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HOMOGENIZATION OF FIRST ORDER EQUATIONS WITH u/ϵ-PERIODIC HAMILTONIAN: RATE OF CONVERGENCE AS $\epsilon \to 0$ AND NUMERICAL METHODS

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We consider homogenization problems for first order Hamilton-Jacobi equations with u^ϵ/ϵ periodic dependence, recently introduced by C. Imbert and R. Monneau, and also studied by G. Barles: this unusual dependence leads to a nonstandard cell problems. We study the rate of convergence of the solution to the solution of the homogenized problem when the parameter ϵ tends to 0. We obtain the same rates as those obtained by I. Capuzzo Dolcetta and H. Ishii for the more usual homogenization problems without the dependence in u^ϵ/ϵ. In a second part, we study Eulerian schemes for the approximation of the cell problems. We prove that when the grid steps tend to zero, the approximation of the effective Hamiltonian converges to the effective Hamiltonian.

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1. Introduction

We consider homogenization problems for first order Hamilton-Jacobi equations with u^ϵ/ϵ periodic dependence, namely

$$\begin{cases} u^\epsilon_t + H\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{u^\epsilon}{\epsilon}, Du^\epsilon\right) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}^N, \\ u^\epsilon(0,x) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

with the following assumptions on the Hamiltonian H:

(H1) Periodicity: for any $(t,x,u,p)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$

$$H(t+1,x+k,u+1,p)=H(t,x,u,p)\quad\text{for any } k\in\mathbb{Z}^N;$$

(H2) Regularity: $H:\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is Lipschitz continuous and there exists a constant $C_1>0$ such that, for almost every $(t,x,u,p)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$

$$|D_{(t,x)}H(t,x,u,p)|\le C_1(1+|p|),\ |D_uH(t,x,u,p)|\le C_1,\ |D_pH(t,x,u,p)|\le C_1;$$

(H3) $H(t,x,u,p)\to+\infty$ as $|p|\to+\infty$ uniformly for $(t,x,u)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}$;

(H4) There exists a constant C such that for almost every $(t,x,u,p)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$

$$|D_pH(t,x,u,p)\cdot p-H(t,x,u,p)|\le C.$$

Typically, our model Hamiltonian is

$$H(t,x,u,p)=a(t,x)|p|+b(t,x,u),$$

where a and b are Lipschitz continuous and periodic functions, with $a>0$;

Problem (1.1) with H independent of t was introduced by Imbert and Monneau [16] as a simplified model for dislocation dynamics in material science. The example proposed immediately above with a and b independent of t is also given in [16]. The complete model is introduced in [17] and leads to nonlocal first order equations of the type

$$u^\epsilon_t+\left(c(\frac{x}{\epsilon})+M^\epsilon(\frac{u^\epsilon}{\epsilon})\right)|Du^\epsilon|+H\left(\frac{u^\epsilon}{\epsilon},Du^\epsilon\right)=0 \tag{1.2}$$

where M^ϵ is a nonlocal jump operator and c is a periodic velocity. In the latter model, the level sets of the solution u^ϵ describe dislocations. By taking $M^\epsilon=0$ in (1.2), c bounded from below by a positive constant, and $H\left(\frac{u^\epsilon}{\epsilon},Du^\epsilon\right)=H\left(\frac{u^\epsilon}{\epsilon}\right)$, we obtain a model in the class studied in the present paper. We cannot say that the model studied here is relevant for dislocation dynamics since the nonlocal term appears very naturally in that context.

