STOCHASTIC HOMOGENIZATION OF A POROUS-MEDIUM TYPE EQUATION

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Abstract. We consider the homogenization problem for the stochastic porous-medium type equation $\partial_t u^\epsilon = \Delta f\left(T\left(\frac{x}{\epsilon}\right)\omega, u^\epsilon\right)$, with a well-prepared initial datum, where $f(T(y)\omega, u)$ is a stationary process, increasing in u, on a given probability space $(\Omega, \mathcal{F}, \mu)$ endowed with an ergodic dynamical system $\{T(y) : y \in \mathbb{R}^N\}$. Differently from the previous literature [3, 13], here we do not assume Ω compact. We first show that the weak solution u^ϵ satisfies a kinetic formulation of the equation, then we exploit the theory of "stochastically two-scale convergence in the mean" developed in [6] to show convergence of the kinetic solution to the kinetic solution of an homogenized problem of the form $\partial_t \overline{u} - \Delta \overline{f}(\overline{u}) = 0$. The homogenization result for the weak solutions then follows.

1. Introduction

In this paper we study the behavior as $\epsilon \to 0$ of the weak solution $u^\epsilon(\cdot, \cdot, \omega)$ of the porous-medium type equation: for any $\omega \in \Omega$

$$\begin{cases} \partial_t u^\epsilon = \Delta f\left(T\left(\frac{x}{\epsilon}\right)\omega, u^\epsilon\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u^\epsilon(0, x) = u_0\left(x, T\left(\frac{x}{\epsilon}\right)\omega\right) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{1.1}$$

where $f(T(y)\omega, u)$ is a stationary process, increasing in u, on a given probability space $(\Omega, \mathcal{F}, \mu)$ endowed with an ergodic dynamical system $\{T(y) : y \in \mathbb{R}^N\}$. Our model example is

$$f(\omega, u) = a(\omega)u|u|^{\gamma(\omega)} + b(\omega) \tag{1.2}$$

with γ, a, b bounded and $\gamma(\omega) \geq \gamma_0 > 0$, $a(\omega) \geq a_0 > 0$ for a.e. $\omega \in \Omega$. The initial datum is assumed to be "well-prepared", i.e., of the form $u_0(x, \omega) = g(\omega, \varphi(x))$, for some $\varphi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, with $g(\omega, \cdot) = f(\omega, \cdot)^{-1}$.

The homogenization problem for porous-medium type equations of the form $\partial_t u^\epsilon - \Delta\left(f\left(\frac{x}{\epsilon}, u^\epsilon\right)\right)$ in the case in which $f(\cdot, u)$ belongs to an ergodic algebra with mean value, has been studied in [3, 13]. An example is when $f(\cdot, u)$ is almost periodic. In this situation it can be proven that the algebra can be identified with the space $C(\Omega)$ for some compact set Ω endowed with a Borel probability measure and a continuous ergodic dynamical system. The porous-medium equation then can be written in the form (1.1). Homogenization is then proven by establishing the existence of multiscale limit Young measures associated with the family of solutions $\{u^\epsilon\}$, and then by showing that such measures are actually Dirac masses concentrated at the solution of an homogenized porous-medium type limit problem. In this setting, the compactification of $\mathbb{R}^N$ provided by the algebras with mean value plays a fundamental role.

There is an extensive literature about the homogenization of non-linear first and second order PDE's in periodic and almost periodic settings, starting from the seminal paper [16]. In more recent years, there has been a resurgence of interest in the homogenization problems in the more general setting of random stationary ergodic media, see e.g. [18] and the references therein. The main difficulty when passing from the periodic or almost periodic setting to the stochastic one is given by the lack of compactness of the probability space, see [18].

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Motivated by these results, the goal of this paper is to solve the homogenization problem (1.1) removing the compactness assumption on Ω. We use a different approach than [3, 13], based on the kinetic formulation for the equation (1.1). The notion of kinetic solutions for hyperbolic homogeneous conservation laws has been introduced by Lions, Perthame and Tadmor [17], and then extended by Chen and Perthame [10] to parabolic laws which include, as special case, the homogeneous porous-medium equation $\partial_t u - \Delta f(u) = 0$. In [11], Dalibard defines a notion of kinetic solutions for heterogeneous parabolic conservation laws of type $\partial_t u + \operatorname{div}(A(x, u(x)) - \Delta u = 0$, where $A(x, u)$ is a given flux, periodic in the space variable x. The kinetic formulation is then used to prove periodic homogenization. In this paper, following this idea, we derive a kinetic formulation for the Cauchy problem associated to the heterogeneous porous-medium type equation of the form

$$\partial_t u - \Delta f(x, u) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N.$$

The corresponding kinetic equation involves an additional variable and its distributional solution is a discontinuous function. Nevertheless, the advantage of using it is that the kinetic equation is linear. We can therefore apply the theory of the "stochastically two-scale convergence in the mean" developed in [6] by Bourgeat Mikelic and S. Wright. These theory is an extension to the stochastic setting of the notion of two-scale convergence previously introduced by Allaire [1] in the context of periodic functions.

We show the existence of the kinetic solution of (1.1) and we prove its convergence, as $\epsilon \to 0$, to the kinetic solution of an homogenized porous-medium equation of the form

$$\partial_t \overline{u} - \Delta \overline{f}(\overline{u}) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N$$

with initial condition $\overline{u}(0, x) = \int_\Omega u_0(x, \omega)\, d\mu$, where $\overline{f}$ is defined by the formula

$$v = \int_\Omega g(\omega, \overline{f}(v)) d\mu, \quad v \in \mathbb{R}$$

and $g(\omega, \cdot) = f(\omega, \cdot)^{-1}$. The proof of the convergence result is inspired by the one of the contraction property of kinetic solutions of parabolic conservation laws, see [10]. Going back from the kinetic to the weak solution of (1.1), we prove the convergence of the weak solution u^ϵ of (1.1) to the weak solution of the homogenized problem.

In order to apply the results in [6], we need to require the space $L^2(\Omega)$ to be separable. In the almost periodic case, or more in general in the case of ergodic algebras with mean value, the corresponding $L^2(\Omega)$ is not separable. However in [8] the authors have been able to extend the theory of two-scale convergence to this setting, we will recall their result in Section 3.3. Therefore, the strategy we adopt here to prove homogenization works also in the context of ergodic algebras, providing a different proof of the analogous results in [3, 13] for the case Ω compact.

1.1. **Organization of the paper.** The paper is organized as follows. In Section 2.1 we derive the ansatz for u^ϵ. The main convergence result, Theorem 2.2, is stated in Section 2.2. In Section 3 we collect some preliminary results concerning the ergodic theory and the stochastically two-scale convergence in the mean that will be used later in the paper. The kinetic formulation for (1.1) is then derived in Section 4. Finally, Section 5 is devoted to the proof of Theorem 2.2.

2. Main result

2.1. **The ansatz.** In order to identify the limit of the solutions $u^\epsilon(t, x)$ of (1.1) as $\epsilon \to 0$, following the classical idea of the two-scale expansion (see [5] for a general presentation of this theory), we are looking for an ansatz of the form $U^0\left(t, x, \frac{x}{\epsilon}\right)$, that is a function such that $u^\epsilon(t, x) - U^0\left(t, x, \frac{x}{\epsilon}\right)$ converges to 0 as $\epsilon \to 0$ in some norm. To simplify the presentation, we suppose that we are in periodic setting, i.e., that u^ϵ is solution of

$$\partial_t u^\epsilon = \Delta f\left(\frac{x}{\epsilon}, u^\epsilon\right) \tag{2.1}$$

with $f(y,u)$ periodic in y and increasing in u. We consider the following two-scale expansion for u^ϵ:

$$u^\epsilon(t,x) = U^0\left(t,x,\frac{x}{\epsilon}\right) + \epsilon U^1\left(t,x,\frac{x}{\epsilon}\right) + \epsilon^2 U^2\left(t,x,\frac{x}{\epsilon}\right) + \dots$$

where $U^0(t,x,y)$, $U^1(t,x,y)$ and $U^2(t,x,y)$ are periodic in y functions . Putting this expression in (2.1) and making a Taylor expansion of $f\left(\frac{x}{\epsilon},\cdot\right)$ around U^0, we get

$$\partial_t U^0 - \Delta_x\left[f\left(\frac{x}{\epsilon},U^0\right) + \partial_u f\left(\frac{x}{\epsilon},U^0\right)(\epsilon U^1 + \epsilon^2 U^2) + \frac{1}{2}\partial^2_{uu} f\left(\frac{x}{\epsilon},U^0\right)\epsilon^2 (U^1)^2\right] + O(\epsilon) = 0,$$

where $O(\epsilon)$ contains all the terms multiplied by a power of ϵ greater or equal than 1. Now, identifying the non-positive powers of ϵ, we derive an equation for U^0. The equation corresponding to ϵ^{-2} is the the following

$$\Delta_y(f(y,U^0(t,x,y))) = 0, \tag{2.2}$$

where we denote $y = x/\epsilon$ and, as usual in deriving an ansatz for homegenization problems, we assume x and y to be independent. By the Liouville Theorem all periodic in y solutions $f(y,U^0(t,x,y))$ of (2.2) in $\mathbb{R}^N$ are constant in y, that is

$$f(y,U^0(t,x,y)) = p(t,x)$$

for any function $p(t,x)$ independent of y. We infer that $U^0(t,x,y)$ is of the following form

$$U^0(t,x,y) = g(y,p(t,x))$$

where

$$g(y,\cdot) = f^{-1}(y,\cdot).$$

The equation corresponding to the power ϵ^0 is

$$\begin{aligned}\partial_t U^0 - \Delta_x(f(y,U^0)) - \mathrm{div}_y \nabla_x(\partial_u f(y,U^0)U^1) - \mathrm{div}_x \nabla_y(\partial_u f(y,U^0)U^1)\\ - \Delta_y(\partial_u f(y,U^0)U^2) - \frac{1}{2}\Delta_y(\partial^2_{uu} f(y,U^0)(U^1)^2) = 0.\end{aligned} \tag{2.3}$$

Assuming that the functions are smooth, we get

$$\begin{aligned}\mathrm{div}_x \nabla_y(\partial_u f(y,U^0)U^1) = \sum_{i=1}^N \partial_{x_i}\partial_{y_i}(\partial_u f(y,U^0)U^1) = \sum_{i=1}^N \partial_{y_i}\partial_{x_i}(\partial_u f(y,U^0)U^1)\\ = \mathrm{div}_y \nabla_x(\partial_u f(y,U^0)U^1).\end{aligned}$$

Thus, averaging (2.3) with respect to y and using that, by periodicity,

$$\int_{\mathcal{T}^N} \mathrm{div}_y \nabla_x(\partial_u f(y,U^0)U^1)\,dy = \int_{\mathcal{T}^N} \Delta_y(\partial_u f(y,U^0)U^2)\,dy = \int_{\mathcal{T}^N} \Delta_y(\partial^2_{uu} f(y,U^0)(U^1)^2)\,dy = 0,$$

with $\mathcal{T}^N$ the N-dimensional torus, yields the evolution equation

$$\partial_t \overline{u} - \Delta p(t,x) = 0,$$

where

$$\overline{u}(t,x) := \int_{\mathcal{T}^N} U^0(t,x,y)dy = \int_{\mathcal{T}^N} g(y,p(t,x))dy, \tag{2.4}$$

and we have used that $f(y,U^0(t,x,y)) = p(t,x)$. The computations above suggest to define the function $\overline{f}$ implicitly in the following way, for any given $u \in \mathbb{R}$,

$$u = \int_{\mathcal{T}^N} g(y,\overline{f}(u))dy. \tag{2.5}$$

Then, from (2.4) and (2.5) we have that $p(t,x) = \overline{f}(\overline{u}(t,x))$. We conclude that if $\overline{u}$ is the solution of

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{f}(\overline{u}) = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ u(0,x) = \int_{\mathcal{T}^N} u_0\,(x,y)\,dy & \text{on} \quad \mathbb{R}^N \end{cases}$$

with $\overline{f}$ defined by (2.5), our guess for U^0 is the following

$$U^0\left(t,x,\frac{x}{\epsilon}\right) = g\left(\frac{x}{\epsilon}, \overline{f}(\overline{u}(t,x))\right).$$

2.2. **Assumptions and main result.** Let us now introduce the mathematical assumptions we make, we refer to Section 3 for the main definitions and some preliminary results. Throughout this paper we assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space with $L^2(\Omega)$ separable, and that $\{T(y) : y \in \mathbb{R}^N\}$ is an ergodic N-dimensional dynamical system on $(\Omega, \mathcal{F}, \mu)$. Moreover, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying, for a.e. $\omega \in \Omega$:

- **(H1)** $f(\omega, \cdot)$ is striclty increasing and locally Lipschitz continuous, uniformly in ω. Moreover, $\lim_{u \to \pm\infty} f(\omega, u) = \pm\infty$, uniformly in ω;
- **(H2)** $f(T(\cdot)\omega, u)$ is continuous and $f(\cdot, u) \in L^\infty(\Omega)$ for all $u \in \mathbb{R}$.
- **(H3)** Let $g(\omega, \cdot) := f(\omega, \cdot)^{-1}$, then for all $p \in \mathbb{R}$ and $\frac{\partial g}{\partial p}(T(\cdot)\omega, \cdot) \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R})$ uniformly in ω.

