MULTIPLE INTERPHASES FOR FRACTIONAL ALLEN-CAHN EQUATION

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1. Introduction

In this paper, we study a nonlocal, reaction-diffusion equation that arrises naturally in the Peierls–Nabarro model for atomic dislocations in crystals. Our initial configuration corresponds to multiple *loop dislocations* with the same orientation. After suitably rescaling the problem from the microscopic scale to the mesoscopic scale, we show that the dislocation loops move independently, according to their mean curvature.

The evolution of *edge dislocations* has been well-studied in the literature, that is, when the dislocations are straight, parallel lines. See [7] for an excellent overview of the subject. In this special setting, the Peierls–Nabarro model reduces to a one-dimensional PDE. We are the first to study the dynamics of dislocation curves that are not edge dislocations and thus the physical model can only be reduced to a *two-dimensional* PDE.

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To be more precise, we are interested in the nonlocal, reaction-diffusion equation

$$\varepsilon\partial_t u^\varepsilon = \frac{1}{\varepsilon\,|\ln\varepsilon|}(\varepsilon\mathcal{I}_n u^\varepsilon - W'(u^\varepsilon)) \quad \text{in } \mathbb{R}^n,\ n \geq 2, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $\mathcal{I}_n$ denotes the fractional Laplacian of order 1 in $\mathbb{R}^n$, and W is a multi-well potential. The nonlocal operator $\mathcal{I}_n$ is given by

$$\mathcal{I}_n u(x) = \mathrm{P.\,V.}\int_{\mathbb{R}^n} (u(x+y) - u(x))\,\frac{dy}{|y|^{n+1}}\,dy$$

where P. V. indicated that the integral is taken in the principal value sense. Up to a multiplicative constant, it can be shown that $\mathcal{I}_n$ satisfies the Fourier transform identity $\widehat{\mathcal{I}_n u}(\xi) = |\xi|\,\widehat{u}(\xi)$, $\xi \in \mathbb{R}^n$. For more further backgound on fractional Laplacians, see for example [6, 14]. Regarding the potential W, we assume that

$$\begin{cases} W \in C^{2,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1 \\ W(u+1) = W(u) & \text{for any } u \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R}\setminus\mathbb{Z} \\ W''(0) > 0. \end{cases}$$

We let u^ε be the solution to (1.1) when the initial condition u_0^ε is a superposition of layer solutions. The layer solution (also called the phase transition) $\phi : \mathbb{R} \to [0,1]$ is the unique solution to the standing wave equation

$$\begin{cases} C_n\mathcal{I}_1[\phi] = W'(\phi) & \text{in } \mathbb{R} \\ \dot{\phi} > 0 & \text{in } \mathbb{R} \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \quad \phi(0) = \frac{1}{2}, \end{cases} \tag{1.2}$$

where $\mathcal{I}_1$ denotes the $1/2$-Laplacian in $\mathbb{R}$ and where the constant $C_n > 0$ (given explicitly in (3.4)) depends only on $n \geq 2$. Further discussion on ϕ will be presented in Section 3.

Let $(\Omega_0^i)_{i=1}^N$ be a sequence of open subsets of $\mathbb{R}^n$ that are both smooth and bounded and that satisfy $\Omega_0^i \subset\subset \Omega_0^{i+1}$. The corresponding boundaries $\Gamma_0^i = \partial\Omega_0^i$ can be understood as the initial dislocation loops in the crystal, see Figure Let $d_i(t,x)$ be the signed distance function associated to Ω_0^i, $i = 1,\dots,N$, given by

$$d_i(x) = \begin{cases} d(x,\Gamma_0^i) & \text{if } x \in \Omega_0^i \\ -d(x,\Gamma_0^i) & \text{otherwise.} \end{cases} \tag{1.3}$$

For our initial condition to be well-prepared, we let u_0^ε be N-fold sum of the layer solutions $\phi(d_i(x)/\varepsilon)$, see Figure

We will show that the dislocation curves $(\Gamma_t^i)_{t\geq 0}$ move according to their mean curvature. Roughly speaking, Γ_t^i is the zero level set of a solution u^i to the mean curvature equation whose initial zero level set is precisely Γ_0^i. Then, we say that $({}^+\Omega_t^i, \Gamma_t^i, {}^-\Omega_t^i)$ denotes the level-set evolution of $(\Omega_0^i, \Gamma_0^i, (\overline{\Omega_0^i})^c)$ where ${}^+\Omega_t^i$ and ${}^-\Omega_t^i$ are the positivity and negativity sets of u^i respectively. See Section 2 for precise definitions and details.

We now present the main result of our paper.

Theorem 1.1. *Let $u^\varepsilon = u^\varepsilon(t,x)$ be the unique solution of the reaction diffusion equation (1.1) with the initial datum $u_0^\varepsilon : \mathbb{R}^n \to [0,N]$ defined by*

$$u_0^\varepsilon(x) = \sum_{i=1}^N \phi\left(\frac{d_i(x)}{\varepsilon}\right).$$

Then, as $\varepsilon \to 0$, the solutions u^ε satisfy

$$\begin{cases} u^\varepsilon \to N & \text{in } {}^+\Omega_t^N, \\ u^\varepsilon \to i & \text{in } {}^+\Omega_t^i \cap {}^-\Omega_t^{i+1}, \quad i = 1, \dots, N-1, \\ u^\varepsilon \to 0 & \text{in } {}^-\Omega_t^1, \end{cases}$$

where $({}^+\Omega_t^i, \Gamma_t^i, {}^-\Omega_t^i)$ denotes the level-set evolution of $(\Omega_0^i, \Gamma_0^i, (\overline{\Omega_0^i})^c)$.

Our result says that u^ε converges to integers between the dislocation curves, see Figure However, due to the degeneracy of the mean curvature equation, the sets Γ_t^i might develop interior and, *a priori*, we cannot say exactly where the jump occurs. More precisely, we say that Γ_t^i does not develop interior if and only if $\Gamma_t^i = \partial({}^+\Omega_t^i) = \partial({}^-\Omega_t^i)$. In this case, the limiting function in Theorem 1.1 satisfies

$$\lim_{\varepsilon\to 0} u^\varepsilon = \frac{N}{2} + \frac{1}{2}\sum_{i=1}^N \left(\mathbb{1}_{{}^+\Omega_t^i} - \mathbb{1}_{(\overline{{}^+\Omega_t^i})^c}\right) \quad \text{in } (0,\infty)\times\mathbb{R}^n \setminus \bigcup_{i=1}^N \Gamma_i.$$

Theorem 1.1 for $N = 1$ was studied by Imbert–Souganidis in the preprint [11]. When $N > 1$, the nonlinearity of the potential W plays more of a role. One of the key tools for the proof is the construction of strict sub/super solutions of the form

$$v^\varepsilon(t,x) \simeq \sum_{i=1}^N \phi\left(\frac{d_i(t,x)}{\varepsilon}\right),$$

where d_i is the signed distance function associated to Γ_t^i. A formal argument for this choice of barrier is presented in Section 4. The difficulty arrises in understanding $v^\varepsilon(t,x)$ when (t,x) is far from the front Γ_t^i since the signed distance function is not smooth at such points. In [11], they use v^ε to interpolate between 0 (outside the curve) and 1 (inside the curve). We found their method to be insufficient when $N > 1$. Instead, we replace d_i with a smooth extension of the signed distance function away from the curve. Then, we are able to use the asymptotic properties of ϕ to show that v^ε is indeed a sub/super solution (see Section 5).

1.1. **The Peierls–Nabarro model for loop dislocations.**

1.2. **Organization of paper.**

The rest of the paper is organized as follows. First, in Section 2, we provide the necessary background pertaining to motion by mean curvature. Section 3 contains preliminary results on the phase transition ϕ and other auxiliary results needed for the rest of the paper. Then, in Section 4, we provide heuristics for the proof of Theorem 1.1 and for the choice of barrier. The construction of barriers is presented in Section 5. Section 6 contains the proof of Theorem 1.1. Lastly, the proofs of some auxiliary lemmas are given in Section 7.

2. Motion by mean curvature

In this section, we introduce the geometric motions of the fronts. For a smooth function $u = u(t,x)$, consider the sets

$$\begin{aligned} {}^{+}\Omega &= \{(t,x) : u(x,t) > 0\} \\ \Gamma &= \{(t,x) : u(x,t) = 0\} \\ {}^{-}\Omega &= \{(t,x) : u(x,t) < 0\}. \end{aligned}$$

Denote the slices in t of ${}^{+}\Omega$ by

$${}^{+}\Omega_t = {}^{+}\Omega \cap (\{t\} \times \mathbb{R}^n).$$

and similarly for Γ_t and ${}^{-}\Omega_t$. Together, these form a set of triples $({}^{+}\Omega_t, \Gamma_t, {}^{-}\Omega_t)_{t\geq 0}$.

Let $d(t,x)$ denote the signed distance function associated to Γ_t:

$$d(t,x) = \begin{cases} d(x,\Gamma_t) & \text{for } x \in \Gamma_t \cup {}^{+}\Omega_t \\ -d(x,\Gamma_t) & \text{for } x \in {}^{-}\Omega_t. \end{cases}$$

Then, as theorized by Osher–Sethian [12] and justified by Evans–Spruck in [8] for viscosity solution, the zero level sets $(\Gamma_t)_{t>0}$ of u move with normal velocity

$$v(t,x,d(x,t)) = -\mu \Delta d(x,t), \quad \mu > 0,$$

if and only if u is a solution to the following nonlinear, degenerate equation

$$\partial_t u = \mu \operatorname{tr}\left((I - \widehat{\nabla u} \otimes \widehat{\nabla u}) D^2 u\right), \tag{2.1}$$

where $\hat{p} = p/|p|$ for $p \in \mathbb{R}^n$ and $\otimes$ denotes the tensor product. That is, the zero level sets of u move according to their mean curvature if and only if u is a solution to the *mean curvature equation* given in (2.1). In fact, the mean curvature equation is a *geometric* equation, so if u solves (2.1), then so does $\Phi(u)$ for any smooth function $\Phi : \mathbb{R} \to \mathbb{R}$. Consequently, u is a solution to the mean curvature equation if and only if every level set of u moves by mean curvature.

For a bounded, open set $\Omega_0 \subset \mathbb{R}^n$, consider the triplet $(\Omega_0, \Gamma_0, (\overline{\Omega_0})^c)$ where $\Gamma_0 = \partial\Omega_0$. Let $u_0(x)$ be such that

$$\Omega_0 = \{x : u_0(x) > 0\} \quad \text{and} \quad \Gamma_0 = \{x : u_0(x) = 0\}.$$

If u is a solution to (2.1) with initial data $u(0,x) = u_0(x)$, then the zero level sets of u move according to their mean curvature and we say that $({}^{+}\Omega_t, \Gamma_t, {}^{-}\Omega_t)_{t\geq 0}$ denotes the *level set evolution* of $(\Omega_0, \Gamma_0, (\overline{\Omega_0})^c)$. Under certain conditions on Ω_0 (such as smooth and convex), the sets $(\Gamma_t)_{t\geq 0}$ do not develop interior, that is, $\Gamma_t = \partial({}^{+}\Omega_t) = \partial({}^{-}\Omega_t)$. However, this is not true in general due to the degeneracy of the mean curvature equation which is why we cannot say anything about the jump sets Γ_t in Theorem 1.1.

Consider the special case in which the curves Γ_t are smooth and do not develop interior for some time. Then, the signed distance function d is smooth and satisfies $|\nabla d| = 1$ in a neighborhood of Γ_t and is a solution to (2.1) on Γ_t. Moreover, as a consequence of the strong maximum principle (see

2.0.1. *Weak solutions.* Due to the underlying geometry of Theorem 1.1, it is helpful to pass the notion of viscosity solutions of the PDE (2.1) to weak solutions of the level sets of the solution u. We use the notion of generalized flows for the mean curvature equation presented in [11]. Let $F(p,X)$ be given by

$$F(p,X) = -\mu \operatorname{tr}\left((I - \widehat{p} \otimes \widehat{p})X\right)$$

and the lower and upper semi-continuous envelopes of F be denoted by F_* and F^* respectively.