Going back to (1.1), it was proved in [16] that, with H independent of t,

- under assumptions (H1) and (H2), there exists a unique bounded continuous viscosity solution of (1.1);
- under assumptions (H1)-(H3), the limit u^0 of u^ϵ as $\epsilon\to0$ exists and it is the unique bounded continuous solution of the homogenized problem

$$\begin{cases}u^0_t+\overline{H}(Du^0)=0,\ (t,x)\in(0,+\infty)\times\mathbb{R}^N,\\ u^0(0,x)=u_0(x),\quad x\in\mathbb{R}^N,\end{cases} \tag{1.3}$$

  where the effective Hamiltonian $\overline{H}$ is uniquely defined by the long time behavior of the solution of

$$\begin{cases}\lambda=v_t+H(x,-\lambda t+p\cdot x+v,p+Dv),\ (t,x)\in(0,+\infty)\times\mathbb{R}^N,\\ v(0,x)=0, \qquad x\in\mathbb{R}^N.\end{cases} \tag{1.4}$$

More precisely, we have the following theorem

Theorem 1.1 (Imbert-Monneau, [16]). *Let H be independent of t. Assume (H1)-(H3) and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Then, as $\epsilon \to 0$, the sequence u^ϵ converges locally uniformly in $(0,+\infty) \times \mathbb{R}^N$ to the solution u^0 of (1.3), where, for any $p \in \mathbb{R}^N$ $\overline{H}(p)$ is defined as the unique number λ for which there exists a bounded continuous viscosity solution of (1.4). Moreover $\overline{H} : \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies the coercivity property*

$$\overline{H}(p) \to +\infty \quad \text{as } |p| \to +\infty.$$

The proof in [16] is rather involved: it uses a *twisted* perturbed test function for a higher dimensional problem posed in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, i.e. a clever modification of the method proposed by Evans in [13,14].
Under the additional assumption (H4), an easier proof of Theorem 1.1 was given by Barles, [3], as a byproduct of a general result on the homogenization of Hamilton-Jacobi equations with non-coercive Hamiltonians.

Remark 1.1. *The hypothesis (H4) which was not used in [16] guarantees the existence of a function H_∞ such that*

$$H_\infty(t,x,u,p) = \lim_{s \to 0^+} sH(t,x,u,s^{-1}p).$$

Moreover H_∞ satisfies (H1)-(H3).
Typically, the results of [16] apply for Hamiltonians of the form $H(t,x,u,p) = a(t,x)(|p|^2+1)^{\beta/2} + b(t,x,u)$, with $a > 0$ and $0 < \beta \le 1$, while here we have to take $\beta = 1$.

In [3], thanks to assumption (H4), the equation for u^ϵ is interpreted as an equation for the motion of a graph: indeed, following [3], for $t \in \mathbb{R}$, $(x,y) \in \mathbb{R}^{N+1}$, $(p_x,p_y) \in \mathbb{R}^{N+1}$, let us introduce the non-coercive Hamiltonian F defined by

$$F(t,x,y,p_x,p_y) = \begin{cases} |p_y| H(t,x,y,|p_y|^{-1}p_x), & \text{if } p_y \neq 0, \\ H_\infty(t,x,y,p_x), & \text{otherwise.} \end{cases} \tag{1.5}$$

The function $U^\epsilon(t,x,y) := u^\epsilon(t,x) - y$ satisfies

$$\begin{cases} U^\epsilon_t + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon+y}{\epsilon}, D_xU^\epsilon, D_yU^\epsilon\right) = 0, & (t,x,y) \in (0,+\infty) \times \mathbb{R}^{N+1}, \\ U^\epsilon(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}. \end{cases} \tag{1.6}$$

In [3] Barles proves that the sequence U^ϵ converges to the solution U^0 of the following problem

$$\begin{cases} U^0_t + \overline{F}(D_xU^0, D_yU^0) = 0, & (t,x,y) \in (0,+\infty) \times \mathbb{R}^{N+1}, \\ U^0(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}, \end{cases} \tag{1.7}$$

where for $(p_x,p_y) \in \mathbb{R}^{N+1}$, $\overline{F}(p_x,p_y)$ is the unique number λ for which the cell problem

$$V_t + F(t,x,y,p_x + D_xV, p_y + D_yV) = \lambda \quad \text{in } \mathbb{R} \times \mathbb{R}^{N+1}. \tag{1.8}$$

admits bounded sub and supersolutions. This result makes it possible to solve the homogenization problem for (1.1):