We assume that the initial data u_0 is "well-prepared", that is of the form

- **(H4)** $u_0(x, \omega) = g(\omega, \varphi(x))$, for some $\varphi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $g(\omega, \varphi) - g(\omega, 0) \in L^1(\Omega; L^1(\mathbb{R}^N))$.

Observe that (H1)-(H3) are satisfied by functions of the form (1.2) with γ, a, b bounded stochastic variables, with $\gamma(T(\cdot))$, $a(T(\cdot))$ and $b(T(\cdot))$ continuous, and $\gamma(\omega) \geq \gamma_0 > 0$, $a(\omega) \geq a_0 > 0$ for a.e. $\omega \in \Omega$. Moreover, in this case (H4) is satisfied for any $\varphi \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Under the assumptions (H1)-(H4), even though $\partial_u f$ can be 0 at some point, (1.1) still belongs to the "non-degenerate" class according to the classification of [7]. Well-posedness of (1.1) in the homogeneous case, i.e. when the coefficients do not depend explicitly on (t, x), was established by Carrillo in [7]. The results of [7] have been extended by Frid and Silva in [13] to the case in which f explicitly depends on x and satisfies (H1)-(H2), these results in particular guarantee that for any fixed $\epsilon > 0$ and a.e. $\omega \in \Omega$ there exists a unique unique weak solution $u^\epsilon(t, x, \omega)$ of (1.1) (see Definition 4.3 for the definition of the weak solution to (2.8)). Assumptions (H3) and (H4) are needed in order to make sense to the kinetic formulation of (1.1) given in Section 4.2.

A L^1 well-posedness theory for the homogeneous anisotropic case, that is for non-diagonal viscosity matrices was established by Chen and Perthame [10] and succesiveley extended to the non-homogeneous case in [9].

Let $\overline{g}$ be the function defined by

$$\overline{g}(p) = \int_\Omega g(\omega, p) d\mu, \quad p \in \mathbb{R}. \tag{2.6}$$

Since $g(\omega, \cdot)$ is strictly increasing, $\overline{g}$ has inverse $\overline{f} := \overline{g}^{-1}$ implicitly defined by the equation

$$v = \int_\Omega g(\omega, \overline{f}(v)) d\mu, \quad v \in \mathbb{R}. \tag{2.7}$$

Lemma 2.1. *Let $\overline{f} : \mathbb{R} \to \mathbb{R}$ be defined by (2.7). Then $\overline{f}$ is strictly increasing and locally Lipschitz continuous in $\mathbb{R}$.*

For the proof of Lemma 2.1 we refer to the proof of Lemma 6.1 in [13]. By the results of [7] and Lemma 2.1, there exists a unique weak solution of the Cauchy problem

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{f}(\overline{u}) = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \\ \overline{u}(0,x) = \int_\Omega u_0\,(x,\omega)\,d\mu & \text{on} \quad \mathbb{R}^N. \end{cases} \tag{2.8}$$

We are now ready to state our main result.

Theorem 2.2. *Let $\overline{u}$ be the unique weak solution of* (2.8)*, where $\overline{f}$ is defined by* (2.7)*. Set*

$$U(t,x,\omega) := g(\omega, \overline{f}(\overline{u}(t,x))),$$

then as $\epsilon \to 0$, we have

$$\int_\Omega \left\| u^\epsilon(t,x,\omega) - U\left(t,x,T\left(\frac{x}{\epsilon}\right)\omega\right) \right\|_{L^1_{loc}(\mathbb{R}^+\times\mathbb{R}^N)} d\mu \to 0, \tag{2.9}$$

and $\int_\Omega u^\epsilon d\mu \to \overline{u}$ in the weak star topology of $L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$.

3. Preliminary results

3.1. **Ergodic theory.** Let us recall some basic facts about the ergodic theory that will be needed in the next sections, we refer to the book [15] for a more complete presentation.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

Definition 3.1. *An N-dimensional dynamical system on $(\Omega, \mathcal{F}, \mu)$ is a family of maps $T(y) : \Omega \to \Omega$, $y \in \mathbb{R}^N$, which satisfies the following conditions:*

(i) **(Group property)** *$T(0) = I$, where I is the identity map on Ω, and $T(y+z) = T(y)T(z)$, $\forall y, z \in \mathbb{R}^N$;*

(ii) **(Invariance)** *The maps $T(y) : \Omega \to \Omega$ are measurable and $\mu(T(y)E) = \mu(E)$, $\forall y \in \mathbb{R}^N$, $\forall E \in \mathcal{F}$;*

(iii) **(Measurability)** *Given any $F \in \mathcal{F}$ the set $\{(y,\omega) \in \mathbb{R}^N \times \Omega : T(y)\omega \in F\} \subset \mathbb{R}^N \times \Omega$ is measurable with respect to the σ-algebra product $\mathcal{L}_N \otimes \mathcal{F}$, where $\mathcal{L}_N$ is the σ-algebra of the Lebesgue measurable sets of $\mathbb{R}^N$.*

Definition 3.2 (Ergodic N-dimensional dynamical system)**.** *A $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$ is called invariant if $f(T(y)\omega) = f(\omega)$ μ-almost everywhere in Ω, for all $y \in \mathbb{R}^N$. A dynamical system is said to be ergodic if every invariant function is μ-equivalent to a constant in Ω.*

Definition 3.3 (Stationary process)**.** *A stochastic process $\tilde{F} : \mathbb{R}^N \times \Omega \to \mathbb{R}$ is called stationary if*

$$\tilde{F}(y+y', \omega) = \tilde{F}(y, T(y')\omega) \quad \text{for all } y, y' \in \mathbb{R}^N \text{ and a.e. } \omega \in \Omega.$$

If the N-dimensional dynamical system $\{T(y) : y \in \mathbb{R}^N\}$ is ergodic, then $\tilde{F}$ is said stationary ergodic.

Remark 3.4. *It is easily checked that a stochastic process $\tilde{F} : \mathbb{R}^N \times \Omega \to \mathbb{R}$ is stationary if and only if there exists a stochastic variable $F : \Omega \to \mathbb{R}$ such that*

$$\tilde{F}(y,\omega) = F(T(y)\omega).$$

Given the probability space $(\Omega, \mathcal{F}, \mu)$, as usual, for $1 \le p < +\infty$, let us denote by $L^p(\Omega) = L^p(\Omega, \mu)$ be the space of the equivalent classes of measurable functions $g : \Omega \to \mathbb{R}$ such that $|g|^p$ is μ-integrable on Ω, and by $L^\infty(\Omega) = L^\infty(\Omega, \mu)$ the space of μ-essentially bounded measurable functions. Let $T(y)$, $y \in \mathbb{R}^N$, be an N-dimensional dynamical system on $(\Omega, \mathcal{F}, \mu)$. If $g \in L^p(\Omega)$, then almost all its realizations $g(T(y)\omega)$ belong to $L^p_{loc}(\mathbb{R}^N)$. Moreover, $T(y)$ induces a group $\{U(y) : y \in \mathbb{R}^N\}$ of unitary operators on $L^2(\Omega)$ defined by

$$(U(y)h)(\omega) = h(T(y)\omega), \quad y \in \mathbb{R}^N, \, \omega \in \Omega, \, h \in L^2(\Omega)$$

which turns out to be strongly continuous in $L^2(\Omega)$.

Let $D_1, ..., D_N$ denote the infinitesimal generators of the group with $\mathcal{D}_1, ..., \mathcal{D}_N$ their respective domains in $L^2(\Omega)$, i.e., for $h \in \mathcal{D}_i$

$$(D_i h)(\omega) := \lim_{\substack{y_i \neq 0,\, y_i \to 0 \\ y_j = 0,\, j \neq i}} \frac{h(T(y)\omega) - h(\omega)}{y_i}, \quad i = 1, \ldots, N$$

in the sense of convergence in $L^2(\Omega)$. Then, for $h \in \mathcal{D}_i$, for a.e. $\omega \in \Omega$, the realization $h(T(y)\omega)$ possesses a weak derivative $\partial_{y_i}(h(T(y)\omega)) \in L^2_{loc}(\mathbb{R}^N)$ and the following equality holds

$$(D_i h)(T(y)\omega) = \partial_{y_i}(h(T(y)\omega)) \quad \text{for a.e. } y \in \mathbb{R}^N. \tag{3.1}$$

The unitary of the group $\{U(y) : y \in \mathbb{R}^N\}$ implies that the operators D_i are skew-symmetric, i.e., for $h,\, g \in \mathcal{D}_i$ we have

$$\int_\Omega D_i h g d\mu = -\int_\Omega h D_i g d\mu \quad i = 1, \dots, N.$$

Define $\mathcal{D}(\Omega) = \cap_{i=1}^N \mathcal{D}_i$ and

$$D^\infty(\Omega) = \{h \in L^\infty(\Omega) \cap \mathcal{D}(\Omega) \, : \, D^\alpha h \in L^\infty(\Omega) \cap \mathcal{D}(\Omega), \text{ for all multi-indeces } \alpha\}. \tag{3.2}$$

For a function $h \in L^2(\Omega)$, the stochastic weak derivative $D^\alpha f$ of f is the linear functional on $\mathcal{D}^\infty(\Omega)$ defined by

$$(D^\alpha f)\varphi = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi d\mu, \quad \varphi \in \mathcal{D}^\infty(\Omega).$$

The following result is proven in [6].

Lemma 3.5 ([6], Lemma 2.3)**.** *Assume the dynamical system $\{T(y) : y \in \mathbb{R}^N\}$ to be ergodic and $L^2(\Omega)$ separable. Let $h \in L^2(\Omega)$ such that $D_i h = 0$ for any $i = 1, \dots, N$, then h is μ-equivalent to a constant in Ω.*

Using Lemma 3.5, we can prove the following Liouville type result that will be needed in Section 5.

