2,\dots,l,$$
$$\vartheta_M := \max_{i=1,\dots,l-1} \overline{x}^\varepsilon_{i+1} - \overline{x}^\varepsilon_i,$$

and $\overline{x}^\varepsilon_1,\dots,\overline{x}^\varepsilon_l$ are given by Theorem 1.5, we obtain, for $|x|<R$ and $t>t_0$,

$$v_\varepsilon(\mathcal{T}^{K-l}_\varepsilon + t, x) \geqslant m - C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}}. \tag{6.17}$$

Estimates (6.16) and (6.17) give (1.26).

Let us now turn to the case $l = 2m+1$. Fix $R > 0$ and let $x \in \mathbb{R}$ such that $|x| \leqslant R$. Then, as before, from (6.3) and (6.5), we infer that exist $t_0 > 0$ and a constant $C > 0$ such that for any $t > t_0$,

$$x_m(t) < x < x_{m+2}(t), \quad |x - x_i(t)| \geqslant C(1+t)^{\frac{1}{2s+1}},\ i \neq m+1$$

and for any $t > 0$

$$x_{m+1}(t) = \underline{y}^{\varepsilon}_{m+1} + \delta(0) - \delta(t) = \underline{y}^{\varepsilon}_{m+1} - (1+2s)(\sigma_\varepsilon + \delta_\varepsilon)[(1+t)^{\frac{1}{1+2s}} - 1] = \underline{x}^{\varepsilon} - \alpha_\varepsilon[(1+t)^{\frac{1}{1+2s}} - 1],$$

where

$$\underline{x}^{\varepsilon} := \underline{y}^{\varepsilon}_{m+1},$$

and

$$\alpha_\varepsilon := (1+2s)(\sigma_\varepsilon + \delta_\varepsilon) = o(1) \quad \text{as } \varepsilon \to 0.$$

We remark that from Theorem 1.5, $\underline{x}^{\varepsilon}$ is bounded with respect to ε. Therefore, from (2.1), we have

$$u\left(\frac{x - x_i(t)}{\varepsilon}\right) \leqslant \begin{cases} 1 + C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}} & \text{if } i = 1, \dots, m \\ C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}} & \text{if } i = m+2, \dots, l. \end{cases}$$

Moreover

$$u\left(\frac{x - x_{m+1}(t)}{\varepsilon}\right) = u\left(\frac{x - \underline{x}^{\varepsilon} + \alpha_\varepsilon[(1+t)^{\frac{1}{1+2s}} - 1]}{\varepsilon}\right)$$

and from (6.2) and (6.3), we see that

$$|c_i(t)| \leqslant C(1+t)^{-\frac{2s}{2s+1}}.$$

As before, the previous estimates, (6.9) and (6.15), imply (1.28). Similarly one can prove (1.27). This concludes the proof of Theorem 1.6.

7. Proof of Corollary 1.7

We argue by contradiction and suppose that there exists a constant solution

$$(x_1(t), \dots, x_N(t)) = (x_1^0, \dots, x_N^0) \tag{7.1}$$

of (1.8) with $\sigma \equiv 0$ and $N \geqslant 2$. Without loss of generality, we suppose that the number of the positive ζ_i's, K, is larger or equal than the number of the negative ones, $N - K$.

Let $R > 0$ be such that $|x_i^0|$, $|\underline{x}_\varepsilon|$, $|\overline{x}_\varepsilon| < R$, for any $i = 1, \dots, N$ and $\varepsilon > 0$, where $\underline{x}_\varepsilon$ and $\overline{x}_\varepsilon$ are given by Theorem 1.6. Pick any point $p < \min\{x_1^0, \underline{x}_\varepsilon\}$ with $|p| < 2R$. Then by (1.26), (1.27) and (1.28), there exists $T_0 > 0$ such that for any $t > T_0$, we have

$$\lim_{\varepsilon \to 0} v_\varepsilon(t, p) = m.$$

On the other hand, since $p < x_1^0$, by Theorem 1.1 in [12] and (7.1), we have that

$$\lim_{\varepsilon \to 0} v_\varepsilon(t, p) = \sum_{i=1}^{N} H(\zeta_i(p - x_i^0)) - (N - K) = 0,$$

where H is the Heaviside function. Therefore, we must have $m = 0$.