Theorem 1.2 (Barles, [3]). *Assume (H1)-(H4). Then the sequence* u^ϵ *converges locally uniformly in* $(0, +\infty) \times \mathbb{R}^N$ *to the solution* u^0 *of* (1.3)*. The function* $\overline{H}(p)$ *in* (1.3) *can be characterized as follows:* $\overline{H}(p) = \overline{F}(p, -1)$*, where, for any* $(p_x, p_y) \in \mathbb{R}^{N+1}$*,* $\overline{F}(p_x, p_y)$ *is the unique number* λ *for which the equation* (1.8) *admits bounded sub and supersolutions in* $\mathbb{R} \times \mathbb{R}^{N+1}$*.*

An important step in the proof of Theorem 1.2 consists of homogenizing the non-coercive level-set equation satisfied by $\mathbb{1}_{\{U^\epsilon \geq 0\}}$.
In this paper, we tackle two questions:

- Is it possible to estimate the rate of convergence of u^ϵ to u^0 when $\epsilon \to 0$?
- Is is possible to approximate numerically the effective Hamiltonian?

The first question was answered by Capuzzo Dolcetta and Ishii, [7] for a more classical homogenization problem: the estimate $\|u^\epsilon - u^0\|_\infty \leq C\epsilon^{\frac{1}{3}}$ was obtained for Hamilton-Jacobi equations of the type

$$u^\epsilon + H\left(x, \frac{x}{\epsilon}, Du^\epsilon\right) = 0,$$

where $(x, y, p) \to H(x, y, p)$ is a coercive Hamiltonian, uniformly Lipschitz continuous for $|p|$ bounded and periodic with respect to y; moreover, if $H(x, y, p)$ does not depend on x, then the convergence is linear in ϵ. We will show that in the present case, it is possible to obtain the same rates of convergence as $\epsilon \to 0$ by adapting the proof in [7] using the arguments contained in [3]. Up to our knowledge, these errors estimates are new for evolution problems also in the case the Hamiltonian H does not depend on u^ϵ. Our main result on this topic is Theorem 2.1 in § 2. The main idea is to approximate U^ϵ (with an error smaller than ϵ) by a discontinuous function $\widetilde{U}^\epsilon$ which takes integer values where U^ϵ has noninteger values and which is a discontinuous viscosity solution of

$$\widetilde{U}^\epsilon_t + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, D_x\widetilde{U}^\epsilon, D_y\widetilde{U}^\epsilon\right) = 0, \qquad (t, x, y) \in (0, +\infty) \times \mathbb{R}^{N+1}.$$

The latter equation has to be compared with (1.6). This approximation $\widetilde{U}^\epsilon$ is obtained as the limit as $\delta \to 0$ of $\phi_\delta(U^\epsilon)$ where $(\phi_\delta)_\delta$ is a sequence of increasing functions. The method of Capuzzo Dolcetta and Ishii [7] can then be applied to $\widetilde{U}^\epsilon$. From the estimates of $\|U^\epsilon - U^0\|_\infty$ we automatically obtain the estimates of $\|u^\epsilon - u^0\|_\infty$.
It is useful to recall that the theory of homogenization for first order Hamilton-Jacobi equations started with the famous unpublished work of Lions, Papanicolaou and Varadhan, [18] for the equation

$$u^\epsilon_t + H\left(\frac{x}{\epsilon}, Du^\epsilon\right) = 0, \tag{1.9}$$

where $(y, p) \to H(y, p)$ is a coercive Hamiltonian, uniformly Lipschitz continuous for $|p|$ bounded and periodic with respect to y. Note also that the method of Capuzzo Dolcetta and Ishii [7] has been recently extended to obtain convergence rates for other periodic homogenization problems, in particular by Camilli and Marchi [5] for fully nonlinear elliptic equations, and by Camilli, Cesaroni and Marchi [6] for fully nonlinear elliptic problems with vanishing viscosity.