Lemma 3.6. *Assume the dynamical system $\{T(y) : y \in \mathbb{R}^N\}$ to be ergodic and $L^2(\Omega)$ separable. Let $h \in L^\infty(\Omega) \cap L^2(\Omega)$ such that $\Delta h = 0$, then h is μ-equivalent to a constant in Ω.*

Proof. Let us introduce a smooth approximation of h. A classical way to do it consists in introducing an even function K such that

$$K \in C_0^\infty(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} K(z) dz = 1, \quad K \geq 0,$$

and set

$$h^\delta(\omega) = \int_{\mathbb{R}^N} K_\delta(z) h(T(z)\omega) dz,$$

where $K_\delta(z) = \delta^{-N} K(\delta^{-1} z)$. It turns out that $h^\delta \in D^\infty(\Omega)$, $U(y)h^\delta$ is infinitely differentiable as a function of $y \in \mathbb{R}^N$ and

$$\lim_{\delta \to 0} \|h^\delta - h\|_{L^2(\Omega)} = 0,$$

see [15]. Moreover h^δ satisfies

$$\int_\Omega \Delta\varphi(\omega) h^\delta(\omega) d\mu = -\sum_{i=1}^n \int_\Omega D_i \varphi(\omega) D_i h^\delta(\omega) d\mu = 0$$

for any $\varphi \in D^\infty(\Omega)$. Lemma 3.5 then implies that $D_i h^\delta(\omega)$ is equivalent to a constant in Ω. In particular, for a.e. $\omega \in \Omega$ and any $y \in \mathbb{R}^N$, $\partial_{y_i}(h^\delta(T(y)\omega)) = (D_i h^\delta)(T(y)\omega)$ is constant. Since in addition $h^\delta \in L^\infty(\Omega)$, we infer that $\partial_{y_i}(h^\delta(T(y)\omega)) = 0$ for a.e. $\omega \in \Omega$, i.e., $h^\delta(T(y)\omega) = h^\delta(\omega)$ for a.e. $\omega \in \Omega$ and every $y \in \mathbb{R}^N$. The ergodicity of the dynamical system $\{T(y) : y \in \mathbb{R}^N\}$ then implies that h^δ is equivalent to a constant in Ω. Passing to the limit as $\delta \to 0$ we conclude that h is equivalent to a constant in Ω. $\square$

3.2. **Stochastically two-scale convergence in the mean.** Following an idea of Nguetseng [19], Allaire [1] defined the notion of two-scale convergence in the periodic setting. Bounded sequence in $L^2(Q)$, where Q is a given domain, are proven to be relatively compact with respect to this type of convergence. The notion of two scale convergence is useful for the homogenization of partial differential equation with periodically oscillating coefficients. In order to threat equations with random coefficients, in [6] Bourgeat et al. extend this theory from the periodic to the stochastic setting, introducing the concept of "stochastically two-scale" convergence in the mean. They prove the following:

Theorem 3.7 ([6], Theorem 3.4)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space such that $L^2(\Omega)$ is separable and let $\{T(y) : y \in \mathbb{R}^N\}$ be a N-dynamical system. Let Q be an open set of $\mathbb{R}^N$ and let $\{w^\epsilon\}$ be a bounded sequence in $L^2(Q \times \Omega)$. Then there exists a subsequence, still denoted by $\{w^\epsilon\}$, and a function $w_0 \in L^2(Q \times \Omega)$ such that*

$$\lim_{\epsilon \to 0} \int_{Q \times \Omega} w^\epsilon(x, \omega)\psi\left(x, T\left(\frac{x}{\epsilon}\right)\omega\right) dx d\mu = \int_{Q \times \Omega} w_0(x, \omega)\psi(x, \omega) dx d\mu$$

for any function ψ such that $\psi(x, T(x)\omega)$ defines an element of $L^2(Q \times \Omega)$. Such a sequence $\{w^\epsilon\}$ is said to "stochastically two-scale" converge in the mean to $w_0(x, y)$.

Remark 3.8. *Not for every element $\psi \in L^2(Q\times\Omega)$, $\psi(x, T(x)\omega)$ defines an element of $L^2(Q\times\Omega)$ but if for example $\psi(x, \omega) = g(x)h(\omega)$ with $g \in L^2(Q)$ and $h \in L^2(\Omega)$, then $(x, \omega) \to \psi(x, T(x)\omega)$ belongs to $L^2(Q \times \Omega)$, see* [6].

We will need to apply the previous result to sequence of functions belonging to $L^\infty(Q \times \Omega)$. With a minor modification of the proof given in [6], the concept of "stochastically two-scale" convergence can be extended to L^∞ functions.

Proposition 3.9. *Under the same assumptions of Theorem 3.7, let $\{w^\epsilon\}$ be a bounded sequence in $L^\infty(Q \times \Omega)$. Then there exists a subsequence, still denoted by $\{w^\epsilon\}$, and a function $w_0 \in L^\infty(Q \times \Omega)$ such that*

$$\lim_{\epsilon \to 0} \int_{Q \times \Omega} w^\epsilon(x, \omega)\psi\left(x, T\left(\frac{x}{\epsilon}\right)\omega\right) dx d\mu = \int_{Q \times \Omega} w_0(x, \omega)\psi(x, \omega) dx d\mu$$

for any function ψ such that $\psi(x, T(x)\omega)$ defines an element of $L^1(Q \times \Omega)$.

3.3. **Ergodic algebras with mean value.** In this subsection we recall the Bohr compactification of the set of almost periodic function on $\mathbb{R}^N$ and more in general of ergodic algebras with mean value. We will then present the two-scale convergence result proven [8].

The set of almost periodic functions on $\mathbb{R}^N$, here denoted by $AP(\mathbb{R}^N)$, is a linear subspace of the space of bounded uniformly continuous functions on $\mathbb{R}^N$, that forms an algebra with mean value. This means that $AP(\mathbb{R}^N)$ satisfies the following conditions:

a) if f, $g \in AP(\mathbb{R}^N)$ then $fg \in AP(\mathbb{R}^N)$;
b) $AP(\mathbb{R}^N)$ with the uniform convergence topology is complete;
c) the constant functions belong to $AP(\mathbb{R}^N)$;
d) $AP(\mathbb{R}^N)$ is invariant under the translations $\tau_y : \mathbb{R}^N \to \mathbb{R}^N$, $\tau_y(x) = x + y$, $y \in \mathbb{R}^N$, that is if $f \in AP(\mathbb{R}^N)$ then $f(\tau_y(\cdot)) \in AP(\mathbb{R}^N)$;
e) any element $f \in AP(\mathbb{R}^N)$ possesses a mean value, that is there exists a number $M(f)$ such that

$$M(f) = \lim_{\epsilon \to 0} \frac{1}{|A|} \int_A f\left(\frac{x}{\epsilon}\right) dx$$

for any Lebesgue measurable bounded set $A \subset \mathbb{R}^N$.

The Besicovitch space of order p, with $1 \le p < +\infty$, denoted by B^p is defined as the closure of $AP(\mathbb{R}^N)$ for the seminorm

$$[f]_p := (M(|f|^p))^{\frac{1}{p}}.$$

The Besicovitch space of order ∞, B^∞, is defined by

$$B^\infty := \{f \in B^1 \,|\, [f]_\infty := \sup_{p \ge 1} [f]_p < +\infty\}.$$

The spaces B^p are seminormed spaces. The quotient of B^p with the kernel of $[\cdot]_p$, denoted by $\mathcal{B}^p$, is a normed space. It is well known, see [12] and [2], that there exists a compact space $\mathbb{G}^N$, called Bohr compactification of $AP(\mathbb{R}^N)$, and an isometric isomorphism $i : AP(\mathbb{R}^N) \to C(\mathbb{G}^N)$ identifying $AP(\mathbb{R}^N)$ with the algebra $C(\mathbb{G}^N)$ of the continuous functions on $\mathbb{G}^N$. If $\mathfrak{m}$ is the Haar measure on $\mathbb{G}^N$ normalized to be a probability measure, then

$$\int_{\mathbb{G}^N} f \, d\mathfrak{m} = M(f).$$

Moreover, the translations τ_y induce a family of homeomorphisms $T(y) : \mathbb{G}^N \to \mathbb{G}^N$, $y \in \mathbb{R}^N$, which is an ergodic continuous N-dimensional dynamical system on $(\mathbb{G}^N, \mathcal{G}, \mathfrak{m})$, with $\mathcal{G}$ the σ-algebra of Borel sets on $\mathbb{G}^N$. Finally, the space $\mathcal{B}^p$, $1 \le p \le +\infty$, is isometrically isomorphic to $L^p(\mathbb{G}^N, \mathfrak{m})$.

More in general, if $\mathcal{A}$ is an algebra with mean value, i.e., satisfies (a)-(e) and $\mathcal{B}^p$ are the generalized Besicovitch spaces associated to $\mathcal{A}$, then

Theorem 3.10 ([3], Theorem 4.1)**.** *The following holds:*

- *i) There exist a compact space K and an isometric isomorphism i identifying $\mathcal{A}$ with the algebra $C(K)$ of continuous functions on K.*
- *ii) The translations τ_y induce a family of homeomorphisms $T(y) : K \to K$, $y \in \mathbb{R}^N$, which is a continuous N-dimensional dynamical system.*
- *iii) The mean value on $\mathcal{A}$ extends to a Radon probability measure $\mathfrak{m}$ on K defined by, for $f \in \mathcal{A}$,*

$$\int_K f \, d\mathfrak{m} = M(f),$$

which is invariant by the group of homeomorphisms $T(y)$.

- *iv) For $1 \le p \le +\infty$, the Besicovitch space $\mathcal{B}^p$ is isometrically isomorphic to $L^p(K, \mathfrak{m})$.*

An algebra with mean value is called ergodic if any function belonging to $\mathcal{B}^2$ and invariant with respect to τ_y is equivalent (in $\mathcal{B}^2$) to a constant. In this case the N-dynamical system given in Theorem 3.10 is ergodic. Lemma 3.5 for ergodic algebras with mean value is proven in [3] (see Lemma 3.2).

Thanks to the Theorem 3.10, equation

$$\partial_t u = \Delta f\left(\frac{x}{\epsilon}, u\right)$$

when $f(\cdot, u)$ is almost periodic or more in general belongs to a linear algebra with mean value, can be written as in (1.1) by setting

$$\Omega := K, \quad \mu := \mathfrak{m}$$

and $T(y)$, $y \in \mathbb{R}^N$, the N-dynamical system induced on K by the translations τ_y of $\mathbb{R}^N$. However, we cannot apply Theorem 3.7 and Proposition 3.9 in this framework as the Besicovitch spaces are in general not separable. In [8] the authors were able to overcome this difficulty and extend the theory of two-scale convergence to generalized Besicovitch spaces, see Definition 4.1 there for the notion of two-scale convergence in this setting.

Theorem 3.11 ([8], Theorem 4.10). *Let Q be an open set of $\mathbb{R}^N$ and let $\{w^\epsilon\}$ be a bounded sequence in $L^p(Q \times \Omega)$, $1 < p \leq \infty$. Then there exists a subsequence, still denoted by $\{w^\epsilon\}$, and a function $w_0 \in L^p(Q; \mathcal{B}^p)$ such that $\{w^\epsilon\}$ two-scale converges to w_0.*

4. The Kinetic formulation

In this section we derive a kinetic formulation for the heterogeneous porous-medium equation (1.1) that will be used in the proof of Theorem 2.2. Let us start by recalling some classical results.

4.1. **Homogeneous porous-medium type equations.** The notion of kinetic solutions for hyperbolic homogeneous conservations laws has been introduced by Lions, Perthame and Tadmor [17], and then extended by Chen and Perthame [10] to parabolic laws that include, as special case, the homogeneous isotropic porous-medium equation

$$\partial_t u - \Delta f(u) = 0 \tag{4.1}$$

(see also [21]). The formulation can be derived from the Kruzkhov's inequalities. Formally, if we multiply the equation by $S'(u)$, where S is a C^2 function, we find the following equation

$$\partial_t(S(u)) - \operatorname{div}(f'(u)\nabla(S(u))) = -S''(u)f'(u)|\nabla u|^2.$$

The choice $S(u) = (u - v)_+$ for any $v \in \mathbb{R}$ as a limiting case of C^2 functions, gives the entropy inequalities

$$\partial_t(u(t,x) - v)_+ - \Delta(f(u(t,x)) - f(v))_+ = -m$$

with

$$m(t,x,v) := \delta_u(v)f'(u)|\nabla u|^2$$

where $\delta_u(v)$ is the Dirac masse at $v = u$. Differentiating with respect to v the previous equation, we find

$$\partial_t\chi_+ - f'(v)\Delta\chi_+ = \frac{\partial}{\partial v}m \tag{4.2}$$

where

$$\chi_+(t,x,v) := \mathbf{1}_{\{u(t,x)>v\}},$$

and $\mathbf{1}_A$ denotes the indicator function of the set A. The same kind of equation holds for $\chi_-(t,x,v) := \mathbf{1}_{\{u(t,x)<v\}}$:

$$\partial_t\chi_- - f'(v)\Delta\chi_- = -\frac{\partial}{\partial v}m.$$

The function that occurs in the kinetic formulation is the function $\chi : \mathbb{R}^2 \to \{-1, 0, 1\}$ defined by

$$\chi(v,u) = \begin{cases} 1 & \text{for} \quad 0 < v < u, \\ -1 & \text{for} \quad u < v < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since (4.2) is linear, we see, at least formally, that the function

$$\chi(t,x,v) := \chi(v, u(t,x)) = \mathbf{1}_{v>0}\chi_+(t,x,v) - \mathbf{1}_{v<0}\chi_-(t,x,v)$$

is still solution of (4.2). We are now ready to give the definition of kinetic solution for (4.1) with initial condition

$$u(0,x) = u_0(x) \in \mathrm{L}^1(\mathbb{R}^N). \tag{4.3}$$

Definition 4.1 (Definition 2.2, [10]). *A kinetic solution of (4.1), (4.3) is a function $u \in L^\infty([0,\infty); L^1(\mathbb{R}^N))$ such that*