Next, we fix $N+1$ points, say $p_1, \dots, p_{N+1}$, with $|p_i| < 2R$ for any $p = 1, \dots, N+1$, such that

$$p_1 < x_1^0 < p_2 < x_2^0 < \dots < x_N^0 < p_{N+1} \tag{7.2}$$

and we denote $P := \{p_1, \ldots, p_{N+1}\}$. By Theorem 1.1 in [12] and (7.1), we have that, for any $p \in P$, and $t > 0$,

$$\lim_{\varepsilon \to 0} v_\varepsilon(t,p) = \sum_{i=1}^{N} H(\zeta_i(p - x_i^0)) - (N-K). \tag{7.3}$$

We remark that the right hand side of (7.3) is the superposition of N Heaviside functions (up to a vertical translation). Accordingly, the values taken by the right hand side of (7.3) have N jumps of size 1 when $p \in P$ (recall (7.2)).

On the other hand, when $l = K - (N-K) = 0$, by (1.26), for any $t > T_0$ and $p \in P$, we have

$$\lim_{\varepsilon \to 0} v_\varepsilon(t,p) = 0$$

which is a contradiction.

When $l = 1$, by (1.27) and (1.28), we must have that, for any $p \in P$,

$$\sum_{i=1}^{N} H(\zeta_i(p - x_i^0)) - (N-K) \in \{0,1\},$$

which means that the particles $(x_1^0, \ldots, x_N^0)$ must have alternate orientation. This is in contradiction with Theorem 1.6 of [12] which states that in the case of alternate dislocations, when $\sigma \equiv 0$, for any initial configuration there is always a collision in finite time, in particular system (1.8) does not admit stationary solutions.

Corollary 1.7 is then proven.

Appendix. Proof of Proposition 6.2

In order to simplify the notation, we set, for $i = 1, \ldots, N$

$$\tilde{u}_i(t,x) := u\left(\frac{x - x_i(t)}{\varepsilon}\right) - H\left(\frac{x - x_i(t)}{\varepsilon}\right), \tag{7.4}$$

where H is the Heaviside function and

$$\psi_i(t,x) := \psi\left(\frac{x - x_i(t)}{\varepsilon}\right).$$

Finally, let

$$I_\varepsilon := \varepsilon(\overline{w}_\varepsilon)_t + \frac{1}{\varepsilon^{2s}} W'(\overline{w}_\varepsilon) - \mathcal{I}_s \overline{w}_\varepsilon. \tag{7.5}$$

We want to find δ_ε such that $I_\varepsilon \geqslant 0$. To do it, we need the following result, which is proven in [14].

Lemma 7.1 (Lemma 8.1 in [14])**.** *For any $(t,x) \in (0,+\infty) \times \mathbb{R}$ we have, for $i = 1, \ldots, N$*

$$\begin{aligned} I_\varepsilon = {} & O(\tilde{u}_i)(\varepsilon^{-2s} \sum_{j \neq i} \tilde{u}_j + \overline{\sigma}_\varepsilon + c_i \eta) + \frac{\delta'}{\gamma} \\ & + \sum_{j=1}^{N} \left\{ O(\varepsilon^{2s+1} \dot{c}_j) + O(\varepsilon^{2s} c_j^2) \right\} \\ & + \sum_{j \neq i} \left\{ O(c_j \psi_j) + O(c_j \tilde{u}_j) + O(\varepsilon^{-2s} \tilde{u}_j^2) \right\} + O(\varepsilon^{2s}), \end{aligned} \tag{7.6}$$

where η and γ are given respectively by (2.4) *and* (1.9).

Let us proceed with the proof of Proposition 6.2. We consider two cases.

Case 1. Suppose that x is close to $x_i(t)$ more than ε^α, for some $i = 1, \dots, N$:

$$|x - x_i(t)| \leqslant \varepsilon^\alpha \quad \text{with } 0 < \alpha < 1. \tag{7.7}$$