The second question was studied in [1] for equation (1.9) where $(y, p) \to H(y, p)$ is a coercive Hamiltonian, uniformly Lipschitz continuous for $|p|$ bounded and periodic with respect to y; in this article, a complete numerical method for solving the homogenized problem was studied, including as a main step the approximation of the effective Hamiltonian by solving discrete cell problems. Error estimates were proved. Up to our knowledge, there are so far few numerical results concerning the homogenization of the local model proposed in [16] and of the nonlocal model presented in [17]. The only indirectly related work that we are aware of is the work by Cacace, Chambolle and Monneau [4] on a posteriori error estimates for the effective Hamiltonian of dislocation dynamics (nonlocal model).

In the present article, we will focus on the case $N = 1$ for simplicity and we will study the approximation of the cell problem (1.8) by Eulerian schemes in $\mathbb{R} \times \mathbb{R}^2$. Since we will look for a periodic solution (with unit period) in the space variables, it will be enough to approximate the solution at discrete times $t_n = n\Delta t$ and at the nodes of a discrete unit torus of $\mathbb{R}^2$.

Remark 1.2. *Note that instead of approximating (1.8), it would have been possible to approximate numerically problem (1.4). Yet, although (1.4) has a lower dimensionality, (indeed (1.4) is posed in $\mathbb{R} \times \mathbb{R}^N$ instead of $\mathbb{R} \times \mathbb{R}^{N+1}$), its solutions may not be periodic (indeed we will see in §4 that (1.4) has periodic solutions only if p has rational coordinates, in which case the period depends on p and may be large). This alternative approach is thus feasible if $N = 1$, but seems difficult to generalize to say, $N = 2$. This is the reason why we have preferred to study the approximation of (1.8).*

In § 3, we prove Theorem 3.1, the discrete analogue of the ergodicity Theorems in [3], i.e. that there exists a unique real number $\lambda_h^{\Delta t}$ such that the discrete analogue of (1.8) has a solution. The arguments in the proof are the discrete counterparts of those in [3]. Then, we prove Proposition 3.2, which states that the discrete effective Hamiltonian converges to the effective Hamiltonian when the grid step of the discrete cell problem tends to zero.

To summarize, the paper is organized as follows: Section 2 is devoted to finding estimates on the rate of convergence as $\epsilon \to 0$. Section 3 is devoted to the numerical approximation of the effective Hamiltonian by Eulerian schemes. Finally, we present some numerical tests in Section 4.

2. An estimate on the rate of convergence when $\epsilon \to 0$

This section is devoted to the estimate of the rate of the uniform convergence of the solutions of (1.1) to the solution of the equation (1.3) in term of ϵ.

2.1. *The main result*

Theorem 2.1. *Assume (H1)-(H4) and* $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. *Let* u^ϵ *and* u^0 *be respectively the viscosity solutions of* (1.1) *and* (1.3). *Then there exists a constant* C, *independent of* $\epsilon \in (0,1)$, *such that for any* $T > 0$

$$\sup_{[0,T]\times\mathbb{R}^N} |u^\epsilon(t,x) - u^0(t,x)| \leq Ce^T \epsilon^{\frac{1}{3}}. \tag{2.1}$$

If u_0 *is affine then*

$$\sup_{\mathbb{R}^+\times\mathbb{R}^N} |u^\epsilon(t,x) - u^0(t,x)| \leq C\epsilon. \tag{2.2}$$

2.2. *Preliminary results*

In this section we recall some results that will be used later to obtain error estimates.

The assumptions (H1)-(H4) on H guarantee that F satisfies

(F1) Periodicity: for any $(t, x, y, p_x, p_y) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$

$$F(t+1, x+k, y+1, p_x, p_y) = F(t, x, y, p_x, p_y) \quad \text{for any } k \in \mathbb{Z}^N;$$

(F2) Regularity: $F : \mathbb{R}\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1} \to \mathbb{R}$ is Lipschitz continuous and there exists a constant $C_1 > 0$ such that, for almost every $(t, x, y, p_x, p_y) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$

$$|D_{(t,x)}F(t,x,y,p_x,p_y)| \leq C_1(|p_x| + |p_y|), \ |D_y F(t,x,y,p_x,p_y)| \leq C_1|p_y|,$$

$$|D_{(p_x,p_y)}F(t,x,y,p_x,p_y)| \leq C_1;$$

(F3) Coercivity: $F(t,x,y,p_x,p_y) \to +\infty$ as $|p_x| \to +\infty$ uniformly for $(t,x,y) \in \mathbb{R} \times \mathbb{R}^{N+1}$, $|p_y| \leq R$, for any $R > 0$;