(i) For any $\xi \in C_c^\infty(\mathbb{R})$, $\xi(u)f'(u)|\nabla u|^2 \in L^1([0,\infty) \times \mathbb{R}^N)$;

(ii) $\chi(t,x,v)=\chi(v,u(t,x))$ *satisfies* (4.2) *in the sense of distributions in* $[0,+\infty)\times\mathbb{R}^N\times\mathbb{R}$, *with initial data* $\chi(0,x,v)=\chi(v,u_0(x))$;

(iii) *If* n *is the positive measure on* $\mathbb{R}$ *defined by*

$$\int_{\mathbb{R}}\xi(v)dn(v):=\int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}}\xi(v)m(t,x,v)\,dt\,dx$$

for $\xi\in C_c(\mathbb{R})$, *then there exists* $\eta\in L^\infty(\mathbb{R})$ *such that* $\eta\to 0$ *as* $|v|\to\infty$ *and*

$$n\le\eta$$

in the sense of distributions in $\mathbb{R}$

Remark that a new real-valued variable, denoted by v, has been added in the kinetic formulation in order to make sense to the derivative $\frac{\partial}{\partial v}m$. In [10] is shown that the notion of kinetic solution is well posed in L^1:

Theorem 4.2 ([10], Theorem 1.2)**.** *Assume* $f\in W^{1,\infty}_{loc}(\mathbb{R}^N)$, $f'\ge 0$ *in* $\mathbb{R}$ *and* $u_0\in L^1(\mathbb{R}^N)$. *Then, there exists a unique kinetic solution* $u\in C([0,\infty);L^1(\mathbb{R}^N))$ *for the Cauchy problem* (4.1), (4.3).

If the initial data belongs to $L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, then the notion of kinetic solution is equivalent to the one of entropy solution, see [10] for the definition of entropy solution and the proof of the equivalence result. However the former is more general than the latter since is well defined in the L^1-setting. The L^1-stability of L^∞-entropy solutions of porous-medium type equations was already proven by Carrillo [7]. In the paper is also shown that if f has continuous inverse, then any weak solution of (4.1), (4.3) is also an entropy (and then a kinetic) solution.

4.2. **Heterogeneous porous-medium type equations.** In [11], Dalibard defines a notion of kinetic solutions for parabolic conservation laws of type $\partial_t u+\operatorname{div}(A(x,u(x))-\Delta u=0$. In the homogeneous case, constants are stationary solutions, while they no longer play a special role in the context of heterogeneous conservation laws. Starting from this remark, already pointed out in [4], she derives a definition of kinetic solution taking in the entropy inequalities $S(u)=(u-v(x))_+$ with v stationary solution.

In this paper, following this idea, we obtain a kinetic formulation for the heterogeneous porous-medium type equations of the form

$$\partial_t u-\Delta f(x,u)=0\quad\text{in }\mathbb{R}^+\times\mathbb{R}^N,\tag{4.4}$$

with initial data

$$u(0,x)=u_0(x)\quad\text{on }\mathbb{R}^N.\tag{4.5}$$

Throughout this section on f we assume:

(f1) $f(x,\cdot)$ is strictly increasing and locally Lipschitz continuous uniformly in x. Moreover, $\lim_{u\to\pm\infty}f(x,u)=\pm\infty$, uniformly in x;

(f2) $f(\cdot,u)$ is continuous and bounded for all $u\in\mathbb{R}$.

Let us recall the notion of weak solution of the Cauchy problem (4.4), (4.5).

Definition 4.3. *A function* $u\in L^\infty(\mathbb{R}^+\times\mathbb{R}^N)$ *is said to be a weak solution of* (4.4), (4.5) *if the following holds:*

(i) $f(x,u(t,x))\in L^2_{loc}(\mathbb{R}^+;H^1_{loc}(\mathbb{R}^N))$;

(ii) *for any* $\phi\in C^\infty_c([0,\infty)\times\mathbb{R}^N)$, *we have*

$$\int_{\mathbb{R}^+\times\mathbb{R}^N}[u\phi_t-\nabla f(x,u)\cdot\nabla\phi]\,dt\,dx+\int_{\mathbb{R}^N}u_0\phi(0,x)\,dx=0.$$

The existence of a unique weak solution of (4.4), (4.5) is proven in [13] under the assumptions (f1) and (f2), see also [3]. Adapting the techniques of [7] in order to handle the explicit dependence on x of f, the authors also show that weak solutions of (4.4) satisfy an L^1-stability property. These results are recalled in the following:

Theorem 4.4 ([13], Theorem 5.1)**.** *Assume (f1), (f2) and $u_0 \in L^\infty(\mathbb{R}^N)$, then we have the following:*

(i) There exists a unique weak solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N) \cap C([0,+\infty); L^1_{loc}(\mathbb{R}^N))$ of (4.4), (4.5).

(ii) If u_1, u_2 are weak solutions of (4.4) with initial data respectively u_{01}, $u_{02} \in L^\infty(\mathbb{R}^N)$, then for all $\phi \in C_c^\infty([0,\infty) \times \mathbb{R}^N)$, $\phi \geq 0$, we have

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} [(u_1(t,x) - u_2(t,x))_+\phi_t + (f(x,u_1(t,x)) - f(x,u_2(t,x)))_+\Delta\phi]\, dt\, dx$$
$$+ \int_{\mathbb{R}^N} (u_{01}(x) - u_{02}(x))_+\phi(0,x)\, dx \geq 0.$$

Moreover (ii) holds true also with the positive part replaced by the negative part.

If u_2 is stationary, inequality (ii) of Theorem 4.4 is a consequence of the next lemma we are going to state and which is proven in [13] (see the proof of Theorem 5.1). The lemma is a central tool in our analysis in order to get a kinetic formulation for the Cauchy problem (4.4), (4.5). Let us first introduce some notation. Let $H_\sigma : \mathbb{R} \to \mathbb{R}$ be the approximation of the Heaviside function given by

$$H_\sigma(s) := \begin{cases} 1, & \text{for} \quad s > \sigma, \\ \frac{s}{\sigma}, & \text{for} \quad 0 < s \leq \sigma, \\ 0, & s \leq 0. \end{cases} \tag{4.6}$$

Moreover, for $k \in \mathbb{R}$, let us define

$$B_\sigma^k(x,\lambda) := \int_k^\lambda H_\sigma(f(x,r) - f(x,k))dr.$$

Lemma 4.5. *Assume (f1), (f2) and let u_1, u_2 be weak solutions of (4.4) with initial data respectively u_{01}, $u_{02} \in L^\infty(\mathbb{R}^N)$. Assume that $u_2 = u_{02}$ is a stationary solution. Then, for all $\phi \in C_c^\infty([0,+\infty) \times \mathbb{R}^N)$ we have*

$$\begin{aligned} &- \int_{\mathbb{R}^+\times\mathbb{R}^N} B_\sigma^{u_2(x)}(x,u_1(t,x))\phi_t\, dt\, dx - \int_{\mathbb{R}^N} B_\sigma^{u_2(x)}(x,u_{01}(x))\phi(0,x))\, dx \\ &+ \int_{\mathbb{R}^+\times\mathbb{R}^N} H_\sigma(f(x,u_1(t,x)) - f(x,u_2(x)))\nabla[f(x,u_1(t,x)) - f(x,u_2(x))] \cdot \nabla\phi\, dt\, dx \\ &= - \int_{\mathbb{R}^+\times\mathbb{R}^N} |\nabla[f(x,u_1(t,x)) - f(x,u_2(x))]|^2 H'_\sigma(f(x,u_1(t,x)) - f(x,u_2(x)))\phi\, dt\, dx. \end{aligned} \tag{4.7}$$

Let $g(x,\cdot) := f^{-1}(x,\cdot)$, then by (f1) and (f2), one can easily show that $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous, $g(\cdot,p) \in L^\infty(\mathbb{R}^N)$ for any fixed $p \in \mathbb{R}$, and $\lim_{p\to\pm\infty} g(x,p) = \pm\infty$ uniformly in x. Moreover $p = f(x,g(x,p)) \in L^2_{loc}(\mathbb{R}^+; H^1_{loc}(\mathbb{R}^N))$. Thus, for any $p \in \mathbb{R}$, the function

$$v(x,p) := g(x,p), \tag{4.8}$$

is a stationary solution of (4.4). We can therefore apply identity (4.7) with $u_1(t,x) = u(t,x)$ the weak solution of (4.4), (4.5), and $u_2(x) = v(x,p)$. The entropy formulation for (4.4), given in the next proposition, is obtained by passing to the limit as $\sigma \to 0$. In order to make sense to the limit of the right-hand side of (4.7), we have to consider $p \in \mathbb{R}$ as a new real-valued variable.

Proposition 4.6. *Assume (f1), (f2) and $u_0 \in L^\infty(\mathbb{R}^N)$. Let u be the weak solution of* (4.4), (4.5) *and $v(x,p)$ be defined as in* (4.8)*, then we have*

$$\partial_t(u(t,x) - v(x,p))_+ - \Delta(f(x,u) - p)_+ = -m, \tag{4.9}$$

in the sense of distributions in $[0,\infty) \times \mathbb{R}^N \times \mathbb{R}$, where

$$m(t,x,p) = |\nabla f(x,u)|^2 \delta_{f(x,u)}(p)$$

is a nonnegative measure on $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$.

Proof. We have to show that for any $\psi \in C_c^\infty([0,+\infty) \times \mathbb{R}^N \times \mathbb{R})$,

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}} \{-(u(t,x) - v(x,p))_+\psi_t + \nabla[(f(x,u(t,x)) - p)_+] \cdot \nabla\psi\}\, dt\, dx\, dp \\
&- \int_{\mathbb{R}^N\times\mathbb{R}} (u_0(x) - v(x,p))_+\psi(0,x,p)\, dx\, dp \\
&= -\int_{\mathbb{R}^+\times\mathbb{R}^N} |\nabla f(x,u(t,x))|^2\psi(t,x,f(x,u))\, dt\, dx.
\end{aligned} \tag{4.10}$$

It suffices to show (4.10) for $\psi(t,x,\xi) = \phi(t,x)\xi(p)$ with $\phi \in C_c^\infty([0,+\infty)\times\mathbb{R}^N)$ and $\xi \in C_c^\infty(\mathbb{R})$. Applying (4.7) with $u_1(t,x) = u(t,x)$, $u_2(x) = v(x,p)$ and then integrating in p, we find

(4.11)
$$\begin{aligned}
&-\int_{\mathbb{R}} \xi(p) \int_{\mathbb{R}^+\times\mathbb{R}^N} B_\sigma^{v(x,p)}(x,u(t,x))\phi_t\, dt\, dx\, dp - \int_{\mathbb{R}} \xi(p) \int_{\mathbb{R}^N} B_\sigma^{v(x,p)}(x,u_0(x))\phi(0,x)\, dx\, dp \\
&+ \int_{\mathbb{R}} \xi(p) \int_{\mathbb{R}^+\times\mathbb{R}^N} H_\sigma(f(x,u(t,x)) - f(x,v(x,p)))\nabla[f(x,u(t,x)) - f(x,v(x,p))] \cdot \nabla\phi\, dt\, dx\, dp \\
&= -\int_{\mathbb{R}} \xi(p) \int_{\mathbb{R}^+\times\mathbb{R}^N} |\nabla[f(x,u(t,x)) - f(x,v(x,p))]|^2 H'_\sigma(f(x,u(t,x)) - f(x,v(x,p)))\phi\, dt\, dx\, dp.
\end{aligned}$$

Remind that $f(x,v(x,p)) = p$. Moreover

$$H'_\sigma(f(x,u) - f(x,v)) = \begin{cases} \frac{1}{\sigma}, & \text{for} \quad f(x,u) - \sigma < p < f(x,u), \\ 0, & \text{for} \quad p < f(x,u) - \sigma \text{ or } p > f(x,u). \end{cases}$$

Hence the right-hand side of (4.11) is equal to the following quantity

$$-\int_{\mathbb{R}^+\times\mathbb{R}^N} |\nabla f(x,u(t,x))|^2\phi(t,x) \int_{f(x,u)-\sigma}^{f(x,u)} \frac{\xi(p)}{\sigma} dpdtdx.$$

Passing to the limit as $\sigma \to 0$ in (4.11), we finally get (4.10). □

We next consider initial data of the form

$$u_0(x) = v(x,\varphi(x))$$

with $\varphi(x) \in L^\infty(\mathbb{R}^N)$. By assumptions (f1) (f2), we have that $v(x,\varphi(x)) \in L^\infty(\mathbb{R}^N)$ if $\varphi(x) \in L^\infty(\mathbb{R}^N)$.