Definition 2.1. A family $(\Omega_t)_{t>0}$ of open (closed) subsets of $\mathbb{R}^n$ is a generalized super-flow (sub-flow) of the mean curvature equation (2.1) if for all $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, $h > 0$, and for all smooth functions $\varphi : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ such that

(i) (Boundedness) There exists $r > 0$ such that
$$\{(t, x) \in [t_0, t_0 + h] \times \mathbb{R}^n : \varphi(t, x) \geq 0\} \subset [t_0, t_0 + h] \times B(x_0, r),$$
(ii) (Strict subsolution) There exists $\delta = \delta(\varphi) > 0$ such that
$$\partial_t \varphi + F^*(D\varphi, D^2\varphi) \leq -\delta \quad \text{in } [t_0, t_0 + h] \times \overline{B}(x_0, r),$$
$$(\partial_t \varphi + F_*(D\varphi, D^2\varphi) \geq \delta)$$
(iii) (Non-degeneracy)
$$D\varphi \neq 0 \quad \text{in } \{(t, x) \in [t_0, t_0 + h] \times \overline{B}(x_0, r) : \varphi(t, x) = 0\},$$
(iv) (Initial condition)
$$\{x \in \overline{B}(x_0, r) : \varphi(t_0, x) \geq 0\} \subset \Omega_{t_0}$$
$$(\{x \in \overline{B}(x_0, r) : \varphi(t_0, x) \leq 0\} \subset \mathbb{R}^n \setminus \Omega_{t_0}),$$

then
$$\{x \in \bar{B}(x_0, r) : \varphi(t_0 + h, x) > 0\} \subset \Omega_{t_0+h}$$
$$(\{x \in \bar{B}(x_0, r) : \varphi(t_0 + h, x) < 0\} \subset \mathbb{R}^n \setminus \Omega_{t_0+h}).$$

For the interested reader, we remark that $(\Omega_t)_{t\geq 0}$ is a generalized super-flow (sub-flow) of (2.1) if and only if $\mathbb{1}_{\Omega_t} - \mathbb{1}_{(\overline{\Omega}_t)^c}$ is a viscosity super (sub) solution of (2.1), see [2, Theorem 2.4]. For an introduction and background on viscosity solutions, see for example [4].

3. The phase transition, the corrector, and the auxiliary functions

In this section, we will introduce the phase transition ϕ and the corrector ψ. Along the way, we will also define the auxiliary functions a_ε and $\bar{a}_\varepsilon$ and exhibit their relationship with fractional Laplacians and the mean curvature equation, respectively.

3.1. **The phase transition ϕ.** Let ϕ be the solution to the standing wave equation (1.2). In [3], they proved existence and uniqueness of the solution ϕ. Asymptotics on the decay of ϕ were established in [13] with finer estimates in [7, 10]. We summarize their results in the next lemma. For convenience in the notation, let c_0 and α be given respectively by

$$c_0^{-1} = \int_{\mathbb{R}} [\dot{\phi}(\xi)]^2 \, d\xi \quad \text{and} \quad \alpha = \frac{W''(0)}{C_n}. \tag{3.1}$$

Lemma 3.1. *There is a unique solution $\phi \in C^{2,\beta}(\mathbb{R})$ of (1.2). Moreover, there exists a constant $C = C(\phi) > 0$ such that*

$$\left| \phi(\xi) - H(\xi) + \frac{1}{\alpha \xi} \right| \leq \frac{C}{|\xi|^2}, \quad |\xi| \geq 1 \tag{3.2}$$

and

$$|\dot{\phi}(\xi)| \leq \frac{C}{|\xi|^2}, \quad |\ddot{\phi}(\xi)| \leq \frac{C}{|\xi|^2}, \quad |\xi| \geq 1. \tag{3.3}$$

The following is an auxiliary lemma that allows us to view one-dimensional fractional Laplacians of $\phi : \mathbb{R} \to \mathbb{R}$ equivalently as n-dimensional fractional Laplacians.

Lemma 3.2. *For a unit vector $e \in \mathbb{S}^n$, let $\phi_e(x) = \phi(e \cdot x) : \mathbb{R}^n \to \mathbb{R}$. Then,*

$$\mathcal{I}_n[\phi_e](x) = C_n \mathcal{I}_1[\phi](e \cdot x).$$

where

$$C_n = \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2 + 1)^{\frac{n+1}{2}}} \, dy. \tag{3.4}$$

Consequently,

$$C_n \mathcal{I}_1[\phi](\xi) = \int_{\mathbb{R}^n} (\phi(\xi + e \cdot z) - \phi(\xi)) \frac{dz}{|z|^{n+1}}, \quad \xi \in \mathbb{R}. \tag{3.5}$$

Proof. Begin by writing

$$\mathcal{I}_n[\phi_e](x) = \int_{\mathbb{R}^n} (\phi_e(x+z) - \phi_e(x)) \frac{dz}{|z|^{n+1}} = \int_{\mathbb{R}^n} (\phi(e \cdot x + e \cdot z) - \phi(e \cdot x)) \frac{dz}{|z|^{n+1}}.$$

We claim that it is enough to prove the result for $e = e_1$. Indeed, let T be a rotation matrix such that $Te = e_1$ and apply the change of variables $Tz = y$ to obtain

$$\begin{aligned}
\mathcal{I}_n[\phi_e](x) &= \int_{\mathbb{R}^n} \left(\phi(e \cdot x + e \cdot T^{-1} y) - \phi(e \cdot x)\right) \frac{dy}{|T^{-1} y|^{n+1}} \\
&= \int_{\mathbb{R}^n} (\phi(e_1 \cdot Tx + e_1 \cdot y) - \phi(e_1 \cdot Tx)) \frac{dy}{|y|^{n+1}} = \mathcal{I}_n[\phi_{e_1}](Tx).
\end{aligned}$$

If $\mathcal{I}_n[\phi_{e_1}](x_0) = C_n \mathcal{I}_1[\phi](e_1 \cdot x_0)$ for any $x_0 \in \mathbb{R}$, then we take $x_0 = Tx$ and notice that

$$\mathcal{I}_n[\phi_e](x) = C_n \mathcal{I}_1[\phi](e_1 \cdot Tx) = C_n \mathcal{I}_1[\phi](Te \cdot Tx) = C_n \mathcal{I}_1[\phi](e \cdot x).$$

Hence, the result holds.

It remains to prove the lemma for $e = e_1$. Observe for $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ that

$$\begin{aligned}
\mathcal{I}_n[\phi_{e_1}](x) &= \int_{\mathbb{R}^n} (\phi(x_1 + z_1) - \phi(z_1)) \frac{dz}{|z|^{n+1}} \\
&= \int_{\mathbb{R}} (\phi(x_1 + z_1) - \phi(z_1)) \left(\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1, z')|^{n+1}} \, dz' \right) dz_1.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1, z')|^{n+1}} \, dz' &= \int_{\mathbb{R}^{n-1}} \frac{1}{(|z'|^2 + z_1^2)^{\frac{n+1}{2}}} \, dz' \\
&= \frac{1}{|z_1|^{n+1}} \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2 + 1)^{\frac{n+1}{2}}} \, |z_1|^{n-1} \, dy = \frac{C_n}{|z_1|^2},
\end{aligned}$$

we have

$$\mathcal{I}_n[\phi_{e_1}](\xi) = C_n \int_{\mathbb{R}} (\phi(x_1 + z_1) - \phi(z_1)) \frac{dz_1}{|z_1|^2} = C_n \mathcal{I}_1[\phi](e_1 \cdot x).$$

To prove (3.5), fix $\xi \in \mathbb{R}$. Let $x \in \mathbb{R}^n$ be such that $\xi = e \cdot x$ and simply observe that

$$\begin{aligned}
C_n \mathcal{I}_1[\phi](\xi) = C_n \mathcal{I}_1[\phi](e \cdot x) &= \mathcal{I}_n[\phi_e](x) \\
&= \int_{\mathbb{R}^n} (\phi(e \cdot x + e \cdot z) - \phi(e \cdot x)) \frac{dz}{|z|^{n+1}} \\
&= \int_{\mathbb{R}^n} (\phi(\xi + e \cdot z) - \phi(\xi)) \frac{dz}{|z|^{n+1}}.
\end{aligned}$$

$\square$

3.2. **The auxiliary functions a_ε and $\bar{a}_\varepsilon$.** Here, we will introduce two auxiliary functions that are necessary for our analysis. Let $d = d(t,x)$ be a given smooth function. Define the function function $a_\varepsilon = a_\varepsilon(\xi; t, x, e)$ by

$$a_\varepsilon = \int_{\mathbb{R}^n} \left(\phi \left(\xi + e \cdot z + \frac{d(t, x + \varepsilon z) - d(t,x) - \nabla d(t,x) \cdot \varepsilon z}{\varepsilon} \right) - \phi\left(\xi + e \cdot z\right) \right) \frac{dz}{|z|^{n+1}},$$

where $(\xi, t, x, e) \in \mathbb{R} \times [0,\infty) \times \mathbb{R}^n \times \mathbb{S}^n$. The corresponding function $\bar{a}_\varepsilon = \bar{a}_\varepsilon(t,x,e)$ is

$$\bar{a}_\varepsilon(t,x,e) = \frac{1}{\varepsilon\,|\ln \varepsilon|} \int_{\mathbb{R}^n} a_\varepsilon\left(\xi; t, x, e\right) \dot{\phi}(\xi)\, d\xi. \tag{3.6}$$

We will be interested in a_ε and $\overline{a}_\varepsilon$ when d is the signed distance function to a front Γ_t. In this case, one of the main results in [11] is that $\bar{a}_\varepsilon$ converges to the mean curvature of d in a neighborhood of Γ_t, see Lemma 3.4. However, we must take care because the signed distance function itself is not smooth everywhere. Throughout the paper, we will use the following smooth extension of the distance function away from Γ_t.

Definition 3.3 (Extension of the signed distance function)**.** Let $\rho > 0$ be such that the signed distance function $\tilde{d}$ associated to a curve Γ_t is smooth in

$$Q_\rho = \{(t,x) : |\tilde{d}(t,x)| \le \rho\}.$$

Consequently, $|\nabla \tilde{d}| = 1$ in Q_ρ. We extend $\tilde{d}(t,x)$ outside of Q_ρ with a smooth, bounded function $d(t,x)$ satisfying

$$d(t,x) \begin{cases} = \tilde{d}(t,x) & \text{in } Q_\rho \\ \ge \rho & \text{in } \{(t,x) : d(t,x) > \rho\} \\ \le -\rho & \text{in } \{(t,x) : d(t,x) < \rho\}. \end{cases}$$

Lemma 3.4 (Lemma 4 in [11])**.** *Let d be as in Definition 3.3. Then,*

$$\lim_{\varepsilon \to 0} c_0 \bar{a}_\varepsilon(t,x,e) = \mu \Delta d(t,x) = \mu \operatorname{tr}\left((I - \widehat{\nabla d} \otimes \widehat{\nabla d}) D^2 d \right)$$

uniformly in $(t,x,e) \in Q_\rho \times \mathbb{S}^{n-1}$.

Remark 3.5. If d is as in Definition 3.3, then since d is smooth and bounded outside of Q_ρ, the function $\bar{a}_\varepsilon$ is bounded independently of ε in $(t,x,e) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{S}^{n-1}$ as a consequence of the proof of [11, Lemma 4].

It is also important to notice that, morally, a_ε is the difference between an n-dimensional and a 1-dimensional fractional Laplacian of $\phi(d/\varepsilon)$. This is seen in the following two lemmas. We delay their proofs until the end of the paper (see Section 7).

Lemma 3.6 (Near the front)**.** *Let d be as in Definition 3.3. If $|d(t,x)| \le \rho$, then*

$$a_\varepsilon \left(\frac{d(t,x)}{\varepsilon}; t, x, \nabla d(t,x) \right) = \varepsilon \mathcal{I}_n \left[\phi \left(\frac{d(t,\cdot)}{\varepsilon} \right) \right](x) - C_n \mathcal{I}_1[\phi] \left(\frac{d(t,x)}{\varepsilon} \right). \tag{3.7}$$

Lemma 3.7 (Far from the front)**.** *Let d be as in Defintion 3.3. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that, for any unit vector e,*

$$\left| a_\varepsilon \left(\frac{d(t,x)}{\varepsilon}; t, x, e \right) - \left[\varepsilon \mathcal{I}_n \left[\phi \left(\frac{d(t,\cdot)}{\varepsilon} \right) \right](x) - C_n \mathcal{I}_1[\phi] \left(\frac{d(t,x)}{\varepsilon} \right) \right] \right| \le \frac{C\varepsilon}{\rho}.$$

3.3. **The corrector ψ.** The linearized operator $\mathcal{L}$ associated to (1.2) is given by

$$\mathcal{L}[\psi] = -C_n\mathcal{I}_1[\psi] + W''(\phi)\psi. \tag{3.8}$$

Let $\psi = \psi(\xi; t, x, e)$ be the solution to the linearized standing wave equation

$$\begin{cases} \mathcal{L}[\psi] = \dfrac{a_\varepsilon(\xi; t, x, e)}{\varepsilon\,|\ln\varepsilon|} + \dot{\phi}(\xi)\, c_0(\sigma - \bar{a}_\varepsilon(t, x, e)) + \tilde{\sigma}\left(W''(\phi(\xi)) - W''(0)\right) & \xi \in \mathbb{R} \\ \psi(\pm\infty; t, x, e) = 0, \end{cases} \tag{3.9}$$

where $\sigma > 0$ is a small positive constant and $\tilde{\sigma} > 0$ is such that $\sigma = W''(0)\tilde{\sigma}$.

Lemma 3.8. *There is a unique solution $\psi = \psi(\xi; t, x, e)$ to (3.9) such that $\psi, \dot{\psi}, \psi_t$, and $D_x^2\psi$ are bounded independently of ξ, t, x, ε.*

Proof. The proof of existence of a unique $\psi \in H_\xi^{\frac{1}{2}}(\mathbb{R})$ follows exactly as in the proof of [10, Theorem 3.2] using a Lax-Milgram argument.

Estimates on the right-hand side of the equation for ψ.