Remark that $F(t,x,y,0,0) = 0$. This and (F2) imply that for every $(t,x,y,p_x,p_y) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$

$$|F(t,x,y,p_x,p_y)| \leq C_1(|p_x| + |p_y|). \tag{2.3}$$

Moreover, by construction, F satisfies the "geometrical" assumption

(F4) For any $(t,x,y,p_x,p_y) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ and any $\lambda > 0$,

$$F(t,x,y,\lambda p_x,\lambda p_y) = \lambda F(t,x,y,p_x,p_y).$$

Assumption (F4) guarantees that the function F in (1.5) is invariant by any nondecreasing change $U \to \varphi(U)$, see [8] and [15], i.e., any function $V = \varphi(U^\epsilon)$, with φ nondecreasing is solution of

$$\begin{cases} V_t + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon + y}{\epsilon}, D_x V, D_y V\right) = 0, & (t,x,y) \in (0,+\infty) \times \mathbb{R}^{N+1}, \\ V(0,x,y) = \varphi(u_0(x) - y), & (x,y) \in \mathbb{R}^{N+1}. \end{cases}$$

Finally, note that (F3) and (F4) imply the existence of a positive constant C_2 such that

$$F(t,x,y,p_x,0) \geq C_2 |p_x| \quad \text{for all } (t,x,y,p_x) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^N. \tag{2.4}$$

In [3], in order to construct sub and supersolutions of (1.8), Barles introduces for $\alpha > 0$ the auxiliary equation

$$W_t^\alpha + F(t,x,y,p_x + D_x W^\alpha, p_y + D_y W^\alpha) + \alpha W^\alpha = 0, \quad (t,x,y) \in \mathbb{R} \times \mathbb{R}^{N+1}, \tag{2.5}$$

with F defined by (1.5), and shows that if (H1)-(H4) hold true, then (2.5) admits a unique continuous periodic viscosity solution. Moreover the limit of $\alpha W^\alpha(t,x,y)$ as $\alpha \to 0^+$ does not depend on (t,x,y) and the half-relaxed limits of $W^\alpha - \min W^\alpha$ provide a bounded subsolution and a bounded supersolution of (1.8), with $\lambda = -\lim_{\alpha\to 0^+} \alpha W^\alpha(t,x,y)$. We use the notation $P = (p_x, p_y) \in \mathbb{R}^{N+1}$ and $W^\alpha(x,y,P)$ for the unique solution of (2.5). We have the following proposition:

Proposition 2.1 (Barles, [3]). *For any $(t,x,y,P) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$, $P = (p_x,p_y)$, the following estimates hold*

(i)

$$\min_{(t,x,y)\in\mathbb{R}\times\mathbb{R}^{N+1}} -F(t,x,y,P) \leq \alpha W^\alpha(t,x,y,P) \leq \max_{(t,x,y)\in\mathbb{R}\times\mathbb{R}^{N+1}} -F(t,x,y,P);$$

(ii) There exists a constant $K_1 > 0$ depending on $\|F(t,x,y,p_x,p_y)\|_\infty$ and C_2 such that

$$\max_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha - \min_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha \leq K_1.$$

Further properties of $W^\alpha(x,y,P)$ are given in the following lemma:

Lemma 2.1. *For any $(t,x,y,P) \in \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ the following estimates hold*

(i) $\alpha |D_P W^\alpha(t,x,y,P)| \leq C_1$*, where C_1 is introduced in (F2);*
(ii) $|\alpha W^\alpha(t,x,y,P) + \overline{F}(P)| \leq \alpha K_1$*, where K_1 is introduced in Proposition 2.1;*
(iii) $W^\alpha(t,x,y,0) \equiv 0$*;*
(iv) $\|D\overline{F}\|_\infty \leq C_1$.