Lemma 4.7. *Assume (f1) (f2) and let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N) \cap C([0,+\infty); L^1_{loc}(\mathbb{R}^N))$ be the weak solution of* (4.4), (4.5) *with $u_0(x) = v(x,\varphi(x))$ for some function $\varphi \in L^\infty(\mathbb{R}^N)$, and $v(x,p)$ defined as in* (4.8)*. Assume in addition that*

$$v(x,\varphi(x)) - v(x,0) \in L^1(\mathbb{R}^N). \tag{4.12}$$

Then the following holds:

(i) For all $t \geq 0$ and $p \geq 0$

$$\int_{\mathbb{R}^N} (u(t,x) - v(x,p))_+ dx \leq \int_{\mathbb{R}^N} (v(x,\varphi(x)) - v(x,p))_+ dx < \infty;$$

for all $t \geq 0$ and $p \leq 0$

$$\int_{\mathbb{R}^N} (u(t,x) - v(x,p))_- dx \leq \int_{\mathbb{R}^N} (v(x,\varphi(x)) - v(x,p))_- dx < \infty.$$

(ii) If $p_1 < 0 < p_2 \in \mathbb{R}$ are such that $p_1 \leq \varphi(x) \leq p_2$ for any $x \in \mathbb{R}^N$, then

$$v(x,p_1) \leq u(t,x) \leq v(x,p_2) \quad \textit{for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N.$$

(iii) Let η be the positive measure on $\mathbb{R}$ defined by

$$\int_{\mathbb{R}} \xi(p) d\eta(p) := \int_{\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}} \xi(p) m(t,x,p) dt dx,$$

for $\xi \in C_c(\mathbb{R})$, then

$$\eta \leq \eta_0 \quad \textit{in } \mathcal{D}'(\mathbb{R})$$

where

$$\eta_0 := \mathbf{1}_{\{p>0\}} \|(v(x,\varphi(x)) - v(x,p))_+\|_{L^1(\mathbb{R}^N)} + \mathbf{1}_{\{p<0\}} \|(v(x,\varphi(x)) - v(x,p))_-\|_{L^1(\mathbb{R}^N)}.$$

Proof. For $p \geq 0$, we have that

$$(v(x,\varphi(x)) - v(x,p))_+ \in L^1(\mathbb{R}^N). \tag{4.13}$$

Indeed, by the monotonicity of $v(x,\cdot)$ and (4.12),

$$0 \leq (v(x,\varphi(x)) - v(x,p))_+ = (v(x,\varphi(x)) - v(x,p)) \mathbf{1}_{\{\varphi(x)>p\}} \leq (v(x,\varphi(x)) - v(x,0))_+ \in L^1(\mathbb{R}^N).$$

Similarly, one can prove that for $p \leq 0$,

$$(v(x,\varphi(x)) - v(x,p))_- \in L^1(\mathbb{R}^N). \tag{4.14}$$

Then, we apply (ii) of Theorem 4.4 with $u_1(t,x) = u(t,x)$, $u_2(t,x) = v(x,p)$, $u_{01}(x) = v(x,\varphi(x))$, $u_{02}(x) = v(x,p)$ with $p \geq 0$. By using that

$$(u_{01} - u_{02})_+ \in L^1(\mathbb{R}^N),$$

choosing $\phi(t,x) = \phi_k(t,x)$ with $\{\phi_k\}$ a sequence of functions in $C_c^\infty([0,+\infty) \times \mathbb{R}^N)$ approximating the function $\mathbf{1}_{[0,t) \times \mathbb{R}^N}$ and letting $k \to +\infty$, we get the first inequality in (i). Similarly, the second inequality is obtained by (ii) of Theorem 4.4 applied to $u_1(t,x) = v(x,p)$ and $u_2(t,x) = u(t,x)$.

The monotonicity of $v(x,\cdot)$ implies that if $p_1 < 0 < p_2$ are such that $p_1 \leq \varphi(x) \leq p_2$, then

$$v(x,p_1) \leq v(x,\varphi(x)) \leq v(x,p_2).$$

Therefore, (ii) is a consequence of (i).

Finally, from (4.9) and (4.13), we infer that for any $\xi \in C_c(\mathbb{R})$

$$\int_0^{+\infty} dp \int_{\mathbb{R}^+ \times \mathbb{R}^N} \xi(p) m(t,x,p) dt dx \leq \int_0^{+\infty} \xi(p) \|(v(x,\varphi(x)) - v(x,p))_+\|_{L^1(\mathbb{R}^N)} dp.$$

Similarly, from (4.9) and (4.14),

$$\int_{-\infty}^0 dp \int_{\mathbb{R}^+ \times \mathbb{R}^N} \xi(p) m(t,x,p) dt dx \leq \int_{-\infty}^0 \xi(p) \|(v(x,\varphi(x)) - v(x,p))_-\|_{L^1(\mathbb{R}^N)} dp.$$

Adding the two previous inequalities we get (iii) and this concludes the proof of the lemma. $\square$

The kinetic formulation for (4.4), (4.5) is finally obtained by deriving (4.9) with respect to p:

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p}(x,p)\chi_+\right) - \Delta\chi_+ = \frac{\partial m}{\partial p}(t,x,p) \\ m(t,x,p) = |\nabla f(x,u)|^2\delta_{f(x,u)}(p) \end{cases} \tag{4.15}$$

where

$$\chi_+(t,x,p) := \mathbf{1}_{\{v(x,p)<u(t,x)\}}. \tag{4.16}$$

Remark that since $v(x,\cdot) = f(x,\cdot)^{-1}$ is monotone increasing, we have

$$\chi_+(t,x,p) = \mathbf{1}_{\{p<f(x,u(t,x))\}}.$$

In the homogeneous case, $v(x,p) = f^{-1}(p)$ does not depend on x. If we make the change of variable $v = f^{-1}(p)$, equation (4.15) becomes (4.2).

In order to make sense to the equation (4.15), we require that $g(x,p) = v(x,p)$ satisfies the following assumption:

(f3) $\frac{\partial g}{\partial p} \in L^1_{loc}(\mathbb{R}^N \times \mathbb{R})$.

We are now ready to give the definition of kinetic solution for (4.4), (4.5).

Definition 4.8. *A kinetic solution of* (4.1), (4.3) *is a function u, with $u(t,x) - v(x,0) \in L^\infty([0,\infty); L^1(\mathbb{R}^N))$, such that*

(i) $f(x,u(t,x)) \in L^2_{loc}(\mathbb{R}^+; H^1_{loc}(\mathbb{R}^N))$;
(ii) $\chi_+(t,x,p) = \mathbf{1}_{\{v(x,p)<u(t,x)\}}$ *satisfies* (4.15) *in the sense of distributions in* $[0,+\infty) \times \mathbb{R}^N \times \mathbb{R}$, *with initial datum* $\chi(0,x,p) = \mathbf{1}_{\{v(x,p)<u_0(x)\}}$;
(iii) If n is the positive measure on $\mathbb{R}$ defined by

$$\int_{\mathbb{R}} \xi(p)dn(p) := \int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}} \xi(p)m(t,x,p)\,dt\,dx$$

for $\xi \in C_c(\mathbb{R})$, then there exists $\eta \in L^\infty(\mathbb{R})$ such that $\eta \to 0$ as $|p| \to \infty$ and

$$n \le \eta$$

in the sense of distributions in $\mathbb{R}$.

We conclude this section by showing that the weak solution of (4.4), (4.5) with suitable initial condition is also kinetic solution. Precisely we have:

Proposition 4.9. *Assume (f1), (f2), (f3) and let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N) \cap C([0,+\infty); L^1_{loc}(\mathbb{R}^N))$ be the weak solution of* (4.4), (4.5) *with $u_0(x) = v(x,\varphi(x))$ for some function $\varphi \in L^\infty(\mathbb{R}^N)$ such that $v(x,\varphi(x)) - v(x,0) \in L^1(\mathbb{R}^N)$. Then the function $\chi_+(t,x,p)$ defined in* (4.16) *is a solution of* (4.15) *in the sense of distributions in $[0,+\infty) \times \mathbb{R}^N \times \mathbb{R}$ with initial condition $\chi_+(0,x,p) = \mathbf{1}_{\{p<\varphi(x)\}}$. In particular there exists a kinetic solution of* (4.4), (4.5).

Proof. Take $\phi = \phi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{R}^N)$ and $\xi = \xi(p) \in C_c^\infty(\mathbb{R})$, then by (4.9)

$$\begin{aligned}
< \frac{\partial m}{\partial p}, \phi\xi > &= - < m, \phi\xi' > \\
&= - \int_{\mathbb{R}^+ \times \mathbb{R}^N} dt\, dx\, \phi_t(t,x) \int_{\mathbb{R}} (u(t,x) - v(x,p))_+ \xi'(p)\, dp \\
&- \int_{\mathbb{R}^N} dx\, \phi(0,x) \int_{\mathbb{R}} (u_0(x) - v(x,p))_+ \xi'(p)\, dp \\
&- \int_{\mathbb{R}^+ \times \mathbb{R}^N} dt\, dx\, \Delta\phi(t,x) \int_{\mathbb{R}} (f(x,u(t,x)) - p)_+ \xi'(p)\, dp \\
&= - \int_{\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}} \left(\phi_t(t,x) \frac{\partial v}{\partial p}(x,p) \chi_+(t,x,p) + \Delta\phi(t,x)) \chi_+(t,x,p) \right) \xi(p)\, dt\, dx\, dp \\
&- \int_{\mathbb{R}^N \times \mathbb{R}} \phi(0,x) \frac{\partial v}{\partial p}(x,p) \mathbf{1}_{\{p < f(x,u_0(x))\}} \xi(p)\, dx\, dp,
\end{aligned}$$

where the last equality is obtained by integrating by parts with respect to p and using that

$$\chi_+(t,x,p) = \mathbf{1}_{\{v(x,p) < u(t,x)\}} = \mathbf{1}_{\{p < f(x,u(t,x))\}}.$$

Since in addition

$$f(x,u_0(x)) = f(x, v(x,\varphi(x))) = \varphi(x),$$

and thus

$$\mathbf{1}_{\{p < f(x,u_0(x))\}} = \mathbf{1}_{\{p < \varphi(x)\}},$$

we conclude that $\chi_+(t,x,p)$ is a solution of (4.15) in the sense of distributions in $[0,+\infty) \times \mathbb{R}^N \times \mathbb{R}$ with initial condition $\chi_+(0,x,p) = \mathbf{1}_{\{p < \varphi(x)\}}$.