- We consider the time derivative of $a_\varepsilon(\xi; t, x, \phi)$:

$$\begin{aligned} \partial_t a_\varepsilon(\xi; t, x, \phi) &= \int_{\mathbb{R}^n} \partial_t \left(\phi\left(\xi + \frac{d(t, x+\varepsilon z) - d(t, x)}{\varepsilon}\right) - \phi\left(\xi + e\cdot z\right)\right) \frac{dz}{|z|^{n+1}} \\ &= \int_{\mathbb{R}^n} \dot{\phi}\left(\xi + \frac{d(t, x+\varepsilon z) - d(t, x)}{\varepsilon}\right) \frac{d_t(t, x+\varepsilon z) - d_t(t, x)}{\varepsilon} \frac{dz}{|z|^{n+1}}. \end{aligned}$$

Therefore,

$$|\partial_t a_\varepsilon(\xi; t, x, \phi)| \le \int_{\mathbb{R}^n} \dot{\phi}\left(\xi + \frac{d(t, x+\varepsilon z) - d(t, x)}{\varepsilon}\right) \frac{|d_t(t, x+\varepsilon z) - d_t(t, x)|}{\varepsilon} \frac{dz}{|z|^{n+1}}.$$

□

With a similar flavor as Lemma 3.7 for ϕ, we conclude this section by stating the following estimate for the n- and 1-dimensional fractional Laplacians of ψ.

Lemma 3.9. *There is a constant $C = C(n, \psi, d) > 0$ such that, for any $(t, x) \in [0, \infty) \times \mathbb{R}^n$ and any unit vector $e = e(t, x)$,*

$$\left| \varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(t, \cdot)}{\varepsilon}; t, \cdot, e(t, \cdot)\right)\right](x) - C_n\mathcal{I}_1[\psi(\cdot; t, x, e(t, x))]\left(\frac{d(t, x)}{\varepsilon}\right)\right| \le C\varepsilon^{1/2}.$$

4. Heurtistics

4.1. **Ansatz for motion by mean curvature.** We believe it is helpful to view the heuristical derivation of the evolution of the fronts Γ_t^i by mean curvature in Theorem 1.1. For this, we consider the simple case of $N = 2$.

For the following formal computations, assume that the signed distance function $d_i(t, x)$ associated to Γ_t^i is smooth and that $|\nabla d_i| = 1$. Moreover, we assume that there is a positive, uniform distance ρ between Γ_t^1 and Γ_t^2.

The ansatz for the solution to the reaction-diffusion equation (1.1) is given by

$$u^\varepsilon(t, x) \simeq \phi\left(\frac{d_1(t, x)}{\varepsilon}\right) + \phi\left(\frac{d_2(t, x)}{\varepsilon}\right). \tag{4.1}$$

Plugging the ansatz into (1.1), the left-hand side gives

$$\varepsilon\partial_t u^\varepsilon \simeq \dot{\phi}\left(\frac{d_1}{\varepsilon}\right)\partial_t d_1 + \dot{\phi}\left(\frac{d_2}{\varepsilon}\right)\partial_t d_2. \tag{4.2}$$

Up to dividing by $\varepsilon\,|\ln\varepsilon|$, we use the equation for ϕ (see (1.2)) and estimates on a_ε (see Lemma 3.6 and Lemma 3.7) to write the right-hand side of (1.1) for the ansatz as

(4.3)

$$\begin{aligned}
&\varepsilon\mathcal{I}_n[u^\varepsilon]-W'(u^\varepsilon)\\
&\quad\simeq \varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_1}{\varepsilon}\right)\right]+\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_2}{\varepsilon}\right)\right]-W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)+\phi\left(\frac{d_2}{\varepsilon}\right)\right)\\
&\quad=\left(\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_1}{\varepsilon}\right)\right]-C_n\mathcal{I}_1[\phi]\left(\frac{d_1}{\varepsilon}\right)\right)+\left(\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_2}{\varepsilon}\right)\right]-C_n\mathcal{I}_1[\phi]\left(\frac{d_2}{\varepsilon}\right)\right)\\
&\qquad+C_n\mathcal{I}_1[\phi]\left(\frac{d_1}{\varepsilon}\right)+C_n\mathcal{I}_1[\phi]\left(\frac{d_2}{\varepsilon}\right)-W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)+\phi\left(\frac{d_2}{\varepsilon}\right)\right)\\
&\quad\simeq a_\varepsilon\left(\frac{d_1}{\varepsilon}\right)+a_\varepsilon\left(\frac{d_2}{\varepsilon}\right)+W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)\right)+W'\left(\phi\left(\frac{d_2}{\varepsilon}\right)\right)-W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)+\phi\left(\frac{d_2}{\varepsilon}\right)\right).
\end{aligned}$$

Freeze a point (t,x) near the front Γ_t^1. Let $\xi=d_1(t,x)/\varepsilon$ and assume separation of scales. That is, assume that ξ and (t,x) are unrelated. In this regard, let $\eta=|d_2(t,x)|\geq\rho$, so that η^{-1} is bounded. Since the ansatz u^ε is a solution to (1.1), we can multiply the equation by by $\dot\phi(\xi)$ and integrate over $\xi\in\mathbb{R}$ to write

$$\int_{\mathbb{R}}\varepsilon\partial_t u^\varepsilon\dot\phi\,d\xi\simeq\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\left(\varepsilon\mathcal{I}_n u^\varepsilon-W'(u^\varepsilon)\right)\dot\phi(\xi)\,d\xi. \tag{4.4}$$

For convenience, we will consider the left and right-hand sides separately again.

First, the left-hand side of (4.4) with (4.2) gives

$$\begin{aligned}
\int_{\mathbb{R}}\varepsilon\partial_t u^\varepsilon\,\dot\phi(\xi)\,d\xi&\simeq\partial_t d_1(t,x)\int_{\mathbb{R}}[\dot\phi\,(\xi)]^2\,d\xi+\dot\phi(\eta)\partial_t d_2(t,x)\int_{\mathbb{R}}\dot\phi(\xi)\,d\xi\\
&\simeq c_0^{-1}\partial_t d_1(t,x)+\frac{C\varepsilon^2}{\eta^2}\partial_t d_2(t,x)\\
&\simeq c_0^{-1}\partial_t d_1(t,x),
\end{aligned}$$

where we used (3.1), (1.2), and the asymptotics on $\dot\phi$ (see (3.3)).

Next, we look at the right-hand side of (4.4) with (4.3). First, using that (1.2) and that W is periodic, we have

$$\begin{aligned}
\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}W'\left(\phi(\xi)\right)\dot\phi(\xi)\,d\xi&=\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\frac{d}{d\xi}[W\left(\phi\left(\xi\right)\right)]\,d\xi\\
&=\frac{1}{\varepsilon\,|\ln\varepsilon|}[W(1)-W(0)]=0.
\end{aligned}$$

Next, we use the asymptotics and properties of ϕ (see (1.2), (3.2)) and Taylor expand W' around the origin to estimate

$$\begin{aligned}
\frac{1}{\varepsilon\,|\ln\varepsilon|}W'\left(\phi(\eta)\right)\int_{\mathbb{R}}\dot\phi(\xi)\,d\xi&=\frac{1}{\varepsilon\,|\ln\varepsilon|}W'\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\\
&\simeq\frac{1}{\varepsilon\,|\ln\varepsilon|}\left(W'(0)+W''(0)\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\right)\\
&\simeq 0+\frac{1}{\varepsilon\,|\ln\varepsilon|}\frac{C\varepsilon}{\eta}\simeq 0.
\end{aligned}$$

For the remaining W' term, we Taylor expand around $\phi(\xi)$ and use similar estimates to obtain

$$\begin{aligned}
&\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} W'\left(\phi\left(\xi\right)+\phi\left(\frac{\eta}{\varepsilon}\right)\right)\dot\phi(\xi)\,d\xi\\
&\quad=\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} W'\left(\phi\left(\xi\right)+\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\dot\phi(\xi)\,d\xi\\
&\quad\simeq\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\left[W'\left(\phi\left(\xi\right)\right)+W''\left(\phi\left(\xi\right)\right)\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\right]\dot\phi(\xi)\,d\xi\\
&\quad=\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} W'\left(\phi\left(\xi\right)\right)\dot\phi(\xi)\,d\xi+\frac{1}{\varepsilon\,|\ln\varepsilon|}\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\int_{\mathbb{R}} W''\left(\phi\left(\xi\right)\right)\dot\phi(\xi)\,d\xi\\
&\quad\simeq 0+\frac{1}{\varepsilon\,|\ln\varepsilon|}\frac{C\varepsilon}{\eta}\int_{\mathbb{R}}\frac{d}{d\xi}[W'\left(\phi\left(\xi\right)\right)]\,d\xi=\frac{C}{\eta\,|\ln\varepsilon|}[W'(1)-W'(0)]=0.
\end{aligned}$$

Lastly, for the nonlocal terms, we first use Corollary 7.1 to justify that

$$\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{\eta}{\varepsilon}\right)\int_{\mathbb{R}}\dot\phi(\xi)\,d\xi\simeq 0$$

and then Lemma 3.4 to conclude that

$$\begin{aligned}
&\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} a_\varepsilon\left(\xi\right)\dot\phi(\xi)\,d\xi\\
&\qquad\qquad=\bar a_\varepsilon(t,x)\simeq c_0^{-1}\mu\,\mathrm{tr}\left((I-\widehat{\nabla d_1(t,x)}\otimes\widehat{\nabla d_1(t,x)})D^2d_1(t,x)\right).
\end{aligned}$$

Combing all these pieces, we conclude that (4.4) for the ansatz gives

$$c_0^{-1}\partial_t d_1(t,x)\simeq\mu c_0^{-1}\,\mathrm{tr}\left((I-\widehat{\nabla d_1(t,x)}\otimes\widehat{\nabla d_1(t,x)})D^2d_1(t,x)\right).$$

The computation for (t,x) frozen near Γ_t^2 is similar. We conclude that the fronts move according to their mean curvature:

$$\begin{cases}\partial_t d_1(t,x)\simeq\mu\,\mathrm{tr}\left((I-\widehat{\nabla d_1(t,x)}\otimes\widehat{\nabla d_1(t,x)})D^2d_1(t,x)\right) & \text{near } \Gamma_t^1\\ \partial_t d_2(t,x)\simeq\mu\,\mathrm{tr}\left((I-\widehat{\nabla d_2(t,x)}\otimes\widehat{\nabla d_2(t,x)})D^2d_2(t,x)\right) & \text{near } \Gamma_t^2.\end{cases}$$

4.2. **Ansatz for corrector.** One of the key ingredients in proving Theorem 1.1 is the construction of strict subsolutions (supersolutions), denoted by $v^\varepsilon=v^\varepsilon(t,x)$. For this, it is necessary to add a small corrector ψ to the ansatz in (4.1). In order to showcase the equation for ψ, we will consider the simplest case in which $N=1$ and assume that $d(t,x)=d_1(t,x)$ is smooth with $|\nabla d|=1$ and satisfies

$$\partial_t d=\mu\Delta d-c_0\sigma\simeq c_0\bar a_\varepsilon(t,x)-c_0\sigma. \tag{4.5}$$

To find the corrector ψ for the barrier, we consider the ansatz

$$v^\varepsilon(t,x)\simeq\phi\left(\frac{d(t,x)}{\varepsilon}\right)+\varepsilon\,|\ln\varepsilon|\,\psi\left(\frac{d(t,x)}{\varepsilon}\right)-\varepsilon\,|\ln\varepsilon|\,\tilde\sigma,$$

where the function ψ is to be determined and $\tilde\sigma>0$ is a small, given constant. Assume for now that ψ is smooth and bounded with bounded derivative.