Then, from (6.3), for $j \neq i$,

$$|x - x_j(t)| \geqslant C(1+t)^{\frac{1}{1+2s}}. \tag{7.8}$$

Here and in what follows we denote by $C > 0$ several constants independent of ε. Hence, from (2.1), (7.4) and (7.8), we get

$$\begin{aligned}
&\left| \frac{\tilde{u}_j(t,x)}{\varepsilon^{2s}} + \frac{1}{2sW''(0)} \frac{x - x_j(t)}{|x - x_j(t)|^{1+2s}} \right| \\
&= \frac{1}{\varepsilon^{2s}} \left| u\left(\frac{x - x_j(t)}{\varepsilon}\right) - H\left(\frac{x - x_j(t)}{\varepsilon}\right) + \frac{\varepsilon^{2s}}{2sW''(0)} \frac{x - x_j(t)}{|x - x_j(t)|^{1+2s}} \right| \\
&\leqslant C \frac{\varepsilon^\kappa}{\varepsilon^{2s}} \frac{1}{|x - x_j(t)|^\kappa} \\
&\leqslant C\varepsilon^{\kappa - 2s}(1+t)^{-\frac{k}{1+2s}},
\end{aligned}$$

where $\kappa > 2s$ is given in Lemma 2.1. Next, a Taylor expansion of the function $\dfrac{x - x_j(t)}{|x - x_j(t)|^{1+2s}}$ around $x_i(t)$, gives

$$\left| \frac{x - x_j(t)}{|x - x_j(t)|^{1+2s}} - \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^{1+2s}} \right| \leqslant \frac{2s}{|\xi - x_j(t)|^{1+2s}} |x - x_i(t)| \leqslant C\varepsilon^\alpha (1+t)^{-1},$$

where ξ is a suitable point lying on the segment joining x to $x_i(t)$. The last two inequalities imply for $j \neq i$

$$\left| \frac{\tilde{u}_j(t,x)}{\varepsilon^{2s}} + \frac{1}{2sW''(0)} \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^{1+2s}} \right| \leqslant C(\varepsilon^{\kappa-2s}(1+t)^{-\frac{k}{1+2s}} + \varepsilon^\alpha (1+t)^{-1}). \tag{7.9}$$

Therefore, from (7.6), we get that

$$\begin{aligned}
I_\varepsilon &= O(\tilde{u}_i) \left(\sum_{j \neq i} -\frac{1}{2sW''(0)} \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^{1+2s}} + \overline{\sigma}_\varepsilon + c_i \eta \right) + \frac{\delta'}{\gamma} \\
&\quad + C\varepsilon^{\kappa-2s}(1+t)^{-\frac{k}{1+2s}} + C\varepsilon^\alpha (1+t)^{-1} \\
&\quad + \sum_{j=1}^{N} \left\{ O(\varepsilon^{2s+1}\dot{c}_j) + O(\varepsilon^{2s} c_j^2) \right\} \\
&\quad + \sum_{j \neq i} \left\{ O(c_j \psi_j) + O(c_j \tilde{u}_j) + O(\varepsilon^{-2s} \tilde{u}_j^2) \right\}.
\end{aligned} \tag{7.10}$$

Now, from (6.12), the definition of η given in (2.4) ($\eta = \frac{1}{\gamma W''(0)}$) and (6.2), we see that

$$\sum_{j \neq i} -\frac{1}{2sW''(0)} \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^{1+2s}} + \overline{\sigma}_\varepsilon + c_i \eta = 0. \tag{7.11}$$

Let us next estimate the remaining terms in (7.10). From the definition of $c_i(t)$ given in (6.12), system (6.2) and estimates (6.3), we have for $j = 1, \dots, N$

$$|c_j| = O((1+t)^{-\frac{2s}{1+2s}}), \tag{7.12}$$

then

$$O(\varepsilon^{2s} c_j^2) = O(\varepsilon^{2s}(1+t)^{-\frac{4s}{1+2s}}). \tag{7.13}$$

Next, differentiating the equations in (6.2) and using (6.3), we get

$$\begin{aligned}
\dot{c}_i &= \gamma \left(-\sum_{j\neq i} \frac{\dot{x}_i - \dot{x}_j}{|x_i - x_j|^{2s+1}} - \delta''(t) \right) \\
&= -\gamma^2 \sum_{j\neq i} |x_i - x_j|^{-2s-1} \left(\sum_{k\neq i} \frac{x_i - x_k}{2s|x_i - x_k|^{1+2s}} - \sum_{l\neq j} \frac{x_j - x_l}{2s|x_j - x_l|^{1+2s}} \right) - \gamma\delta''(t) \\
&= O((1+t)^{-\frac{4s+1}{2s+1}}).
\end{aligned}$$