We have shown that the weak solution u satisfies (i) and (ii) of Definition 4.8. By (i) of Lemma 4.7 and $v(x,\varphi(x)) - v(x,0) \in L^1(\mathbb{R}^N)$, we also have that $u(t,x) - v(x,0) \in L^\infty([0,\infty); L^1(\mathbb{R}^N))$. Finally (iii) of Lemma 4.7 implies that (iii) of Definition 4.8 holds true. Thus u is kinetic solution of (4.4), (4.5). □

5. Proof of Theorem 2.2

Let $u^\epsilon(t,x,\omega)$ be the weak solution of (1.1) whose existence is guaranteed by Theorem 4.4. Let us define

$$v(\omega,p) := g(\omega,p), \quad \text{where } g(\omega,\cdot) = f^{-1}(\omega,\cdot), \tag{5.1}$$

$$\chi_+^\epsilon(t,x,p,\omega) := \mathbf{1}_{\{v(T(\frac{x}{\epsilon})\omega,p) < u^\epsilon(t,x,\omega)\}} = \mathbf{1}_{\{p < f(T(\frac{x}{\epsilon})\omega, u^\epsilon(t,x,\omega))\}}$$

and

$$\chi_-^\epsilon(t,x,p,\omega) := \mathbf{1}_{\{v(T(\frac{x}{\epsilon})\omega,p) > u^\epsilon(t,x,\omega)\}} = \mathbf{1}_{\{p > f(T(\frac{x}{\epsilon})\omega, u^\epsilon(t,x,\omega))\}}.$$

Then, by Proposition 4.9, for a.e. $\omega \in \Omega$, χ_+^ϵ and χ_-^ϵ are respectively solutions in the sense of distributions in $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$ of

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\chi_+^\epsilon\right) - \Delta\chi_+^\epsilon = \frac{\partial m^\epsilon}{\partial p}(t,x,p,\omega) \\ \chi_+^\epsilon(0,x,p) = \mathbf{1}_{\{p<\varphi(x)\}} \end{cases} \tag{5.2}$$

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\chi_-^\epsilon\right) - \Delta\chi_-^\epsilon = -\frac{\partial m^\epsilon}{\partial p}(t,x,p,\omega) \\ \chi_-^\epsilon(0,x,p) = \mathbf{1}_{\{p>\varphi(x)\}} \end{cases}$$

where

$$m^\epsilon(t,x,p,\omega) = \left|\nabla\left[f\left(T\left(\frac{x}{\epsilon}\right)\omega, u^\epsilon\right)\right]\right|^2 \delta_{f(T(\frac{x}{\epsilon})\omega,u^\epsilon)}(p).$$

Moreover, by Lemma 4.7 we have the following estimates:

- If $p_1 < 0 < p_2 \in \mathbb{R}$ are such that $p_1 \le \varphi(x) \le p_2$ for any $x \in \mathbb{R}^N$, then for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$ and a.e. $\omega \in \Omega$

$$v\left(T\left(\frac{x}{\epsilon}\right)\omega, p_1\right) \le u^\epsilon(t,x,\omega) \le v\left(T\left(\frac{x}{\epsilon}\right)\omega, p_2\right). \tag{5.3}$$

- Let n^ϵ be the positive measure on $\mathbb{R}$ defined by

$$\int_{\mathbb{R}} \xi(p)\, dn^\epsilon(p) := \int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} \xi(p) m^\epsilon(t,x,p,\omega)\, dt\, dx\, d\mu$$

for $\xi \in C_c(\mathbb{R})$, then

$$n^\epsilon \le \eta \tag{5.4}$$

in the sense of distributions in $\mathbb{R}$, where $\eta \in L^\infty(\mathbb{R})$ with compact support, is defined by

$$\begin{aligned}\eta :=& \mathbf{1}_{\{p>0\}} \int_{\mathbb{R}^N\times\Omega} \left[v\left(T\left(\frac{x}{\epsilon}\right)\omega, \varphi(x)\right) - v\left(T\left(\frac{x}{\epsilon}\right)\omega, p\right)\right]_+ dx\, d\mu \\ &+ \mathbf{1}_{\{p<0\}} \int_{\mathbb{R}^N\times\Omega} \left[v\left(T\left(\frac{x}{\epsilon}\right)\omega, \varphi(x)\right) - v\left(T\left(\frac{x}{\epsilon}\right)\omega, p\right)\right]_- dx\, d\mu \\ =& \mathbf{1}_{\{p>0\}} \int_{\mathbb{R}^N\times\Omega} \left[v(\omega, \varphi(x)) - v(\omega, p)\right]_+ dx\, d\mu \\ &+ \mathbf{1}_{\{p<0\}} \int_{\mathbb{R}^N\times\Omega} \left[v(\omega, \varphi(x)) - v(\omega, p)\right]_- dx\, d\mu,\end{aligned} \tag{5.5}$$

by using the invariance of T with respect to μ.

Remark that the fact that η has compact support is a consequence of the monotonicity of $v(\omega,\cdot)$ and the assumption $\varphi \in L^\infty(\mathbb{R}^N)$.

Lemma 5.1. *There exists a subsequence of the measures $\{m^\epsilon\}$, still denoted by $\{m^\epsilon\}$ and a measure m^0 on $\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega$ such that m^ϵ converges to m^0 weakly in the sense of measures in $\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega$. Moreover, if n^0 is the positive measure on $\mathbb{R}$ defined by*

$$\int_{\mathbb{R}} \xi(p)\, dn^0(p) := \int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} \xi(p) m^0(t,x,p,\omega)\, dt\, dx\, d\mu$$

for $\xi \in C_c(\mathbb{R}^N)$, then

$$n^0 \le \eta$$

in the sense of distributions in $\mathbb{R}$, where $\eta \in L^\infty(\mathbb{R})$ with compact support.

Proof. The lemma is an immediate consequence of (5.4) and (5.5). □

The functions $\{\chi_+^\epsilon\}_\epsilon$ and $\{\chi_-^\epsilon\}_\epsilon$ are obviously bounded uniformly in ϵ, then by Proposition 3.9, there exist $\chi_+^0, \chi_-^0 \in L^\infty(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega)$ such that, up to subsequence, $\{\chi_+^\epsilon\}$ and $\{\chi_-^\epsilon\}$ stochastically two-scale converge in the mean respectively to χ_+^0 and χ_-^0 as $\epsilon \to 0$.

Lemma 5.2. *The functions χ_+^0 and χ_-^0 are independent of $\omega \in \Omega$.*

Proof. In (5.2), take as test function $\phi(t,x)\epsilon^2\psi\left(T\left(\frac{x}{\epsilon}\right)\omega\right)\xi(p)$, where $\phi \in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^N)$, $\xi \in C_c^\infty(\mathbb{R})$ and $\psi \in D^\infty(\Omega)$, $D^\infty(\Omega)$ being the set defined in (3.2). Using that for a.e. $\omega \in \Omega$, a.e. $x \in \mathbb{R}^N$,

$$\frac{\partial}{\partial x_i}\left(\psi\left(T\left(\frac{x}{\epsilon}\right)\omega\right)\right) = \frac{1}{\epsilon} D_i\psi\left(T\left(\frac{x}{\epsilon}\right)\omega\right),$$

we get

$$\int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} \chi_+^\epsilon \left[\epsilon^2\psi\xi\frac{\partial v}{\partial p}\partial_t\phi + \epsilon^2\psi\xi\Delta_x\phi + 2\epsilon\xi\nabla_x\phi D_\omega\psi + \phi\xi\Delta_\omega\psi\left(T\left(\frac{x}{\epsilon}\right)\omega\right)\right] dt\,dx\,dp\,d\mu$$
$$= \int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} \epsilon^2\phi\psi m^\epsilon \xi'(p)\,dt\,dx\,dp\,d\mu.$$

By Lemma 5.1, the right hand-side of the identity above goes to 0 as $\epsilon \to 0$. Hence, passing to the limit as $\epsilon \to 0$, we find that for almost every $(t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$, $\chi_+^0(t,x,p,\omega)$ is solution of

$$\Delta_\omega \chi_+^0 = 0$$

in the sense of distributions. Then, Lemma 3.6 implies that χ_+^0 is independent of ω. Similarly, we can prove that χ_-^0 independent of ω. □

Lemma 5.3. *For a.e.* $(x,p) \in \mathbb{R}^N \times \mathbb{R}$, $\chi_+^0(0,x,p) = \mathbf{1}_{\{p<\varphi(x)\}}$ *and* $\chi_-^0(0,x,p) = \mathbf{1}_{\{p>\varphi(x)\}}$.

Proof. Since χ_+^ϵ is solution of (5.2), for any $\psi \in C_c^\infty([0,+\infty)\times\mathbb{R}^N\times\mathbb{R})$,

$$\int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} \chi_+^\epsilon \left[\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega, p\right)\partial_t\psi + \Delta_x\psi\right] dt\,dx\,dp\,d\mu$$

$$= -\int_{\mathbb{R}^N\times\mathbb{R}\times\Omega} \mathbf{1}_{\{p<\varphi(x)\}}\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega, p\right)\psi(0,x,p)\,dx\,dp\,d\mu + \int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} m^\epsilon\partial_p\psi\,dt\,dx\,dp\,d\mu.$$

Passing to the limit as $\epsilon \to 0$, by the two-scale convergence of χ_+^ϵ to χ_+^0, Lemma 5.1 and the identity

$$\int_\Omega \frac{\partial v}{\partial p}(\omega,p)\,d\mu = \overline{g}'(p),$$

we get

$$\int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}} \chi_+^0\left[\overline{g}'(p)\partial_t\psi + \Delta_x\psi\right] dt\,dx\,dp$$
$$= -\int_{\mathbb{R}^N\times\mathbb{R}} \mathbf{1}_{\{p<\varphi(x)\}}\overline{g}'(p)\ \psi(0,x,p)\,dx\,dp + \int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} m^0\partial_p\psi\,dt\,dx\,dp\,d\mu.$$

Since $\overline{g}'(p) > 0$ for a.e. $p \in \mathbb{R}$, this implies $\chi_+^0(0,x,p) = \mathbf{1}_{\{p<\varphi(x)\}}$. Similarly, we get $\chi_-^0(0,x,p) = \mathbf{1}_{\{p>\varphi(x)\}}$. □

Now, we want to identify χ_+^0 and χ_-^0. Let us denote

$$\overline{\chi}_+(t,x,p) := \mathbf{1}_{\{p<\overline{f}(\overline{u}(t,x))\}}$$

and

$$\overline{\chi}_-(t,x,p) := \mathbf{1}_{\{p>\overline{f}(\overline{u}(t,x))\}}$$

where $\overline{u}$ is the weak solution of (2.8). Since $u_0(x,\omega) = g(\omega,\varphi(x))$, recalling the definition (2.6) of $\overline{g}(p) = \overline{f}^{-1}(p)$, we see that $\overline{u}$ satisfies the initial condition

$$\overline{u}(x,0) = \int_\Omega g(\omega,\varphi(x))\,d\mu = \overline{g}(\varphi(x)),$$

with $\varphi \in L^\infty(\mathbb{R}^N)$. Moreover, by Assumption (H4),

$$\overline{g}(\varphi(x)) - \overline{g}(0) \in L^1(\mathbb{R}^N).$$

Thus, by Proposition 4.9, we know that $\overline{\chi}_+$ and $\overline{\chi}_-$ are respectively solution in the sense of distribution in $[0,+\infty)\times\mathbb{R}^N\times\mathbb{R}$ of

$$\begin{cases}\frac{\partial}{\partial t}\left(\overline{g}'(p)\overline{\chi}_+\right)-\Delta\overline{\chi}_+=\frac{\partial\overline{m}}{\partial p}(t,x,p)\\ \overline{\chi}_+(0,x,p)=\mathbf{1}_{\{p<\varphi(x)\}}\end{cases}\tag{5.6}$$

$$\begin{cases}\frac{\partial}{\partial t}\left(\overline{g}'(p)\overline{\chi}_-\right)-\Delta\overline{\chi}_-=-\frac{\partial\overline{m}}{\partial p}(t,x,p)\\ \overline{\chi}_-(0,x,p)=\mathbf{1}_{\{p>\varphi(x)\}}\end{cases}\tag{5.7}$$

where

$$\overline{m}(t,x,p)=\left|\nabla\left[\overline{f}(\overline{u})\right]\right|^2\delta_{\overline{f}(\overline{u})}(p).$$

Lemma 5.4. *We have*

$$\chi^0_+\overline{\chi}_-=0 \textit{ and } \chi^0_-\overline{\chi}_+=0 \quad \textit{for a.e. } (t,x,p)\in\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}.$$

Let us first give the heuristic proof of the lemma.