Since v^ε is a supersolution to (1.1), then heuristically, there is a $\sigma>0$ such that

$$\varepsilon\partial_t v^\varepsilon=\frac{1}{\varepsilon\,|\ln\varepsilon|}(\varepsilon\mathcal{I}_n v^\varepsilon-W'(v^\varepsilon))-\sigma. \tag{4.6}$$

Plugging the ansatz into (4.6), the left-hand side gives

$$\varepsilon\partial_t v^\varepsilon \simeq \dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d + \varepsilon\left|\ln\varepsilon\right|\dot{\psi}\left(\frac{d}{\varepsilon}\right)\partial_t d \simeq \dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d, \tag{4.7}$$

where we use that $\dot{\psi}$ and $\partial_t d$ are bounded. Next, we look at the right-hand side of (4.6) for the ansatz. First, we use the equation for ϕ (see (1.2)) and estimates on a_ε (see Lemmas 3.6, 3.7, 3.9) to find that

$$\begin{aligned}
\frac{\varepsilon}{\varepsilon\left|\ln\varepsilon\right|}\mathcal{I}_n[v^\varepsilon] &\simeq \frac{\varepsilon}{\varepsilon\left|\ln\varepsilon\right|}\mathcal{I}_n\left[\phi\left(\frac{d}{\varepsilon}\right)\right] + \varepsilon\mathcal{I}_n\left[\psi\left(\frac{d}{\varepsilon}\right)\right]\\
&= \frac{1}{\varepsilon\left|\ln\varepsilon\right|}\left(\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d}{\varepsilon}\right)\right] - C_n\mathcal{I}_1[\phi]\left(\frac{d}{\varepsilon}\right)\right) + \frac{1}{\varepsilon\left|\ln\varepsilon\right|}C_n\mathcal{I}_1[\phi]\left(\frac{d}{\varepsilon}\right)\\
&\quad + \left(\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d}{\varepsilon}\right)\right] - C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right)\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right)\\
&\simeq \frac{1}{\varepsilon\left|\ln\varepsilon\right|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) + \frac{1}{\varepsilon\left|\ln\varepsilon\right|}W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right).
\end{aligned} \tag{4.8}$$

On the other hand, we do a Taylor expansion for W' around $\phi(d/\varepsilon)$ to estimate

$$\begin{aligned}
\frac{1}{\varepsilon\left|\ln\varepsilon\right|}W'(v^\varepsilon) &\simeq \frac{1}{\varepsilon\left|\ln\varepsilon\right|}\left[W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + W''\left(\phi\left(\frac{d)}{\varepsilon}\right)\right)\left(v^\varepsilon - \phi\left(\frac{d}{\varepsilon}\right)\right)\right]\\
&\simeq \frac{1}{\varepsilon\left|\ln\varepsilon\right|}\left[W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\left(\varepsilon\left|\ln\varepsilon\right|\psi\left(\frac{d}{\varepsilon}\right) - \varepsilon\left|\ln\varepsilon\right|\tilde{\sigma}\right)\right].
\end{aligned} \tag{4.9}$$

Equating (4.7) with (4.8) and (4.9), the equation for the ansatz gives

$$\begin{aligned}
\dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d &\simeq \frac{1}{\varepsilon\left|\ln\varepsilon\right|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right)\\
&\quad - W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\psi\left(\frac{d}{\varepsilon}\right) + \tilde{\sigma}W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right) - \sigma.
\end{aligned} \tag{4.10}$$

Rearranging and using (4.5), we have

$$\begin{aligned}
&-C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right) + W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\psi\left(\frac{d}{\varepsilon}\right)\\
&\qquad\simeq \frac{1}{\varepsilon\left|\ln\varepsilon\right|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) - \dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d + \tilde{\sigma}W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right) - \sigma\\
&\qquad\simeq \frac{1}{\varepsilon\left|\ln\varepsilon\right|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) - \dot{\phi}\left(\frac{d}{\varepsilon}\right)c_0\bar{a}_\varepsilon + c_0\sigma\dot{\phi}\left(\frac{d}{\varepsilon}\right) + \tilde{\sigma}W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right) - \sigma.
\end{aligned}$$

We let ψ be the solution to this equation. In particular, let $\mathcal{L}$ be the linearized operator in (3.8). Then, that corrector ψ satisfies the equation

$$\begin{aligned}
\mathcal{L}[\psi]\left(\frac{d(t,x)}{\varepsilon}\right) &= \frac{1}{\varepsilon\left|\ln\varepsilon\right|}a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}\right) - \dot{\phi}\left(\frac{d(t,x)}{\varepsilon}\right)c_0\bar{a}_\varepsilon(t,x)\\
&\quad + c_0\sigma\dot{\phi}\left(\frac{d(t,x)}{\varepsilon}\right) + \tilde{\sigma}W''\left(\phi\left(\frac{d(t,x)}{\varepsilon}\right)\right) - \sigma,
\end{aligned} \tag{4.11}$$

as desired. See (3.9) with $\sigma = W''(0)\tilde{\sigma}$.

In order to check the validity equation (4.11), at least formally, we freeze a point (t,x) near Γ_t^1. Let $\xi = d(t,x)/\varepsilon$ and assume separation of scales. We multiply both sides of (4.11) by $\dot\phi(\xi)$ and integrate over $\mathbb{R}$ to write

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}[\psi]\,(\xi)\,\dot\phi(\xi)\,d\xi &= \int_{\mathbb{R}} \left(\frac{1}{\varepsilon\,|\ln\varepsilon|} a_\varepsilon\,(\xi) - \dot\phi\,(\xi)\,c_0\bar a_\varepsilon(t,x)\right)\dot\phi(\xi)\,d\xi \\ &\quad + \int_{\mathbb{R}} \left(c_0\sigma\dot\phi\,(\xi) + \tilde\sigma W''\,(\phi\,(\xi)) - \sigma\right)\dot\phi(\xi)\,d\xi.\end{aligned}$$

Since $\mathcal{I}_1$ is self-adjoint and ϕ satisfies (1.2), the left-hand side of the equation gives

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}[\psi]\dot\phi\,d\xi &= \int_{\mathbb{R}} \left(-C_n\mathcal{I}_1[\dot\phi] + W''\,(\phi)\,\dot\phi\right)\psi\,d\xi \\ &= \int_{\mathbb{R}} \frac{d}{d\xi}\left(-C_n\mathcal{I}_1[\phi] + W'\,(\phi)\right)\psi\,d\xi = 0.\end{aligned}$$

To show that the right-hand side is also zero, we first use the definitions of $\bar a_\varepsilon$ and c_0 to find

$$\int_{\mathbb{R}} \left(\frac{1}{\varepsilon\,|\ln\varepsilon|} a_\varepsilon\,(\xi) - \dot\phi\,(\xi)\,c_0\bar a_\varepsilon(t,x)\right)\dot\phi(\xi)\,d\xi = \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} a_\varepsilon\,(\xi)\,\dot\phi(\xi)\,d\xi - \bar a_\varepsilon(t,x) = 0.$$

Then, we use that W' is periodic to find that

$$\int_{\mathbb{R}} \tilde\sigma W''\,(\phi(\xi))\,\dot\phi(\xi)\,d\xi = \tilde\sigma\int_{\mathbb{R}} \frac{d}{d\xi}[W'\,(\phi(\xi))]\,d\xi = \tilde\sigma[W'(1) - W'(0)] = 0$$

and the definition of c_0 to see that

$$\int_{\mathbb{R}} \left(c_0\sigma\dot\phi(\xi) - \sigma\right)\dot\phi(\xi)\,d\xi = c_0\sigma\int_{\mathbb{R}} [\dot\phi(\xi)]^2\,d\xi - \sigma = 0,$$

as desired.

Remark 4.1. Notice that ψ depends on the distance function $d(t,x)$. Hence, when $N > 1$, we have a finite sequence of correctors, denoted by $\psi_1,\dots,\psi_N$, depending on the signed distance function $d_i(t,x)$ to the front Γ_t^i, $i = 1,\dots,N$.

Remark 4.2. To see that $\sigma = W''(0)\tilde\sigma$, assume that $d(t,x) << -1$ and $\psi \equiv 0$. Then, $(t,x) \in {}^-\Omega_t^1$ is far from the front Γ_t^1 which implies $\phi(d(t,x)/\varepsilon) \approx 0$ and $a_\varepsilon((t,x)/\varepsilon) \simeq 0$ (which is a consequence of Corollary 7.1). Therefore, in (4.10), we have

$$0 \simeq 0 + \tilde\sigma W''\,(0) - \sigma.$$

5. Construction of barriers

The main challenge in proving Theorem 1.1 is the construction of strict subsolutions (supersolutions) to (1.1). In particular, we will use barriers to prove that a sequence of sets are generalized super(sub)-flows. We will focus on the construction of subsolutions as the construction of supersolutions is similar.

Let $\varphi_i(t,x)$, $i = 1,\dots,N$, be smooth functions satisfying *(i),(ii),(iii)* in Definition 2.1. Moreover, we assume that

$$\{(t,x) : \varphi_{i+1}(t,x) > 0\} \subset\subset \{(t,x) : \varphi_i(t,x) > 0\} \quad \text{for } i = 1,\dots,N-1. \tag{5.1}$$

Let $\tilde d_i(t,x)$ be the signed distance function associated to $\{x : \varphi_i(t,x) > 0\}$. Then, we can denote the zero level set of φ_i by $\Gamma_t^i = \{x : \tilde d_i(t,x) = 0\}$. As a consequence of *(iii)* in Definition 2.1, there is a $\rho > 0$ such that $\tilde d_i(t,x)$ is smooth in the set

$$Q_\rho^i = \{(t,x) : |\tilde d_i(t,x)| \le \rho\}$$

and $|\nabla \tilde{d}_i| = 1$ in Q^i_ρ. Moreover, by (5.1), and perhaps making ρ smaller, we can assume that $Q^i_\rho \cap Q^j_\rho = \emptyset$ for $i \neq j$. Let d_i be a smooth, bounded extension of $\tilde{d}_i$ as defined in Definition 3.3. Similarly, let $e_i(t,x) \in \mathbb{S}^n$ be such that $e_i(t,x) = \nabla d_i(t,x)$ in Q^i_ρ and is smooth and bounded outside of Q^i_ρ.

As a consequence of *(ii)* in Definition 2.1, for $\sigma > 0$ sufficiently small,

$$\partial_t d_i \leq \mu \operatorname{tr}\left((I - \widehat{\nabla d_i} \otimes \widehat{\nabla d_i}) D^2 d_i\right) - c_0\sigma = \mu \Delta d_i - c_0 \sigma \quad \text{in } Q^i_\rho. \tag{5.2}$$

Let $\tilde{\sigma} > 0$ be such that $\sigma = W''(0)\tilde{\sigma}$. Then, we define smooth barrier $v^\varepsilon(t,x)$ by

$$v^\varepsilon(t,x) = \sum_{i=1}^N \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^N \psi_i\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}; t, x, e_i(t,x)\right) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,. \tag{5.3}$$

Lemma 5.1. *For sufficiently small ε, v^ε is a strict subsolution to*

$$\varepsilon \partial_t v^\varepsilon - \frac{1}{\varepsilon\,|\ln\varepsilon|}\left(\varepsilon \mathcal{I}_n[v^\varepsilon] - W'(v^\varepsilon)\right) < -\frac{\sigma}{2}. \tag{5.4}$$

Moreover, for ε sufficiently small, v^ε satisfies

$$-2\tilde{\sigma}\varepsilon\,|\ln\varepsilon| \leq v^\varepsilon(t,x) - \sum_{i=1}^N \mathbb{1}_{\{d_i(t,\cdot) \geq \tilde{\sigma}/2\}}(x) \leq -\frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,. \tag{5.5}$$

Proof. We will break the proof into four main steps. First, we estimate the equation for $v^\varepsilon(t,x)$ for any (t,x). Then, we will show that $v^\varepsilon(t,x)$ satisfies (5.4) when (t,x) is near a front $\Gamma^{i_0}_t$ and then when (t,x) is far from all fronts Γ^i_t, $i = 1, \ldots, N$. Lastly, we establish (5.5) for all (t,x).

For convenience, we use the following notation throughout the proof.

$$\begin{aligned}
\phi_i &:= \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)\\
\psi_i &:= \psi_i\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}; t, x, e_i\right)\\
\tilde{\phi}_i &:= \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) - H\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)\\
\bar{a}^i_\varepsilon &:= \bar{a}_\varepsilon\left(t, x, e_i\right)\\
a^i_\varepsilon &:= a_\varepsilon\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}; t, x, e_i\right)\\
b^i_\varepsilon &:= \varepsilon \mathcal{I}_n\left[\phi\left(\frac{d_i(t,\cdot)-\tilde{\sigma}}{\varepsilon}\right)\right](x) - C_n \mathcal{I}_1[\phi]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right).
\end{aligned} \tag{5.6}$$

We note that it will be important for the reader to remember the dependence of ψ_i and a_ε on the variables t, x, e as well as $\xi = d_i(t,x)$.

Step 1. Computation for $v^\varepsilon(t,x)$ in (1.1) for an arbitrary $(t,x) \in [0,\infty) \times \mathbb{R}^n$.