Proof. Let us fix $Q \in \mathbb{R}^{N+1}$. The Lipschitz continuity of F, i.e. (F2), implies that the function $W(t,x,y) = W^\alpha(t,x,y,P+Q)$ satisfies

$$W_t + F(t,x,y,P+DW) + \alpha W \leq C_1 |Q|$$

and then, by comparison

$$\alpha W(t,x,y) \le \alpha W^\alpha(t,x,y,P) + C_1|Q|.$$

A similar argument shows that $\alpha W(t,x,y) \ge \alpha W^\alpha(t,x,y,P) - C_1|Q|$. It then follows

$$\alpha|W^\alpha(t,x,y,P+Q) - W^\alpha(t,x,y,P)| \le C_1|Q|,$$

which proves (i).

Let us turn out to (ii). We claim that

$$\mu := \alpha \max_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha \ge -\overline{F}(P).$$

Indeed, $W^\alpha(t,x,y,P)$ is a supersolution of

$$W^\alpha_t + F(t,x,y,P+DW^\alpha) = -\mu.$$

Let V be a bounded subsolution of (1.8), then by comparison between $W^\alpha + \mu t$ and $V - \overline{F}(P)t$, we have

$$V(t,x,y) - W^\alpha(t,x,y) \le V(0,x,y) - W^\alpha(0,x,y) + t(\overline{F}(P) + \mu).$$

Since V and W^α are bounded, dividing by $t > 0$ and letting t tend to $+\infty$, we obtain $\mu \ge -\overline{F}(P)$. Then from (ii) of Proposition 2.1, for $(t,x,y) \in \mathbb{R}\times\mathbb{R}^{N+1}$,

$$\alpha W^\alpha(t,x,y,P) \ge \alpha \min_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha \ge \alpha \max_{\mathbb{R}\times\mathbb{R}^{N+1}} W^\alpha - \alpha K_1 \ge -\overline{F}(P) - \alpha K_1.$$

A similar argument shows that

$$\alpha W^\alpha(t,x,y,P) + \overline{F}(P) \le \alpha K_1;$$

this concludes the proof of (ii).

Property (iii) follows from $F(t,x,y,0,0) = 0$ and the uniqueness of the periodic solution of (2.5).

Finally, (iv) is an immediate consequence of

$$\overline{F}(P) - \overline{F}(Q) \le 2\alpha K_1 + \alpha\|D_P W^\alpha\|_\infty |P-Q|$$

and of (i). □

We conclude this section by recalling some properties of the solutions u^0 and u^ϵ.

Proposition 2.2. *There exist constants $C_T, L > 0$ such that for any $(t,x),(s,y) \in [0,T]\times\mathbb{R}^N$*

$$|u^\epsilon(t,x)|,\ |u^0(t,x)| \le C_T, \tag{2.6}$$

$$|u^0(t,x) - u^0(s,y)| \le L(|t-s| + |x-y|). \tag{2.7}$$

Moreover, for any $t \in [0,T]$, the Lipschitz constant of $u^0(t,\cdot)$ is the Lipschitz constant of the initial datum u_0.

Proof. By comparison

$$|u^\epsilon(t,x) - u_0(x)| \leq C_0 t$$

where $C_0 = \max_{x,y,|p|\leq |u_0|_{1,\infty}} |H(x,y,p)|$. This implies (2.6) for u^ϵ. Similarly can be showed the same estimate for u^0.

The Lipschitz continuity of u^0 follows from the comparison principle for (1.3), see [2], Theorem III.3.7 and Remark III.3.8. □

2.3. *Proof of the main result*

This section is devoted to the proof of Theorem 2.1. We are going to show that for any $T > 0$

$$\sup_{[0,T]\times\mathbb{R}^{N+1}} |U^\epsilon(t,x,y) - U^0(t,x,y)| \leq Ce^T \epsilon^{\frac{1}{3}},$$

where C does not depend on T. Since $U^\epsilon(t,x,y) = u^\epsilon(t,x) - y$ and $U^0(t,x,y) = u^0(t,x) - y$, this estimate automatically gives (2.1).