Heuristic proof. We want to prove that

$$\frac{d}{dt}\int_{\mathbb{R}^N\times\mathbb{R}}\overline{g}'(p)\chi^0_+\overline{\chi}_-dxdp\le 0.\tag{5.8}$$

Indeed, the previous inequality and

$$\chi^0_+(0,x,p)\overline{\chi}_-(0,x,p)=\mathbf{1}_{\{p<\varphi(x)\}}\mathbf{1}_{\{p>\varphi(x)\}}=0$$

imply that for $t>0$

$$\int_{\mathbb{R}^N\times\mathbb{R}}\overline{g}'(p)\chi^0_+\overline{\chi}_-\,dx\,dp\le 0.$$

Since $\overline{g}'>0$, we infer that $\chi^0_+\overline{\chi}_-=0$ a.e. in $\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}$.

Multiplying equation (5.2) by $\overline{\chi}_-$ and equation (5.7) by χ^ϵ_+, and integrating by parts with respect to (x,p), we get respectively

$$\int_{\mathbb{R}^N\times\mathbb{R}}\left\{\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\partial_t\chi^\epsilon_+\overline{\chi}_-+\nabla\chi^\epsilon_+\cdot\nabla\overline{\chi}_-\right\}dx\,dp=-\int_{\mathbb{R}^N\times\mathbb{R}}m^\epsilon\partial_p\overline{\chi}_-\,dx,$$

$$\int_{\mathbb{R}^N\times\mathbb{R}}\left\{\overline{g}'(p)\partial_t\overline{\chi}_-\chi^\epsilon_++\nabla\chi^\epsilon_+\cdot\nabla\overline{\chi}_-\right\}dx\,dp=\int_{\mathbb{R}^N\times\mathbb{R}}\overline{m}\partial_p\chi^\epsilon_+\,dx.$$

Summing the two previous inequalities, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}}\left\{\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\partial_t\chi^\epsilon_+\overline{\chi}_-+\overline{g}'(p)\partial_t\overline{\chi}_-\chi^\epsilon_+\right\}dx\,dp\\ &=\int_{\mathbb{R}^N\times\mathbb{R}}\left\{-2\nabla\chi^\epsilon_+\cdot\nabla\overline{\chi}_--m^\epsilon\partial_p\overline{\chi}_-+\overline{m}\partial_p\chi^\epsilon_+\right\}dx\,dp\\ &=\int_{\mathbb{R}^N\times\mathbb{R}}\left\{2\nabla\left[f\left(T\left(\frac{x}{\epsilon}\right)\omega,u^\epsilon\right)\right]\cdot\nabla(\overline{f}(\overline{u}))\right.\\ &\left.-\left|\nabla\left[f\left(T\left(\frac{x}{\epsilon}\right)\omega,u^\epsilon\right)\right]\right|^2-\left|\nabla\left[\overline{f}(\overline{u})\right]\right|^2\right\}\delta_{f\left(T\left(\frac{x}{\epsilon}\right)\omega,u^\epsilon\right)}(p)\delta_{\overline{f}(\overline{u})}(p)\,dx\\ &\le 0.\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N\times\mathbb{R}}\left\{\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\partial_t\chi^\epsilon_+\overline{\chi}_-+\overline{g}'(p)\partial_t\overline{\chi}_-\chi^\epsilon_+\right\}dx\,dp\le 0.$$

Integrating the previous inequality with respect to ω and passing to the limit as $\epsilon \to 0$, using the fact that χ_+^0 does not depend on ω, and that by assumption (H3)

$$\int_\Omega \frac{\partial v}{\partial p}(\omega, p)\, d\mu = \overline{g}'(p),$$

we get

$$\int_{\mathbb{R}^N\times\mathbb{R}} (\overline{g}'(p)\partial_t \chi_+^0 \overline{\chi}_- + \overline{g}'(p)\partial_t \overline{\chi}_- \chi_+^0)\, dx\, dp \le 0,$$

which is (5.8).

Proof of Lemma 5.4

Let us prove that $\chi_+^0\overline{\chi}_- = 0$ a.e. in $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$. We use the Kruzhkov's doubling variables method [14]. Let $0 \le \phi \in C_c^\infty([0,\infty)\times\mathbb{R}^N)$ and let ψ_m, θ_n be classical smooth, compactly supported, approximations of identity in $\mathbb{R}^N$ and $\mathbb{R}$. For $(t,x,s,y) \in (\mathbb{R}^+\times\mathbb{R}^N)^2$, let us define

$$\Phi(t,x,s,y) := \phi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\psi_m\left(\frac{x-y}{2}\right)\theta_n\left(\frac{t-s}{2}\right).$$

The functions ψ_m and θ_n satisfy respectively,

$$\int_{\mathbb{R}^N} g(y)\psi_m\left(\frac{x-y}{2}\right) dy \to g(x) \quad \text{as } m \to +\infty$$

for a.e. x and for any $g \in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}} h(s)\theta_n\left(\frac{t-s}{2}\right) ds \to h(t), \quad \text{as } n \to +\infty$$

for a.e. t and for any $h \in L^1(\mathbb{R})$. Let $H_\sigma(s)$ be the approximation of the Heaviside function given by (4.6). Now take $H_\sigma(p - \overline{f}(\overline{u}(s,y)))\Phi(t,x,s,y)$ as test function in (5.2), and integrate first in $(t,x,p) \in \mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}$ and then in $(s,y) \in \mathbb{R}^+\times\mathbb{R}^N$. Remark that even if $H_\sigma(p-\overline{f}(\overline{u}(s,y)))$ does not have compact support, by (5.3) $\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))$ has compact support as a function of p, for (t,x) and (s,y) belonging to compact subsets of $[0,+\infty)\times\mathbb{R}^N$. For simplicity of notation, in what follows, we will not write the domains of integration, and we will skip the dependence on $\omega \in \Omega$ denoting

$$v\left(\frac{x}{\epsilon}, p\right) := v\left(T\left(\frac{x}{\epsilon}\right)\omega, p\right), \quad f\left(\frac{x}{\epsilon}, u^\epsilon\right) := f\left(T\left(\frac{x}{\epsilon}\right)\omega, u^\epsilon\right).$$

Then, a.e. in Ω, we have

$$\begin{aligned}
&-\int \frac{\partial v}{\partial p}\left(\frac{x}{\epsilon},p\right)\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\partial_t\Phi\, dp\, dt\, dx\, ds\, dy \\
&-\int \frac{\partial v}{\partial p}\left(\frac{x}{\epsilon},p\right)\mathbf{1}_{\{p<\varphi(x)\}}H_\sigma(p-\overline{f}(\overline{u}(s,y)))\Phi(0,x,s,y)\, dp\, dx\, ds\, dy \\
&= \int \chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\Delta_x\Phi dp\, dt\, dx\, ds\, dy \\
&-\int H_\sigma'\left(f\left(\frac{x}{\epsilon}, u^\epsilon(t,x)\right) - \overline{f}(\overline{u}(s,y))\right)\left|\nabla\left[f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)\right]\right|^2\Phi\, dt\, dx\, ds\, dy.
\end{aligned} \tag{5.9}$$

It is easy to check that

$$\Delta_x\Phi + \Delta_y\Phi + 2\mathrm{div}_y\nabla_x\Phi = \theta_n\psi_n\Delta\phi.$$

Hence, the first term in the right-hand side of (5.9), becomes

$$\begin{aligned}
&\int \chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\Delta_x\Phi\, dp\, dt\, dx\, ds\, dy \\
&= \int dp\, dt\, dx\, ds\, \chi_+^\epsilon(t,x,p)\int H_\sigma(p-\overline{f}(\overline{u}(s,y)))(-2\mathrm{div}_y\nabla_x\Phi - \Delta_y\Phi + \theta_n\psi_n\Delta\phi)\, dy
\end{aligned}$$

and by integrating by parts

$$
\begin{aligned}
&\int dp\,dt\,dx\,ds\,\chi_+^\epsilon(t,x,p)\int H_\sigma(p-\overline{f}(\overline{u}(s,y)))(-2\mathrm{div}_y\nabla_x\Phi)\,dy\\
&=-\int dp\,dt\,dx\,ds\,\chi_+^\epsilon(t,x,p)\int 2H'_\sigma(p-\overline{f}(\overline{u}(s,y)))\nabla_y(\overline{f}(\overline{u}(s,y)))\cdot\nabla_x\Phi\,dy\\
&=-\int dt\,dx\,ds\,dy 2\nabla_y(\overline{f}(\overline{u}(s,y)))\cdot\nabla_x\Phi\int\chi_+^\epsilon(t,x,p)H'_\sigma(p-\overline{f}(\overline{u}(s,y)))\,dp\\
&=-\int dtdsdy\int 2H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)\nabla_y(\overline{f}(\overline{u}(s,y)))\cdot\nabla_x\Phi\,dx\\
&=\int 2\mathrm{div}_x\left[H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)\nabla_y(\overline{f}(\overline{u}(s,y)))\right]\Phi\,dt\,dx\,ds\,dy\\
&=\int 2H'_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)\nabla_x\left[f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)\right]\cdot\nabla_y(\overline{f}(\overline{u}(s,y)))\Phi\,dt\,dx\,ds\,dy.
\end{aligned}
$$

We infer that

$$
\begin{aligned}
&\int\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\Delta_x\Phi\,dp\,dt\,dx\,ds\,dy\\
&=\int 2H'_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)\nabla_x\left[f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)\right]\cdot\nabla_y(\overline{f}(\overline{u}(s,y)))\Phi\,dt\,dx\,ds\,dy\\
&+\int\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))(-\Delta_y\Phi+\theta_n\psi_n\Delta\phi)\,dp\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{5.10}
$$

From (5.9) and (5.10) we conclude that

$$
\begin{aligned}
&-\int\frac{\partial v}{\partial p}\left(\frac{x}{\epsilon},p\right)\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\partial_t\Phi\,dp\,dt\,dx\,ds\,dy\\
&-\int\frac{\partial v}{\partial p}\left(\frac{x}{\epsilon},p\right)\mathbf{1}_{\{p<\varphi(x)\}}H_\sigma(p-\overline{f}(\overline{u}(s,y)))\Phi(0,x,s,y)\,dp\,dx\,ds\,dy\\
&=\int 2H'_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)\nabla_x\left[f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)\right]\cdot\nabla_y(\overline{f}(\overline{u}(s,y)))\Phi\,dt\,dx\,ds\,dy\\
&+\int\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))(-\Delta_y\Phi+\theta_n\psi_n\Delta\phi)\,dp\,dt\,dx\,ds\,dy\\
&-\int H'_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)\left|\nabla_x\left[f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)\right]\right|^2\Phi\,dt\,dx\,ds\,dy.
\end{aligned}
\tag{5.11}
$$

Next, take $H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)\Phi(t,x,s,y)$, as test function for the equation (5.7) and integrate first in (s,y,p) then in (t,x). We have

$$
\begin{aligned}
&-\int\overline{g}'(p)\overline{\chi}_-(s,y,p)H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)\partial_s\Phi\,dp\,ds\,dy\,dt\,dx\\
&-\int\overline{g}'(p)\mathbf{1}_{\{p>\varphi(y)\}}H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)\Phi(t,x,0,y)\,dp\,dy\,dt\,dx\\
&=\int\overline{\chi}_-(s,y,p)H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)\Delta_y\Phi\,dp\,ds\,dy\,dt\,dx\\
&-\int H'_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)|\nabla_y[\overline{f}(\overline{u}(s,y))]|^2\Phi\,ds\,dy\,dt\,dx.
\end{aligned}
\tag{5.12}
$$