First, the time derivative of v^ε at (t,x) is given by

$$\varepsilon \partial_t v^\varepsilon(t,x) = \sum_{i=1}^N \dot{\phi}_i\, \partial_t d_i(t,x) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^N \left[(\psi_i)_\xi \partial_t d_i(t,x) + \varepsilon(\psi_i)_t + \varepsilon(\psi_i)_e \partial_t e_i(t,x)\right].$$

Since

$$\varepsilon\partial_t v^\varepsilon = \sum_{i=1}^{N} \dot{\phi}_i\, \partial_t d_i(t,x) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|).$$

Next, we consider the nonlocal term. For each $i = 1, \ldots, N$, we use that ϕ satisfies (1.2) to find

$$\varepsilon\mathcal{I}_n[\phi_i](x) = \varepsilon\mathcal{I}_n[\phi_i](x) - C_n\mathcal{I}_1[\phi]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + W'\left(\phi_i\right).$$

Also, using that ψ satisfies (3.9) and Lemma 3.9, we find that

$$\begin{aligned}\varepsilon\mathcal{I}_n[\psi_i](x) &= \varepsilon\mathcal{I}_n[\psi_i](x) - C_n\mathcal{I}_1[\psi_i]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) - \mathcal{L}[\psi]\left(\frac{d_i(t,\cdot)-\tilde{\sigma}}{\varepsilon}\right) + W''\left(\phi_i\right)\psi_i \\ &= \mathcal{O}(\varepsilon^{1/2}) - \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon^i + \dot{\phi}_i c_0\left(\bar{a}_\varepsilon^i - \sigma\right) - \tilde{\sigma}\left(W''\left(\phi_i\right) - W''(0)\right) + W''\left(\phi_i\right)\psi_i.\end{aligned}$$

Therefore, the $1/2$-Laplacian of v^ε can be written as

$$\begin{aligned}\varepsilon\mathcal{I}_n[v^\varepsilon](x) = \sum_{i=1}^{N}&\left[\varepsilon\mathcal{I}_n[\phi_i](x) - C_n\mathcal{I}_1[\phi]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + W'\left(\phi_i\right)\right] \\ &+ \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\Bigg[\mathcal{O}(\varepsilon^{1/2}) - \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon^i + \dot{\phi}_i c_0\left(\bar{a}_\varepsilon^i - \sigma\right) \\ &\qquad - \tilde{\sigma}\left(W''\left(\phi_i\right) - W''(0)\right) + W''\left(\phi_i\right)\psi_i\Bigg].\end{aligned}$$

Recall the definitions of $\tilde{\phi}_i$ and b_ε^i introduced in (5.6). Since W' is periodic, we have that $W'(\phi_i) = W'(\tilde{\phi}_i)$ and similarly for $W''(\phi_i) = W''(\tilde{\phi}_i)$. Using this, rearranging, and utilizing the notation b_ε^i, we can equivalently write

$$\begin{aligned}\varepsilon\mathcal{I}_n[v^\varepsilon](x) &= \sum_{i=1}^{N}\left[(b_\varepsilon^i - a_\varepsilon^i) + W'\left(\tilde{\phi}_i\right)\right] \\ &\quad + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\left[\mathcal{O}(\varepsilon^{1/2}) + W''(\tilde{\phi}_i)\psi_i + \dot{\phi}_i c_0\left(\bar{a}_\varepsilon^i - \sigma\right) - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right].\end{aligned}$$

Then, the equation for v^ε at (t,x) can be written as

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &:= \varepsilon\partial_t v^\varepsilon(t,x) - \frac{1}{\varepsilon\,|\ln\varepsilon|}\left(\varepsilon\mathcal{I}_n[v^\varepsilon(t,\cdot)](x) - W'(v^\varepsilon(t,x))\right) \\ &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \sum_{i=1}^{N}\dot{\phi}_i\partial_t d_i(t,x) \\ &\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\Bigg\{\sum_{i=1}^{N}\left[(b_\varepsilon^i - a_\varepsilon^i) + W'(\tilde{\phi}_i)\right] \\ &\quad + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\left[\mathcal{O}(\varepsilon^{1/2}) + W''(\tilde{\phi}_i)\psi_i + \dot{\phi}_i c_0\left(\bar{a}_\varepsilon^i - \sigma\right) - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right] \\ &\quad - W'\left(\sum_{i=1}^{N}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\Bigg\}.\end{aligned}$$

Grouping the error terms, the $\dot{\phi}_i$ terms, and the nonlinear terms together, we have

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2}) - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^{N}(b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^{N}\dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right]\\
&\quad + \frac{1}{\varepsilon\,|\ln\varepsilon|}\Bigg\{W'\left(\sum_{i=1}^{N}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\\
&\qquad - \sum_{i=1}^{N}\left(W'(\tilde{\phi}_i) + \varepsilon\,|\ln\varepsilon|\left[W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right]\right)\Bigg\}.\end{aligned}$$

Fix an index $i_0 \in \{1,\dots,N\}$. For the remainder of Step 1, we will conveniently isolate every term indexed with i_0 to help with Step 2. First, we do a Taylor expansion for W' around $\tilde{\phi}_{i_0}$ to obtain

$$\begin{aligned}&W'\left(\sum_{i=1}^{N}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\\
&\qquad = W'(\tilde{\phi}_{i_0}) + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\\
&\qquad\quad + \mathcal{O}\left(\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)^2\right).\end{aligned}$$

By

$$\frac{1}{\varepsilon\,|\ln\varepsilon|}\mathcal{O}\left(\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)^2\right) = \sum_{i\neq i_0}\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|).$$

Hence, we have that

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)\\
&\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^{N}(b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^{N}\dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right]\\
&\quad + \frac{1}{\varepsilon\,|\ln\varepsilon|}\Bigg\{W'(\tilde{\phi}_{i_0}) + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{N}\psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\\
&\qquad - \left(W'(\tilde{\phi}_{i_0}) + \varepsilon\,|\ln\varepsilon|\left[W''(\tilde{\phi}_{i_0})\psi_{i_0} - \tilde{\sigma}\left(W''(\tilde{\phi}_{i_0}) - W''(0)\right)\right]\right)\\
&\qquad - \sum_{i\neq i_0}\left(W'(\tilde{\phi}_i) + \varepsilon\,|\ln\varepsilon|\left[W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right]\right)\Bigg\}\end{aligned}$$

where in the last two lines we extracted the i_0 term. Cancelling the $W'(\tilde{\phi}_{i_0})$ and $W''(\tilde{\phi}_{i_0})\psi_{i_0}$ terms then distributing $1/(\varepsilon\,|\ln\varepsilon|)$, we simplify to

$$\begin{aligned}
\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)\\
&\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right]\\
&\quad + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} + \sum_{i\neq i_0}\psi_i - \tilde{\sigma}\right) + \tilde{\sigma}\left(W''(\tilde{\phi}_{i_0}) - W''(0)\right)\\
&\quad - \sum_{i\neq i_0}\left[\frac{W'(\tilde{\phi}_i)}{\varepsilon\,|\ln\varepsilon|} + W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right].
\end{aligned}$$

Next, we do a Taylor expansion for W' around 0 and recall that $W'(0) = 0$ to write

$$W'(\tilde{\phi}_i) = W'(0) + W''(0)\tilde{\phi}_i + \mathcal{O}((\tilde{\phi}_i)^2) = W''(0)\tilde{\phi}_i + \mathcal{O}((\tilde{\phi}_i)^2).$$

With this, we now have that

$$\begin{aligned}
\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)\\
&\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right]\\
&\quad + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} + \sum_{i\neq i_0}\psi_i - \tilde{\sigma}\right) + \tilde{\sigma}\left(W''(\tilde{\phi}_{i_0}) - W''(0)\right)\\
&\quad - \sum_{i\neq i_0}\left[\frac{W''(0)\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} + W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right].
\end{aligned}$$

We rearrange to group the terms with $W''(\tilde{\phi}_{i_0}) - W''(0)$ together

$$\begin{aligned}
\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)\\
&\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right]\\
&\quad + \left(W''(\tilde{\phi}_{i_0}) - W''(0)\right)\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} - \tilde{\sigma}W''(0)\\
&\quad + \sum_{i\neq i_0}\left[\left(W''(\tilde{\phi}_{i_0}) - W''(\tilde{\phi}_i)\right)\psi_i + \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right].
\end{aligned}$$

Looking at the last line, we Taylor expand W'' around 0 to find, for $i \neq i_0$,

$$\begin{aligned}
(W''(\tilde{\phi}_{i_0}) - W''(\tilde{\phi}_i))\psi_i + \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right) &= \mathcal{O}(\psi_i) - \tilde{\sigma}\left(W'''(0)\tilde{\phi}_i + \mathcal{O}(\tilde{\phi}_i)^2\right)\\
&= \mathcal{O}(\psi_i) + \mathcal{O}(\tilde{\phi}_i)
\end{aligned}$$

and also

$$W''(\tilde{\phi}_{i_0}) - W''(0) = W'''(0)\tilde{\phi}_{i_0} + \mathcal{O}((\tilde{\phi}_{i_0})^2) = \mathcal{O}(\tilde{\phi}_{i_0}).$$

Therefore, we have

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\left[\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(\tilde{\phi}_i) + \mathcal{O}(\psi_i)\right]\\ &\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b^i_\varepsilon - a^i_\varepsilon) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}^i_\varepsilon - \sigma\right)\right]\\ &\quad + \mathcal{O}(\tilde{\phi}_{i_0})\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} - \sigma\end{aligned}$$

where we also used that $\sigma = W''(0)\tilde{\sigma}$. Since d is smooth, with Remark 3.5 we have that $\partial_t d_i - c_0\bar{a}^i_\varepsilon + c_0\sigma$ is bounded independently of ε. Hence, we conclude that

(5.7)

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\left(\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(\tilde{\phi}_i) + \mathcal{O}(\dot{\phi}_i) + \mathcal{O}(\psi_i) + \frac{\mathcal{O}(\tilde{\phi}_{i_0})\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}\right)\\ &\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N(b^i_\varepsilon - a^i_\varepsilon) + \dot{\phi}_{i_0}[\partial_t d_{i_0}(t,x) - c_0(\bar{a}^{i_0}_\varepsilon - \sigma)] - \sigma\end{aligned}$$

Step 2. $v^\varepsilon(t,x)$ satisfies (5.4) when (t,x) is near the front $\Gamma^{i_0}_t$.

Assume that $|d_{i_0}(t,x) - \sigma| \le |\ln\varepsilon|^{-1/2}$ for some index $1 \le i_0 \le N$. Then, for ε sufficiently small,

$$|d_i(t,x) - \sigma| \ge |\ln\varepsilon|^{-1/2} \quad \text{for all } i \neq i_0.$$

We begin by estimating the error terms in (5.7) for $i \neq i_0$. First, we use (3.2) to estimate

$$\begin{aligned}|\tilde{\phi}_i| &\le \left|\phi\left(\frac{d_i(t,x)-\sigma}{\varepsilon}\right) - H\left(\frac{d_i(t,x)-\sigma}{\varepsilon}\right) + \frac{\varepsilon}{\alpha(d_i(t,x)-\sigma)}\right| + \left|\frac{\varepsilon}{\alpha(d_i(t,x)-\sigma)}\right|\\ &\le \frac{C\varepsilon^2}{|d_i(t,x)-\sigma|^2} + \frac{\varepsilon}{\alpha\,|d_i(t,x)-\sigma|}\\ &\le C\varepsilon^2\,|\ln\varepsilon| + \frac{\varepsilon\,|\ln\varepsilon|^{1/2}}{\alpha}\\ &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|^{1/2}),\end{aligned}$$

from which it follows that

(5.8)
$$\frac{|\tilde{\phi}_i|}{\varepsilon\,|\ln\varepsilon|} \le \mathcal{O}(|\ln\varepsilon|^{-1/2}) \quad \text{and} \quad \frac{|\tilde{\phi}_i|^2}{\varepsilon\,|\ln\varepsilon|} \le \mathcal{O}(\varepsilon).$$

Next, we use (3.3) to find that

(5.9)
$$|\dot{\phi}_i| \le \frac{C\varepsilon^2}{|d_i(t,x)-\sigma|^2} \le \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|)$$

By Lemma [estimates on ψ], $\mathcal{O}(\psi_i) = o(1)$. Combining the above estimates in view of (5.7), we have

$$\sum_{i\neq i_0}\left(\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(\tilde{\phi}_i) + \mathcal{O}(\dot{\phi}_i) + \mathcal{O}(\psi_i) + \frac{\mathcal{O}(\tilde{\phi}_{i_0})\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|}\right)$$
$$\leq \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|^{1/2}) + \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|) + o(1) + \mathcal{O}(|\ln\varepsilon|^{-1/2})$$

To check the mean curvature term, we use that $\dot{\phi}_{i_0} \geq 0$, (5.2), and Lemma 3.4 to estimate

$$\begin{aligned}\dot{\phi}_{i_0}[\partial_t d_{i_0}(t,x) - c_0(\bar{a}_\varepsilon^{i_0} - \sigma)] &= \dot{\phi}_{i_0}\left([\partial_t d_{i_0}(t,x) - \mu\Delta d_{i_0}(t,x) + c_0\sigma] + [\mu\Delta d_{i_0}(t,x) - c_0\bar{a}_\varepsilon^{i_0}]\right)\\ &\leq \dot{\phi}_{i_0}(0 + o(1)) = o(1).\end{aligned}$$

Lastly, by Lemma 3.6 and Lemma 3.7,

$$\begin{aligned}\frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^{N}|b_\varepsilon^i - a_\varepsilon^i| &= \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i\neq i_0}|b_\varepsilon^i - a_\varepsilon^i|\\ &\leq \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i\neq i_0}\frac{C\varepsilon}{|\ln\varepsilon|^{-1/2}} = \mathcal{O}(|\ln\varepsilon|^{-1/2}).\end{aligned} \tag{5.10}$$

Consequently, in (5.7), we have that

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) \leq{}& \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2})\\ &+ \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|^{1/2}) + \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|) + o(1) + \mathcal{O}(|\ln\varepsilon|^{-1/2}) - \sigma.\end{aligned}$$

Taking ε sufficiently small, (5.3) holds.

Step 3. $v^\varepsilon(t,x)$ satisfies (5.4) when (t,x) is away from all fronts Γ_t^i.

Assume that for all $i = 1,\dots,N$,

$$|d_i(t,x) - \sigma| \geq |\ln\varepsilon|^{-1/2}.$$

Then, estimating $\dot{\phi}_{i_0}$ exactly as in (5.9) and using Remark 3.5 to note that $\partial_t d_{i_0} - c_0\bar{a}_\varepsilon^{i_0} + c_0\sigma$ is bounded, we find that

$$\left|\dot{\phi}_{i_0}\left(\partial_t d_{i_0}(t,x) - c_0\bar{a}_\varepsilon^{i_0} + c_0\sigma\right)\right| \leq \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|).$$

With this, (5.8), and (5.10) (without dropping the $i = i_0$ term), we have that (5.7) gives

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) \leq{}& \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(\varepsilon^{1/2})\\ &+ \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|^{1/2}) + \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|) + o(1) + \mathcal{O}(|\ln\varepsilon|^{-1/2}) - \sigma.\end{aligned}$$

Taking ε sufficiently small, (5.3) holds.