Then

$$O(\varepsilon^{2s+1} \dot{c}_j) = O(\varepsilon^{2s+1}(1+t)^{-\frac{4s+1}{2s+1}}). \tag{7.14}$$

Next, from (2.1) and (7.8), we have for $j \neq i$

$$|\tilde{u}_j| \leqslant C\varepsilon^{2s}|x - x_j|^{-2s} \leqslant C\varepsilon^{2s}(1+t)^{-\frac{2s}{2s+1}} \tag{7.15}$$

then using (7.12), we get for $j \neq i$

$$O(c_j \tilde{u}_j) = O(\varepsilon^{2s}(1+t)^{-\frac{4s}{2s+1}}), \tag{7.16}$$

and

$$O(\varepsilon^{-2s} \tilde{u}_j^2) = O(\varepsilon^{2s}(1+t)^{-\frac{4s}{2s+1}}). \tag{7.17}$$

Next, from (2.5) we know that for $|x| \geqslant \varepsilon^{-1}C(1+t)^{\frac{1}{1+2s}}$

$$|\psi(x)| \leqslant \left| \psi\left(\varepsilon^{-1} C (1+t)^{\frac{1}{1+2s}} \right) \right| + C\varepsilon^{2s}(1+t)^{-\frac{2s}{1+2s}}.$$

Therefore, from (7.8) and (7.12) we get

$$O(c_j \psi_j) = O\left((1+t)^{-\frac{2s}{1+2s}} \psi\left(\varepsilon^{-1} C (1+t)^{\frac{1}{1+2s}} \right) \right) + O(\varepsilon^{2s}(1+t)^{-\frac{4s}{1+2s}}). \tag{7.18}$$

Let us choose δ_ε such that

$$\varepsilon^\alpha,\ \varepsilon^{2s},\ \psi(\varepsilon^{-1}),\ \varepsilon^{\kappa-2s} = o(\delta_\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{7.19}$$

Then, from (7.10), (7.11), (7.13), (7.14), (7.16), (7.17), (7.18), (7.19) and the definition of δ given in (6.11), we obtain

$$\begin{aligned}
I_\varepsilon &= o(\delta_\varepsilon)(1+t)^{-\frac{2s}{1+2s}} + \frac{1+2s}{\gamma}(\sigma_\varepsilon + \delta_\varepsilon)(1+t)^{-\frac{2s}{1+2s}} \\
&\geqslant o(\delta_\varepsilon)(1+t)^{-\frac{2s}{1+2s}} + \frac{1+2s}{\gamma}\delta_\varepsilon(1+t)^{-\frac{2s}{1+2s}}.
\end{aligned} \tag{7.20}$$

being $\sigma_\varepsilon \geqslant 0$.

Case 2. Suppose that for any $i = 1, \dots, N$ we have

$$|x - x_i(t)| \geqslant \varepsilon^\alpha.$$

If $x_i(t)$ is the closest particle to x, then from (6.3), for $j \neq i$, we have that

$$|x - x_j(t)| \geqslant C(1+t)^{1+2s}.$$

Then estimates (7.12), (7.13), (7.14), (7.15), (7.16), (7.17) and (7.18) hold. Moreover, using (2.1), we have

$$|\tilde{u}_i| \leqslant C\varepsilon^{2s}|x - x_i|^{-2s} \leqslant C\varepsilon^{2s(1-\alpha)},$$

and as a consequence, using in addition (7.15), for $j \neq i$

$$O(\tilde{u}_i)(\varepsilon^{-2s}\tilde{u}_j) = O(\varepsilon^{2s(1-\alpha)}(1+t)^{-\frac{2s}{1+2s}}).$$

Finally from (7.12), we have

$$O(\tilde{u}_i)c_i = O(\varepsilon^{2s(1-\alpha)}(1+t)^{-\frac{2s}{1+2s}}).$$

Then, if in addition to (7.19), we choose δ_ε such that

$$\varepsilon^{2s(1-\alpha)} = o(\delta_\varepsilon) \quad \text{as } \varepsilon \to 0,$$

from (7.6), we obtain again (7.20).

Now, in both cases, from (7.20), for ε small enough we obtain that

$$I_\varepsilon \geqslant 0$$

and the proposition is proven.

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