Summing (5.11) and (5.12), we get

$$
\begin{aligned}
&-\int \frac{\partial v}{\partial p}\left(\frac{x}{\epsilon},p\right)\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\partial_t\Phi\,dp\,dt\,dx\,ds\,dy\\
&-\int \overline{g}'(p)\overline{\chi}_-(s,y,p)H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)\partial_s\Phi\,dp\,dt\,dx\,ds\,dy\\
&-\int \frac{\partial v}{\partial p}\left(\frac{x}{\epsilon},p\right)\mathbf{1}_{\{p<\varphi(x)\}}H_\sigma(p-\overline{f}(\overline{u}(s,y)))\Phi(0,x,s,y)\,dp\,dx\,ds\,dy\\
&-\int \overline{g}'(p)\mathbf{1}_{\{p>\varphi(y)\}}H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)\Phi(t,x,0,y)\,dp\,dt\,dx\,dy\\
&=\int\left\{\overline{\chi}_-(s,y,p)H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)-\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\right\}\Delta_y\Phi\,dp\,dt\,dx\,ds\,dy\\
&-\int H_\sigma'\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-\overline{f}(\overline{u}(s,y))\right)\left\{\left|\nabla_x\left[f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)\right]\right|^2+|\nabla_y[\overline{f}(\overline{u}(s,y))]|^2\right.\\
&\left.-2\nabla_x\left[f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)\right]\cdot\nabla_y(\overline{f}(\overline{u}(s,y)))\right\}\Phi\,dt\,dx\,ds\,dy\\
&+\int \chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\theta_n\psi_n\Delta\phi\,dp\,dt\,dx\,ds\,dy\\
&\le\int\left\{\overline{\chi}_-(s,y,p)H_\sigma\left(f\left(\frac{x}{\epsilon},u^\epsilon(t,x)\right)-p\right)-\chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\right\}\Delta_y\Phi\,dp\,dt\,dx\,ds\,dy\\
&+\int \chi_+^\epsilon(t,x,p)H_\sigma(p-\overline{f}(\overline{u}(s,y)))\theta_n\psi_n\Delta\phi\,dp\,dt\,dx\,ds\,dy.
\end{aligned}
$$

Then, letting δ go to 0, and integrating over Ω, we get

$$
\begin{aligned}
&-\int \frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\chi_+^\epsilon(t,x,p,\omega)\overline{\chi}_-(s,y,p)\partial_t\Phi\,dp\,dt\,dx\,ds\,dy\,d\mu\\
&-\int \overline{g}'(p)\overline{\chi}_-(s,y,p)\chi_+^\epsilon(t,x,p,\omega)\partial_s\Phi\,dp\,dy\,ds\,dx\,dt\,d\mu\\
&\le\int \frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\mathbf{1}_{\{p<\varphi(x)\}}\overline{\chi}_-(s,y,p)\Phi(0,x,s,y)\,dp\,dx\,ds\,dy\,d\mu\\
&+\int \overline{g}'(p)\mathbf{1}_{\{p>\varphi(y)\}}\chi_+^\epsilon(t,x,p,\omega)\Phi(t,x,0,y)\,dp\,dt\,dx\,dy\,d\mu\\
&+\int \chi_+^\epsilon(t,x,p,\omega)\overline{\chi}_-(s,y,p)\theta_n\psi_n\Delta\phi\,dp\,dt\,dx\,ds\,dy\,d\mu\\
&=:I_1+I_2+\int \chi_+^\epsilon(t,x,p,\omega)\overline{\chi}_-(s,y,p)\theta_n\psi_n\Delta\phi\,dp\,dt\,dx\,ds\,dy\,d\mu.
\end{aligned}
\tag{5.13}
$$

Let us estimate the right hand-side of (5.13). Recalling that $\overline{\chi}_-(s,y,p)=\mathbf{1}_{\{p>\overline{f}(\overline{u}(s,y))\}}$ and using (5), we can estimate the first term in the right hand-side of (5.13) as follows

$$
\begin{aligned}
I_1&=\int \frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\mathbf{1}_{\{p<\varphi(x)\}}\overline{\chi}_-(s,y,p)\Phi(0,x,s,y)\,dp\,dx\,ds\,dy\,d\mu\\
&=\int dp\,dx\,ds\,dy\,\mathbf{1}_{\{p<\varphi(x)\}}\overline{\chi}_-(s,y,p)\Phi(0,x,s,y)\int \frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)d\mu\\
&=\int dx\,ds\,dy\,\Phi(0,x,s,y)\int_{\overline{f}(\overline{u}(s,y))}^{\varphi(x)}\overline{g}'(p)\,dp\\
&=\int(\overline{g}(\varphi(x))-\overline{u}(s,y)))_+\Phi(0,x,s,y)\,dx\,ds\,dy.
\end{aligned}
$$

Thus I_1 is actually independent of ϵ. Recalling that $\overline{u}(t,x)-\overline{g}(0)\in C([0,+\infty);L^1(\mathbb{R}^N))$ and $\overline{u}(0,x)=\overline{g}(\varphi(x))$, we see that letting $n,m\to+\infty$,

$$I_1\to 0. \tag{5.14}$$

Next, by letting first $\epsilon\to 0$ and then $n,m\to+\infty$, by Lemma 5.3, we get

$$I_2\to\int\overline{g}'(p)\mathbf{1}_{\{p>\varphi(x)\}}\mathbf{1}_{\{p<\varphi(x)\}}\phi(0,x)\,dp\,dt\,dx=0. \tag{5.15}$$

Then, from (5.13), we have

$$\begin{aligned}
&-\int\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\chi_+^\epsilon(t,x,p,\omega)\overline{\chi}_-(s,y,p)\psi_m\left(\frac{1}{2}\phi_t\theta_n+\frac{1}{2}\phi\theta_n'\right)\,dp\,dt\,dx\,ds\,dy\,d\mu\\
&-\int\overline{g}'(p)\overline{\chi}_-(s,y,p)\chi_+^\epsilon(t,x,p,\omega)\psi_m\left(\frac{1}{2}\phi_t\theta_n-\frac{1}{2}\phi\theta_n'\right)\,dp\,dt\,dx\,ds\,dy\,d\mu\\
&\le I_1+I_2+\int\chi_+^\epsilon(t,x,p,\omega)\overline{\chi}_-(s,y,p)\theta_n\psi_n\Delta\phi\,dp\,dt\,dx\,ds\,dy\,d\mu.
\end{aligned} \tag{5.16}$$

Letting first $\epsilon\to 0$ then $n,m\to\infty$, by the stochastically two-scale convergence of χ_+^ϵ to χ_+^0, using that χ_+^0 is independent of ω and recalling that

$$\int_\Omega\frac{\partial v}{\partial p}(\omega,p)\;d\mu=\overline{g}'(p),$$

we obtain

$$-\int\overline{g}'(p)\chi_+^0(t,x,p)\overline{\chi}_-(t,x,p)\partial_t\phi(t,x)dpdtdx\le C\int|\Delta\phi(t,x)|dt\,dx.$$

Finally, taking $\phi(t,x)=\phi_1(t)\phi_2(x)$ with $\phi_1'<0$ in $[0,t]$, ϕ_2 with compact support and converging to 1 in $C^2(\mathbb{R}^N)$, we get that a.e.,

$$0\le-\overline{g}'(p)\chi_+^0(t,x,p)\overline{\chi}_-(t,x,p)\le 0.$$

The previous inequalities and $\overline{g}'>0$ imply that $\chi_+^0\overline{\chi}_-=0$ a.e.

Similarly, we can prove that $\chi_-^0\overline{\chi}_+$ a.e., and this concludes the proof of the lemma. □

Corollary 5.5. *We have*

$$\chi_+^0=\overline{\chi}_+\quad\textit{and}\quad\chi_-^0=\overline{\chi}_-\quad\textit{for a.e. }(t,x,p)\in\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}.$$

Proof. We know that a.e.

$$0\le\chi_+^0,\chi_-^0\le 1,\quad\chi_+^0+\chi_-^0=1,$$

and, by definition of $\overline{\chi}_+$ and $\overline{\chi}_-$

$$\overline{\chi}_+^2=\overline{\chi}_+,\quad\overline{\chi}_-^2=\overline{\chi}_-,\quad\overline{\chi}_++\overline{\chi}_-=1.$$

Using Lemma 5.4 and the previous properties, we get

$$(\chi_+^0-\overline{\chi}_+)^2=(\chi_+^0)^2+(\overline{\chi}_+)^2-2\chi_+^0\overline{\chi}_+=(\chi_+^0)^2+(\overline{\chi}_+)^2-2\chi_+^0(1-\overline{\chi}_-)\le\overline{\chi}_+-\chi_+^0.$$

On the other hand

$$(\chi_+^0-\overline{\chi}_+)^2=(\chi_+^0)^2+(\overline{\chi}_+)^2-(1-\chi_-^0)2\overline{\chi}_+\le-(\overline{\chi}_+-\chi_+^0).$$

Therefore we get $(\chi_+^0-\overline{\chi}_+)^2=0$, i.e., $\chi_+^0=\overline{\chi}_+$. In the same way we can prove that $\chi_-^0=\overline{\chi}_-$. □

We are now ready to conclude the proof of Theorem 2.2.

Take $\phi = \phi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ and fix $p_0 > 0$. Corollary 5.5 and the uniqueness of the kinetic solution of the limit problem (2.8) proven in [10], imply that the whole sequence $\chi_+^\epsilon(t,x,p,\omega)$ two-scale converges to $\chi_+^0 = \mathbf{1}_{\{p<\overline{f}(\overline{u}(t,x))\}}$. Therefore,

$$\int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} \mathbf{1}_{\{\overline{f}(\overline{u}(t,x))<p<p_0\}}\chi_+^\epsilon(t,x,p,\omega)\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\phi(t,x)dtdxdpd\mu \to 0$$

as $\epsilon \to 0$. The left-hand side above can be rewritten as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}\times\Omega} \mathbf{1}_{\{\overline{f}(\overline{u}(t,x))<p<p_0\}}\chi_+^\epsilon(t,x,p,\omega)\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)\phi(t,x)dt\,dx\,dp\,d\mu \\
&= \int_{\mathbb{R}^+\times\mathbb{R}^N\times\Omega} dt\,dx\,d\mu\,\phi(t,x)\int_{\overline{f}(\overline{u})}^{\min\left[f\left(T\left(\frac{x}{\epsilon}\right)\omega,u^\epsilon\right),p_0\right]}\frac{\partial v}{\partial p}\left(T\left(\frac{x}{\epsilon}\right)\omega,p\right)dp \\
&= \int_{\mathbb{R}^+\times\mathbb{R}^N\times\Omega}\left[\min\left[u^\epsilon(t,x,\omega),v\left(T\left(\frac{x}{\epsilon}\right)\omega,p_0\right)\right]-v\left(T\left(\frac{x}{\epsilon}\right)\omega,\overline{f}(\overline{u}(t,x))\right)\right]_+\phi(t,x)dtdxd\mu.
\end{aligned}$$

Now, from (5.3) we infer that $\min\left[u^\epsilon(t,x,\omega),v\left(T\left(\frac{x}{\epsilon}\right)\omega,p_0\right)\right] = u^\epsilon(t,x,\omega)$, provide $p_0 > |\varphi|_\infty$. We deduce that, up to subsequence

$$\int_{\mathbb{R}^+\times\mathbb{R}^N\times\Omega}\left[u^\epsilon(t,x,\omega)-v\left(T\left(\frac{x}{\epsilon}\right)\omega,\overline{f}(\overline{u}(t,x))\right)\right]_+\phi(t,x)dtdxd\mu \to 0$$

as $\epsilon \to 0$.

Similarly, using the two-scale convergence of $\chi_-^\epsilon(t,x,p,\omega)$ to $\chi_-^0 = \mathbf{1}_{\{p>\overline{f}(\overline{u}(t,x))\}}$, we can prove that the previous limit with the positive part replaced by the negative one holds true. Hence, we get

$$\int_{\mathbb{R}^+\times\mathbb{R}^N\times\Omega}\left|u^\epsilon(t,x,\omega)-v\left(T\left(\frac{x}{\epsilon}\right)\omega,\overline{f}(\overline{u}(t,x))\right)\right|\phi(t,x)dtdxd\mu \to 0$$

as $\epsilon \to 0$ and this proves (2.9).

Now, let us show the weak star convergence of $\int_\Omega u^\epsilon d\mu$ to $\overline{u}$. For any $\psi \in C_c(\mathbb{R}^+\times\mathbb{R}^N)$, we have

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_K\int_\Omega u^\epsilon(t,x,\omega)\psi(t,x)d\mu dxdt &= \lim_{\epsilon\to 0}\int_\Omega\int_K v\left(T\left(\frac{x}{\epsilon}\right)\omega,\overline{f}(\overline{u}(t,x))\right)\psi(t,x)dtdxd\mu \\
&= \int_K dtdx\psi(t,x)\int_\Omega v(\omega,\overline{f}(\overline{u}(t,x))d\mu \qquad (5.17)\\
&= \int_K \overline{u}(t,x)\psi(t,x)dtdx.
\end{aligned}$$

This concludes the proof of Theorem 2.2.

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