Step 4. $v^\varepsilon(t,x)$ satisfies (5.5).

It is enough to show v^ε satisfies the following.

(1) In the set $\{d_N(t,x) \geq \tilde{\sigma}/2\}$:

$$N - 2\tilde{\sigma}\varepsilon\,|\ln\varepsilon| \leq v^\varepsilon \leq N - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|.$$

(2) In the set $\{d_{i+1}(t,x) < \tilde{\sigma}/2 \leq d_i(t,x)\}$ for $i = 1,\dots,N-1$:

$$i - 2\tilde{\sigma}\varepsilon\,|\ln\varepsilon| \leq v^\varepsilon \leq i - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|.$$

(3) In the set $\{d_1(t,x) < \tilde{\sigma}/2\}$:

$$-2\tilde{\sigma}\varepsilon\,|\ln\varepsilon| \le v_\varepsilon \le -\frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,.$$

We begin with (2). For a fixed $1 \le i_0 \le N-1$, let (t,x) be such that $d_{i_0+1}(t,x) < \frac{\tilde{\sigma}}{2} \le d_{i_0}(t,x)$. Note that

$$-\frac{1}{d_i(t,x)-\tilde{\sigma}} < \frac{2}{\tilde{\sigma}} \quad \text{for all } i_0+1 \le i \le N.$$

Then, by (3.2) and [estimate for ψ], for ε small we have

(5.11)

$$\begin{aligned}
v^\varepsilon(t,x) &= \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&= \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \sum_{i=i_0+1}^{N} \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&\le \sum_{i=1}^{i_0} 1 + \sum_{i=i_0+1}^{N} \left(0 - \frac{\varepsilon}{\alpha(d_i(t,x)-\tilde{\sigma})} + \frac{C\varepsilon^2}{(d_i(t,x)-\tilde{\sigma})^2}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&\le i_0 + \sum_{i=1_0+1}^{N} \left(\frac{\varepsilon}{\alpha(\tilde{\sigma}/2)} + \frac{C\varepsilon^2}{(\tilde{\sigma}/2)^2}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&\le i_0 - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,.
\end{aligned}$$

On the other hand,

$$-\frac{1}{d_i(t,x)-\tilde{\sigma}} \ge \frac{2}{\tilde{\sigma}} \ge 0 \quad \text{for all } 1 \le i \le i_0.$$

Hence, for ε small, we similarly estimate from below using that $\phi \ge 0$ to find

(5.12)

$$\begin{aligned}
v^\varepsilon(t,x) &= \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&= \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \sum_{i=i_0+1}^{N} \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&\ge \sum_{i=1}^{i_0} \left(1 - \frac{\varepsilon}{\alpha(d_i(t,x)-\tilde{\sigma})} - \frac{C\varepsilon^2}{(d_i(t,x)-\tilde{\sigma})^2}\right) + \sum_{i=i_0+1}^{N} 0 + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&\ge i_0 + \sum_{i=1}^{i_0} \left(\frac{\varepsilon}{\alpha(\tilde{\sigma}/2)} + \frac{C\varepsilon^2}{(\tilde{\sigma}/2)^2}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\
&\ge i_0 - 2\tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,.
\end{aligned}$$

We now look at (1). Let (t,x) be such that $d_N(t,x) \ge \frac{\tilde{\sigma}}{2}$. Then, using that $\phi \le 1$, [estimate for ψ], and taking ε sufficiently small,

$$v^\varepsilon(t,x) = \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{N} \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|$$

$$\leq \sum_{i=1}^{N} 1 + \varepsilon \left|\ln \varepsilon\right| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon \left|\ln \varepsilon\right|$$
$$\leq N - \frac{\tilde{\sigma}}{2}\varepsilon \left|\ln \varepsilon\right|.$$

On the other hand, since $d_i(t,x) \geq \tilde{\sigma}/2$ for all $1 \leq i \leq N$, we have

$$-\frac{1}{d_i(t,x) - \tilde{\sigma}} \geq \frac{2}{\tilde{\sigma}} \geq 0 \quad \text{for all } 1 \leq i \leq N.$$

Estimating as in (5.12), we find that, for ε small,

$$v^\varepsilon(t,x) \geq N - 2\tilde{\sigma}\varepsilon \left|\ln \varepsilon\right|.$$

Finally, we show (3). Let (t,x) be such that $d_1(t,x) < \frac{\tilde{\sigma}}{2}$. Note that

$$-\frac{1}{d_i(t,x) - \tilde{\sigma}} < \frac{2}{\tilde{\sigma}} \quad \text{for all } 1 \leq i \leq N.$$

Estimating as in (5.11) for sufficiently small ε, we get

$$v^\varepsilon(t,x) \leq 0 - \frac{\tilde{\sigma}}{2}\varepsilon \left|\ln \varepsilon\right|.$$

On the other hand, using that $\phi \geq 0$, for ε small,

$$v^\varepsilon(t,x) = \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon \left|\ln \varepsilon\right| \sum_{i=1}^{N} \psi_i - \tilde{\sigma}\varepsilon \left|\ln \varepsilon\right|$$
$$\geq \sum_{i=1}^{N} 0 + \varepsilon \left|\ln \varepsilon\right| \sum_{i=1}^{N} \mathcal{O}(\psi_i) - \tilde{\sigma}\varepsilon \left|\ln \varepsilon\right|$$
$$\geq -2\tilde{\sigma}\varepsilon \left|\ln \varepsilon\right|.$$

This completes the proof. □

6. Proof of Theorem 1.1

We apply an adaptation of the abstract method introduced in [2], see also [1].
We define the families of open sets $(D^i)_{i=1}^N$ and $(E^i)_{i=1}^N$ by

$$D^i = \operatorname{Int}\left\{(t,x) \in (0,\infty) \times \mathbb{R}^n : \liminf_{\varepsilon \to 0}{}_* \frac{u^\varepsilon - i}{\varepsilon \left|\ln \varepsilon\right|} \geq 0\right\} \subset (0,\infty) \times \mathbb{R}^n$$

$$E^i = \operatorname{Int}\left\{(t,x) \in (0,\infty) \times \mathbb{R}^n : \limsup_{\varepsilon \to 0}{}_* \frac{u^\varepsilon - (i-1)}{\varepsilon \left|\ln \varepsilon\right|} \leq 0\right\} \subset (0,\infty) \times \mathbb{R}^n.$$

To define the traces of D^i and E^i, we first define the functions $\underline{\chi}^i, \overline{\chi}^i : (0,\infty) \times \mathbb{R}^n \to [-1,1]$, respectively, by

$$\underline{\chi}^i = \mathbb{1}_{D_i} - \mathbb{1}_{(D_i)^c} \quad \text{and} \quad \overline{\chi}^i = \mathbb{1}_{(E_i)^c} - \mathbb{1}_{E_i}.$$

Since D^i is open, $\underline{\chi}^i$ is lower semicontinuous, and since $(E_i)^c$ is closed, $\overline{\chi}^i$ is upper semicontinuous. To ensure that $\overline{\chi}^i$ and $\underline{\chi}^i$ remain lower and upper semicontinuous, respectively, at $t = 0$, we set

$$\underline{\chi}^i(0,x) = \liminf_{t \to 0,\ y \to x} \underline{\chi}^i(t,y) \quad \text{and} \quad \overline{\chi}^i(0,x) = \limsup_{t \to 0,\ y \to x} \overline{\chi}^i(t,y).$$

Define the traces D_0^i and E_0^i by

$$D_0^i = \{x \in \mathbb{R}^n : \underline{\chi}^i(0,x) = 1\} \quad \text{and} \quad E_0^i = \{x \in \mathbb{R}^n : \overline{\chi}^i(0,x) = -1\}.$$

To apply the abstract method, we need the following propositions. We delay their proofs.

Proposition 6.1 (Initialization)**.** *For each $i = 1, \ldots, N$,*

$$\Omega_0^i \subset D_0^i \quad \textit{and} \quad (\overline{\Omega}_0^i)^c \subset E_0^i.$$

Proposition 6.2 (Propagation)**.** *For each $i = 1, \ldots, N$, the set D^i is a generalized super-flow, and the set $\overline{E^i}$ is a generalized sub-flow.*

For $t > 0$, define the sets D_t^i and E_t^i by

$$D_t^i = D^i \cap (\{t\} \times \mathbb{R}^n) \quad \text{and} \quad E_t^i = E^i \cap (\{t\} \times \mathbb{R}^n).$$

By the abstract method (see [1,2]), it follows from Propositions 6.1 and 6.2 that

$$^+\Omega_t^i \subset D_t^i \subset {}^+\Omega_t^i \cup \Gamma_t^i \quad \text{and} \quad {}^-\Omega_t^i \subset E_t^i \subset {}^-\Omega_t^i \cup \Gamma_t^i.$$

The conclusion readily follows; we provide the details for completeness.

First, since $^+\Omega_t^i \subset D_t^i$, we use the definition of D_t^i to see that

$$\liminf_{\varepsilon \to 0} {}_* u^\varepsilon(t,x) \geq i \quad \text{for } x \in {}^+\Omega_t^i. \tag{6.1}$$

Using that $^-\Omega_t^{i+1} \subset E_t^{i+1}$, we similarly get

$$\limsup_{\varepsilon \to 0} {}_* u^\varepsilon(t,x) \leq (i+1) - 1 = i \quad \text{for } x \in {}^-\Omega_t^{i+1}. \tag{6.2}$$

Therefore, for $i = 1, \ldots, N-1$,

$$\lim_{\varepsilon \to 0} u^\varepsilon(t,x) = i \quad \text{in } {}^+\Omega_t^i \cap {}^-\Omega_t^{i+1}.$$

Next, by the comparison principle, $0 \leq u^\varepsilon \leq N$. Consequently,

$$0 \leq \liminf_{\varepsilon \to 0} {}_* u^\varepsilon \quad \text{and} \quad \limsup_{\varepsilon \to 0} {}_* u^\varepsilon \leq N.$$

Hence, together with (6.1) we have

$$\lim_{\varepsilon \to 0} u^\varepsilon(t,x) = N \quad \text{in } {}^+\Omega_t^N,$$

and with (6.2) we have

$$\lim_{\varepsilon \to 0} u^\varepsilon(t,x) = 0 \quad \text{in } {}^-\Omega_t^1.$$

It remains to prove Propositions 6.1 and 6.2. We begin with the initialization.

6.1. **Proof of Proposition 6.1.**

Proof. We will prove that $\Omega_0^{i_0} \subset D_0^{i_0}$ for all $1 \leq i_0 \leq N$. The proof of $(\overline{\Omega}_0^{i_0})^c \subset E_0^{i_0}$ is similar.

Fix i_0, a point $x_0 \in \Omega_0^{i_0}$, and a small constant $\tilde{\sigma} > 0$. To prove that $x_0 \in D_0^{i_0}$, it is enough to show that, for all (t,x) in a neighborhood of $(0,x_0)$,

$$\liminf_{\varepsilon \to 0} {}_* \frac{u^\varepsilon(t,x) - i_0}{\varepsilon\,|\ln \varepsilon|} \geq 0.$$

For this, we will use (5.3) to construct a suitable subsolution $v^\varepsilon \leq u^\varepsilon$, depending on $\tilde{\sigma}$.

We begin by defining smooth functions φ_i for each $i = 1, \ldots, i_0$ that satisfy conditions *(i),(ii),(iii)* in Definition 2.1. For this, we first let $r_i > 0$ be given by

$$r_i = d_i(x_0) - \frac{\tilde{\sigma}}{2}, \quad i = 1, \ldots, i_0,$$

where d_i is given in (1.3). Note that $B_{r_i}(x_0) \subset\subset \Omega_0^i$ and $r_i - r_{i+1} = d_i(x_0) - d_{i+1}(x_0) \geq d(\Gamma_0^i, \Gamma_0^{i+1})$. Define the smooth functions $\varphi_i(x)$, $i = 1, \dots, i_0$, by

$$\varphi_i(t,x) = (r_i - Ct)_+^2 - |x - x_0|^2$$

for a large constant $C > 0$, to be determined. It is easy to check that the signed distance function $\tilde{d}_i(t,x)$ associated to $\{x : \varphi_i(t,x) > 0\}$ is

$$\tilde{d}_i(t,x) = r_i - Ct - |x - x_0| \tag{6.3}$$

and that

$$\{(t,x) : \varphi_i(t,x) > 0\} = \bigcup_{t \geq 0} \{t\} \times B_{r_i - Ct}(x_0).$$

Hence, *(i)* in Definition 2.1 is satisfied. Next, we see that

$$\nabla \varphi_i(t,x) = (-2C(r_i - Ct), -2(x - x_0))$$

and, for $t < r_{i_0}/(2C)$, we have

$$\begin{aligned} \partial_t \varphi_i - \mu \operatorname{tr}\left((I - \widehat{\nabla_x \varphi_i} \otimes \widehat{\nabla_x \varphi_i}) D_x^2 \varphi_i\right) &= -2C(r_i - Ct) + 2\mu\,(n-1) \\ &\leq -2C(r_{i_0} - Ct) + 2\mu\,(n-1) \\ &\leq -Cr_{i_0} + 2\mu\,(n-1) \\ &\leq -c_0 \sigma \end{aligned}$$

for $C > 0$ sufficiently large. Hence, *(ii),(iii)* in Definition 2.1 are satisfied.

Let ρ and d_i be such that d_i is a smooth, bounded extension of $\tilde{d}_i$ outside of Q_ρ^i as in Definition 3.3. Similarly, let $e_i(t,x) \in \mathbb{S}^n$ be such that $e_i(t,x) = \nabla d_i(t,x)$ in Q_ρ^i and is smooth and bounded outside of Q_ρ^i. Let $v^\varepsilon = v^\varepsilon(t,x)$ be given by

$$v^\varepsilon(t,x) = \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln \varepsilon| \sum_{i=1}^{i_0} \psi_i\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}, t, x, e_i\right) - \tilde{\sigma}\varepsilon\,|\ln \varepsilon|\,.$$

By Lemma 5.1 (with $N = i_0$), we have that v^ε is a subsolution to (5.4) in $[0, r_{i_0}/(2C)] \times \mathbb{R}^n$.

We claim that $v^\varepsilon \leq u^\varepsilon$ in a neighborhood $\mathcal{N}(0,x_0) \subset \{d_{i_0}(t,x) \geq \tilde{\sigma}/2\}$. Let x be such that $d_{i_0}(0,x) \geq \tilde{\sigma}/2$. Then $d_i(x) \geq \tilde{\sigma}/2$ for all $i = 1, \dots, i_0$, and we use (3.2) to estimate

$$\begin{aligned} u^\varepsilon(0,x) &\geq \sum_{i=1}^{i_0} \phi\left(\frac{d_i(x)}{\varepsilon}\right) \\ &\geq \sum_{i=1}^{i_0} \left(1 - \frac{\varepsilon}{\alpha d_i(x)} - \frac{C\varepsilon^2}{(d_i(x))^2}\right) \\ &\geq \sum_{i=1}^{i_0} \left(1 - \frac{2\varepsilon}{\alpha\tilde{\sigma}} - \frac{4C\varepsilon^2}{\tilde{\sigma}^2}\right) \\ &= i_0 - \left(\frac{2\varepsilon}{\alpha\tilde{\sigma}} + \frac{4C\varepsilon^2}{\tilde{\sigma}^2}\right) i_0 \\ &\geq i_0 - \left(\frac{2\varepsilon}{\alpha\tilde{\sigma}} + \frac{4C\varepsilon^2}{\tilde{\sigma}^2}\right) N \\ &\geq i_0 - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln \varepsilon|\,, \end{aligned}$$

for ε sufficiently small. For each $1 \le i < i_0$, we can similarly show that

$$u^\varepsilon(0,x) \ge i - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \quad \text{in } \left\{x : d_i(0,x) \ge \frac{\tilde{\sigma}}{2}\right\}.$$

On the other hand, when $d_1(0,x) < \frac{\tilde{\sigma}}{2}$, we simply have

$$u^\varepsilon(0,x) = \sum_{i=1}^{N} \phi\left(\frac{d_i(x)}{\varepsilon}\right) \ge 0 > -\frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,.$$

Therefore, by the second inequality in (5.5), we have

$$u^\varepsilon(0,x) \ge \sum_{i=1}^{i_0} \mathbb{1}_{\{d_i(0,\cdot)\ge\tilde{\sigma}/2\}}(x) - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \ge v^\varepsilon(0,x).$$

By the comparison principle, the claim holds. Consequently, by the first inequality in (5.5), we have

$$\liminf_{\varepsilon\to0}{}_*\frac{u^\varepsilon(t,x)-i_0}{\varepsilon\,|\ln\varepsilon|} \ge \liminf_{\varepsilon\to0}{}_*\frac{v^\varepsilon(t,x)-i_0}{\varepsilon\,|\ln\varepsilon|} \ge -2\tilde{\sigma} \quad \text{in } \mathcal{N}(0,x_0).$$

Letting $\tilde{\sigma} \to 0$, the result follows. □

6.2. **Proof of Proposition 6.2.**

Proof. Fix $1 \le i_0 \le N$. We will show that D^{i_0} is a generalized super-flow. The proof that $\overline{E^{i_0}}$ is a generalized sub-flow is similar.

Let $(t_0,x_0) \in (0,\infty)\times\mathbb{R}^n$, $h > 0$, and φ_{i_0} be a smooth function satisfying *(i)-(iv)* in Definition 2.1. For $i = 1,\dots,i_0-1$, define the smooth functions $\varphi_i : [t_0,t_0+h]\times\mathbb{R}^n \to \mathbb{R}$ by

$$\varphi_i(t,x) := \varphi_{i_0}(t,x) + c_0\left(\frac{i_0-i}{i_0-1}\right),$$

where $c_0 > 0$ is a small constant. Since the mean curvature equation is geometric, φ_i satisfies *(ii)* in Definition 2.1. Notice that

$$\{(t,x)\in[t_0,t_0{+}h]{\times}\mathbb{R}^n : \varphi_i(t,x)\ge0\} = \left\{(t,x)\in[t_0,t_0{+}h]{\times}\mathbb{R}^n : \varphi_{i_0}(t,x) \ge -c_0\left(\frac{i_0-i}{i_0-1}\right)\right\}.$$

Since φ_{i_0} is smooth, we can take c_0 sufficiently small to guarantee that φ_i, $i = 1,\dots,i_0-1$, also satisfies *(i),(iii)* in Definition 2.1. Moreover, (5.1) holds. Lastly, since $D^i \subset\subset D^j$ for all $i < j$, we have that

$$\{x\in\overline{B}(x,r) : \varphi_i(t_0,x)\ge0\} = \left\{x\in\overline{B}(x,r) : \varphi_{i_0}(t_0,x) \ge -c_0\left(\frac{i_0-i}{i_0-1}\right)\right\} \subset D^i_{t_0}$$

by making c_0 smaller, if necessary. Therefore, φ_i also satisfies *(iv)* in Definition 2.1.

Let $\tilde{d}_i(t,x)$ be the signed distance function associated to $\{(t,x) : \varphi_i(t,x)\ge0\}$. Let ρ and d_i be such that d_i is a smooth, bounded extension of $\tilde{d}_i$ outside of Q^i_ρ as in Definition 3.3. Similarly, let $e_i(t,x)\in\mathbb{S}^n$ be such that $e_i(t,x) = \nabla d_i(t,x)$ in Q^i_ρ and is smooth and bounded outside of Q^i_ρ.

By the initial condition *(iv)*, we have that

$$\{x : d_i(t_0,x)\ge0\} = \{x : \varphi_i(t_0,x)\ge0\} \subset D^i_{t_0} = \left\{x : \liminf_{\varepsilon\to0}{}_*\frac{u^\varepsilon(t_0,x)-i}{\varepsilon\,|\ln\varepsilon|} \ge 0\right\}.$$

Therefore,

$$\{x : d_i(t_0,x) \geq \tilde{\sigma}/2\} \subset \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0,x)-i}{\varepsilon\,|\ln\varepsilon|} \geq 0\right\},$$

which further gives that

$$u^\varepsilon(t_0,x) \geq i - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \quad \text{in } \{x : d_i(t_0,x) \geq \tilde{\sigma}/2\}.$$

In particular,

$$u^\varepsilon(t_0,x) \geq \sum_{i=1}^{i_0} \mathbb{1}_{\{d_i(t_0,\cdot)\geq\tilde{\sigma}/2\}}(x) - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,.$$

Let $v^\varepsilon = v^\varepsilon(t,x)$ be given by

$$v^\varepsilon(t,x) = \sum_{i=1}^{i_0}\phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^{i_0}\psi_i\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon};t,x,e_i\right) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,.$$

By Lemma 5.1, we have that v^ε is a subsolution to (5.4) in $[t_0,t_0+h]\times\mathbb{R}^n$. Moreover, by the second inequality in (5.5), we have that

$$u^\varepsilon(t_0,x) \geq \sum_{i=1}^{i_0} \mathbb{1}_{\{d_i(t_0,\cdot)\geq\tilde{\sigma}/2\}}(x) - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \geq v^\varepsilon(t_0,x).$$

By the comparison principle, $u^\varepsilon \geq v^\varepsilon$ on $[t_0,t_0+h]\times\mathbb{R}^n$. By the first inequality in (5.5), we have that

$$\frac{u^\varepsilon(t_0+h,x)-i}{\varepsilon\,|\ln\varepsilon|} \geq \frac{v^\varepsilon(t_0+h,x)-i}{\varepsilon\,|\ln\varepsilon|} \geq -2\tilde{\sigma} \quad \text{in } \{d_i(t_0+h,x) > \tilde{\sigma}/2\}.$$

Taking $\tilde{\sigma}\to 0$, it follows that

$$\{x : \varphi_i(t_0+h,x)\geq 0\} = \{x : d_i(t_0+h,x)\geq 0\} \subset \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0+h,x)-i}{\varepsilon\,|\ln\varepsilon|} \geq 0\right\}.$$

as desired. □

7. Appendix

In this section, we prove the estimates stated in Section 3.

First, we will prove Lemmas 3.6 and 3.7. That is, we will establish the relationship between a_ε and fractional Laplacians of ϕ. It will be convenience to first split a_ε into two terms

$$\begin{aligned} a_\varepsilon = &\int_{\mathbb{R}^n}\left(\phi\left(\xi + e\cdot z + \frac{d(t,x+\varepsilon z)-d(t,x)-\nabla d(t,x)\cdot\varepsilon z}{\varepsilon}\right) - \phi(\xi)\right)\frac{dz}{|z|^{n+1}} \\ &-\int_{\mathbb{R}^n}\left(\phi\left(\xi+e\cdot z\right)-\phi(\xi)\right)\frac{dz}{|z|^{n+1}} \end{aligned} \tag{7.1}$$

where, by Lemma 3.2, the second integral is exactly $C_n\mathcal{I}_1[\phi](\xi)$.

Proof of Lemma 3.6. From (7.1) with $\xi = d(t,x)/\varepsilon$ and $e = \nabla d(t,x)$, we write

$$\begin{aligned} &a_\varepsilon\left(\frac{d(t,x)}{\varepsilon};t,x,\nabla d(t,x)\right) \\ &= \int_{\mathbb{R}^n}\left(\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right)-\phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}} - C_n\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(t,x+z)}{\varepsilon}\right) - \phi\left(\frac{d(t,x)}{\varepsilon}\right)\right) \frac{dz}{|z|^{n+1}} - C_n \mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right) \\
&= \varepsilon \mathcal{I}_n \left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_n \mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right).
\end{aligned}
$$

□

Proof of Lemma 3.7. Recalling (7.1), we want to estimate

$$
\begin{aligned}
&a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}; t,x,e\right) - \left[\varepsilon \mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_n\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right] \\
&= \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(t,x+\varepsilon z) + (e - \nabla d(t,x))\cdot \varepsilon z}{\varepsilon}\right) - \phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}} - \varepsilon \mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) \\
&= \int_{\mathbb{R}^n} \left(\phi\left(\frac{d(t,x+\varepsilon z) + (e - \nabla d(t,x))\cdot \varepsilon z}{\varepsilon}\right) - \phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}}.
\end{aligned}
$$

Let $0 < \delta < 1$ be a small constant, to be determined. We split the integral into two pieces

$$
\begin{aligned}
I &= \int_{|z|<\delta\rho/\varepsilon} \Bigg(\phi\left(\frac{d(t,x+\varepsilon z) + (e - \nabla d(t,x))\cdot \varepsilon z}{\varepsilon}\right) - \phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right) \\
&\qquad\qquad - \dot\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right)(e - \nabla d(t,x))\cdot z\Bigg)\frac{dz}{|z|^{n+1}} \\
II &= \int_{|z|>\delta\rho/\varepsilon} \left(\phi\left(\frac{d(t,x+\varepsilon z) + (e - \nabla d(t,x))\cdot \varepsilon z}{\varepsilon}\right) - \phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}}.
\end{aligned}
$$

For the long-range interactions, we simply estimate

$$
|II| \le 2\|\phi\|_\infty \int_{|z|>\delta\rho/\varepsilon} \frac{dz}{|z|^{n+1}} = \frac{C\varepsilon}{\delta\rho},
$$

where $C = C(n,\phi)$. Consider the short-range interactions in I. For each $|z| < \delta\rho/\varepsilon$, there is a $0 \le \tau \le 1$ such that

$$
\begin{aligned}
&\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon} + (e - \nabla d(t,x))\cdot z\right) - \phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right) - \dot\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right)(e - \nabla d(t,x))\cdot z \\
&\qquad = \ddot\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon} + \tau(e - \nabla d(t,x))\cdot z\right)|(e - \nabla d(t,x))\cdot z|^2.
\end{aligned}
$$

For all $0 \le \tau \le 1$ and $|z| < \delta\rho/\varepsilon$, we notice that

$$
\tau\,|(e - \nabla d(t,x))\cdot z| \le |e - \nabla d(t,x)|\,\frac{\delta\rho}{\varepsilon}.
$$

and

$$
\left|\frac{d(t,x+\varepsilon z)}{\varepsilon}\right| \ge \frac{|d(t,x)|}{\varepsilon} - |z| \ge \frac{\rho}{\varepsilon} - \frac{\delta\rho}{\varepsilon} = (1-\delta)\frac{\rho}{\varepsilon}.
$$

Consequently,

$$
\begin{aligned}
\left|\frac{d(t,x+\varepsilon z)}{\varepsilon} + \tau(e - \nabla d(t,x))\cdot z\right| &\ge \left|\frac{d(t,x+\varepsilon z)}{\varepsilon}\right| - \tau\,|(e - \nabla d(t,x))\cdot z| \\
&\ge (1-\delta)\frac{\rho}{\varepsilon} - |e - \nabla d(t,x)|\,\frac{\delta\rho}{\varepsilon} \\
&= (1/\delta - (1 + |e - \nabla d(t,x)|))\,\frac{\delta\rho}{\varepsilon} = \frac{\rho}{2\varepsilon}
\end{aligned}
$$

when we choose $\delta = \delta(e,d)$ to be

$$\delta = \frac{1}{2}\frac{1}{1+|e-\nabla d(t,x)|}.$$

Note that $|e-\nabla d(t,x)| \leq 1 + |d(t,x)|$ implies that

$$\frac{1}{2(2+|\nabla d(t,x)|)} \leq \delta \leq \frac{1}{2}.$$

In particular, δ can be bounded from above and below independently of e. By (3.3), we have

$$\sup_{0\leq\tau\leq 1}\left|\ddot{\phi}\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}+\tau(e-\nabla d(t,x))\cdot z\right)\right| \leq \frac{C}{\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}+\tau(e-\nabla d(t,x))\cdot z\right)^2} \leq \frac{C\varepsilon^2}{\rho^2}.$$

We can now estimate

$$\begin{aligned}
&\left|\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}+(e-\nabla d(t,x))\cdot z\right)-\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right)-\dot{\phi}\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right)(e-\nabla d(t,x))\cdot z\right| \\
&\qquad\leq \sup_{0\leq\tau\leq 1}\left|\ddot{\phi}\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}+\tau(e-\nabla d(t,x))\cdot z\right)\right| |(e-\nabla d(t,x))\cdot z|^2 \\
&\qquad\leq \frac{C\varepsilon^2}{\rho^2}|(e-\nabla d(t,x))\cdot z|^2 \\
&\qquad\leq \frac{C\varepsilon^2}{\rho^2}(1+|\nabla d(t,x)|^2)\,|z|^2.
\end{aligned}$$

Finally, we estimate the short-range interactions,

$$|I| \leq \frac{C\varepsilon^2}{\rho^2}\int_{|z|<\delta\rho/\varepsilon}|z|^2\frac{dz}{|z|^{n+1}} = \frac{C\varepsilon^2}{\rho^2}\left(\frac{\delta\rho}{\varepsilon}\right) \leq C\frac{\varepsilon}{\rho}.$$

Combing the estimates, we conclude that

$$|I|+|II| \leq C\left(\frac{\varepsilon}{\delta\rho}+\frac{\varepsilon}{\rho}\right) \leq C\frac{\varepsilon}{\rho},$$

which completes the proof. □

The following corollary is used to justify some of the formal computations in Section 4.

Corollary 7.1. *Let d be as in Definition 3.3. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that, for any unit vector e,*

$$\left|a_\varepsilon\left(\frac{d(t,x)}{\varepsilon};t,x,e\right)\right| \leq \frac{C\varepsilon}{\rho}.$$

Proof. By Lemma 3.7, we have that

$$\begin{aligned}
\left|a_\varepsilon\left(\frac{d(t,x)}{\varepsilon};t,x,e\right)\right| &\leq \left|a_\varepsilon\left(\frac{d(t,x)}{\varepsilon};t,x,e\right)-\left[\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x)-C_n\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right]\right| \\
&\quad+\left|\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x)\right|+C_n\left|\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right| \\
&\leq \frac{C\varepsilon}{\rho}+\left|\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x)\right|+C_n\left|\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right|.
\end{aligned}$$

Using the same techniques as in the proof of Lemma 3.7, we can show that both fractional Laplacian terms are controlled by ε/ρ. □

The next lemma is a more general estimate on a_ε which is used to establish properties of ψ.

Lemma 7.2. *There is a constant* $C = C(n, \phi, d) > 0$ *such that*

$$|a_\varepsilon(\xi; t, x, e)| \le C\varepsilon^{1/2}.$$

Proof. Begin by writing

$$\begin{aligned} a_\varepsilon = &\int_{|z|<1/\varepsilon^{1/2}} \left(\phi\left(\xi + e\cdot z + \frac{d(t,x+\varepsilon z) - d(t,x) - \nabla d(t,x)\cdot \varepsilon z}{\varepsilon} \right) - \phi(\xi + e\cdot z) \right) \frac{dz}{|z|^{n+1}} \\ &+ \int_{|z|>1/\varepsilon^{1/2}} \left(\phi\left(\xi + e\cdot z + \frac{d(t,x+\varepsilon z) - d(t,x) - \nabla d(t,x)\cdot \varepsilon z}{\varepsilon} \right) - \phi(\xi + e\cdot z) \right) \frac{dz}{|z|^{n+1}} \\ &=: I + II. \end{aligned}$$

For the long-range interactions,

$$|II| \le 2\|\phi\|_\infty \int_{|z|>1/\varepsilon^{1/2}} \frac{dz}{|z|^{n+1}} = C\varepsilon^{1/2}.$$

For the short range interactions, we use the mean value theorem and Taylor's theorem to estimate

$$\begin{aligned} |I| &\le \|\dot\phi\|_\infty \int_{|z|<1/\varepsilon^{1/2}} \frac{|d(t,x+\varepsilon z) - d(t,x) - \nabla d(t,x)\cdot\varepsilon z|}{\varepsilon} \frac{dz}{|z|^{n+1}} \\ &\le \|\dot\phi\|_\infty \left\|D^2 d\right\|_\infty \int_{|z|<1/\varepsilon^{1/2}} \varepsilon |z|^2 \frac{dz}{|z|^{n+1}} \\ &= \frac{C\varepsilon}{\varepsilon^{1/2}} = C\varepsilon^{1/2}. \end{aligned}$$

The conclusion follows. ☐

We end the paper with the proof of Lemma 3.9 for ψ. if we need it...

Proof of Lemma 3.9. For convenience, we drop the t notation. We begin by using Lemma 3.2 for ψ and a change of variables to write a single integral expression

$$\begin{aligned} &\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(t,\cdot)}{\varepsilon}, t, \cdot, e(t,\cdot)\right)\right](x) - C_n\mathcal{I}_1[\psi(\cdot; t, x, e)]\left(\frac{d(t,x)}{\varepsilon}\right) \\ &= \varepsilon\int_{\mathbb{R}^n} \left(\psi\left(\frac{d(x+y)}{\varepsilon}; x+y, e(x+y) \right) - \psi\left(\frac{d(x)}{\varepsilon}; x, e(x) \right) \right) \frac{dy}{|y|^{n+1}} \\ &\quad - \int_{\mathbb{R}^n} \left(\psi\left(\frac{d(t,x)}{\varepsilon} + e(x)\cdot z; x, e(x) \right) - \psi\left(\frac{d(t,x)}{\varepsilon}; x, e(x) \right) \right) \frac{dz}{|z|^{n+1}} \\ &= \varepsilon\int_{\mathbb{R}^n} \left(\psi\left(\frac{d(x+y)}{\varepsilon}; x+y, e(x+y) \right) - \psi\left(\frac{d(x)}{\varepsilon}; x, e(x) \right) \right) \frac{dy}{|y|^{n+1}} \\ &\quad - \varepsilon\int_{\mathbb{R}^n} \left(\psi\left(\frac{d(x) + e(x)\cdot y}{\varepsilon}; x, e(x) \right) - \psi\left(\frac{d(x)}{\varepsilon}; x, e(x) \right) \right) \frac{dy}{|y|^{n+1}} \\ &= \varepsilon\int_{\mathbb{R}^n} \left(\psi\left(\frac{d(x+y)}{\varepsilon}; x+y, e(x+y) \right) - \psi\left(\frac{d(x) + e(x)\cdot y}{\varepsilon}; x, e(x) \right) \right) \frac{dy}{|y|^{n+1}}. \end{aligned}$$

Furthermore, we can write this expression as

$$\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(t,\cdot)}{\varepsilon}; t, \cdot, e(t,\cdot)\right)\right](x) - C_n\mathcal{I}_1[\psi(\cdot; t, x, e)]\left(\frac{d(t,x)}{\varepsilon}\right)$$

$$
\begin{aligned}
&= \varepsilon \int_{\mathbb{R}^n} \left(\psi\left(\frac{d(x+y)}{\varepsilon}; x+y, e(x+y)\right) - \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x+y)\right)\right) \frac{dy}{|y|^{n+1}} \\
&\quad + \varepsilon \int_{\mathbb{R}^n} \left(\psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x+y)\right) - \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x)\right)\right) \frac{dy}{|y|^{n+1}} \\
&\quad + \varepsilon \int_{\mathbb{R}^n} \left(\psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x)\right) - \psi\left(\frac{d(x)+e(x)\cdot y}{\varepsilon}; x, e(x)\right)\right) \frac{dy}{|y|^{n+1}} \\
&=: I + II + III.
\end{aligned}
$$

We will show that $I, II = \mathcal{O}(\varepsilon)$ and $III = \mathcal{O}(\varepsilon^{1/2})$.

First, observe that

$$
\begin{aligned}
|I| &\leq \varepsilon \int_{|y|<1} \left| \psi\left(\frac{d(x+y)}{\varepsilon}; x+y, e(x+y)\right) - \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x+y)\right) \right. \\
&\qquad\qquad \left. - \nabla_x \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x+y)\right)\cdot y \right| \frac{dy}{|y|^{n+1}} \\
&\quad + \varepsilon \int_{|y|\geq 1} \left| \psi\left(\frac{d(x+y)}{\varepsilon}; x+y, e(x+y)\right) - \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x+y)\right)\right| \frac{dy}{|y|^{n+1}} \\
&\leq \varepsilon \left\|D_x^2\psi\right\|_\infty \int_{|y|<1} |y|^2 \frac{dy}{|y|^{n+1}} + 2\varepsilon \left\|\psi\right\|_\infty \int_{|y|\geq 1} \frac{dy}{|y|^{n+1}} \\
&\leq C_{n,\psi}\varepsilon
\end{aligned}
$$

and similarly

$$
\begin{aligned}
|II| &\leq \varepsilon \int_{|y|<1} \left| \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x+y)\right) - \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x)\right) \right. \\
&\qquad\qquad \left. - \psi_e\left(\frac{d(x+y)}{\varepsilon}; x, e(x)\right)\cdot y \right| \frac{dy}{|y|^{n+1}} \\
&\quad + \varepsilon \int_{|y|\geq 1} \left| \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x+y)\right) - \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x)\right)\right| \frac{dy}{|y|^{n+1}} \\
&\leq \varepsilon \left\|\psi_{ee}\right\|_\infty \int_{|y|<1} |y|^2 \frac{dy}{|y|^{n+1}} + 2\varepsilon \left\|\psi\right\|_\infty \int_{|y|\geq 1} \frac{dy}{|y|^{n+1}} \\
&\leq C_{n,\psi}\varepsilon.
\end{aligned}
$$

For III, we use the mean value theorem and Taylor's theorem to estimate

$$
\begin{aligned}
|III| &\leq \varepsilon \int_{|y|<\sqrt{\varepsilon}} \left| \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x)\right) - \psi\left(\frac{d(x)-e(x)\cdot y}{\varepsilon}; x, e(x)\right)\right| \frac{dy}{|y|^{n+1}} \\
&\quad + \varepsilon \int_{|y|\geq\sqrt{\varepsilon}} \left| \psi\left(\frac{d(x+y)}{\varepsilon}; x, e(x)\right) - \psi\left(\frac{d(x)+e(x)\cdot y}{\varepsilon}; x, e(x)\right)\right| \frac{dy}{|y|^{n+1}} \\
&\leq \varepsilon \|\dot{\psi}\|_\infty \int_{|y|<\sqrt{\varepsilon}} \frac{|d(x+y)-d(x)-e(x)\cdot y|}{\varepsilon} \frac{dy}{|y|^{n+1}} + 2\varepsilon \left\|\psi\right\|_\infty \int_{|y|\geq\sqrt{\varepsilon}} \frac{dy}{|y|^{n+1}} \\
&\leq \|\dot{\psi}\|_\infty \left\|D^2 d\right\|_\infty \int_{|y|<\sqrt{\varepsilon}} |y|^2 \frac{dy}{|y|^{n+1}} + C_{n,\psi}\frac{\varepsilon}{\sqrt{\varepsilon}} \\
&= C_{n,\psi,d}\sqrt{\varepsilon}.
\end{aligned}
$$

$\square$

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