Let us consider a function $\zeta : \mathbb{R} \to \mathbb{R}$ with the following properties

$$\begin{cases} \zeta'(s) > 0, & \text{for any } s \in \mathbb{R}, \\ \lim\limits_{s\to+\infty} \zeta(s) = 1, & \lim\limits_{s\to-\infty} \zeta(s) = 0, \\ |\zeta(s) - \chi(s)|,\ |\zeta'(s)| \leq \frac{K_2}{1+s^2}, & \text{for any } s \in \mathbb{R}, \end{cases} \tag{2.8}$$

where we have denoted by $\chi(s)$ the Heaviside function defined by

$$\chi(s) = \begin{cases} 1, \text{ for } s \geq 0, \\ 0, \text{ for } s < 0. \end{cases}$$

For $n \in \mathbb{N}$, ϵ, $\delta > 0$, let us define the function

$$\varphi_\epsilon^{n,\delta}(s) := \sum_{i=-n}^{n} \epsilon\zeta\left(\frac{s - i\epsilon}{\delta}\right) - \epsilon(n+1).$$

Then we have:

Lemma 2.2. *Assume* (2.8)*. Then for any* $s \in \mathbb{R}$*, the limit* $\lim_{n\to+\infty} \varphi_\epsilon^{n,\delta}(s)$ *exists and the function* φ_ϵ^δ*:*

$$\varphi_\epsilon^\delta(s) := \lim_{n\to+\infty} \varphi_\epsilon^{n,\delta}(s)$$

is of class C^1 *with* $(\varphi_\epsilon^\delta)'(s) > 0$ *for any* $s \in \mathbb{R}$*. Moreover*

$$\lim_{\delta\to0^+} \varphi_\epsilon^\delta(s) = \begin{cases} (i-1)\epsilon + \zeta(0)\epsilon, & \text{if } s = i\epsilon, \\ i\epsilon, & \text{if } i\epsilon < s < (i+1)\epsilon. \end{cases} \tag{2.9}$$

Proof. To show that the sequence is convergent it suffices to show that for any $s \in \mathbb{R}$ $\varphi_\epsilon^{n,\delta}(s)$ is a Cauchy sequence. Fix $s \in \mathbb{R}$ and let $i_0 \in \mathbb{Z}$ be the closest integer to s, i.e., $s = i_0\epsilon + \gamma\epsilon$, with $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. Let $k > m > |i_0|$, then, by assumptions (2.8) we have

$$\begin{aligned}\varphi_\epsilon^{k,\delta}(s) - \varphi_\epsilon^{m,\delta}(s) &= \sum_{i=-k}^{-m-1} \epsilon\zeta\left(\frac{s-\epsilon i}{\delta}\right) + \sum_{i=m+1}^{k} \epsilon\zeta\left(\frac{s-\epsilon i}{\delta}\right) - \epsilon(k-m)\\ &= \sum_{i=-k}^{-m-1} \epsilon\left[\zeta\left(\frac{s-\epsilon i}{\delta}\right) - 1\right] + \sum_{i=m+1}^{k} \epsilon\zeta\left(\frac{s-\epsilon i}{\delta}\right)\\ &\le \epsilon K_2\delta^2 \sum_{i=-k}^{-m-1} \frac{1}{(s-\epsilon i)^2} + \epsilon K_2\delta^2 \sum_{i=m+1}^{k} \frac{1}{(s-\epsilon i)^2}\\ &= K_2\frac{\delta^2}{\epsilon} \sum_{i=-k}^{-m-1} \frac{1}{(i_0 - i + \gamma)^2} + K_2\frac{\delta^2}{\epsilon} \sum_{i=m+1}^{k} \frac{1}{(i_0 - i + \gamma)^2}.\end{aligned}$$

Similarly, it can be showed that

$$\varphi_\epsilon^{k,\delta}(s) - \varphi_\epsilon^{m,\delta}(s) \ge -K_2\frac{\delta^2}{\epsilon} \sum_{i=-k}^{-m-1} \frac{1}{(i_0 - i + \gamma)^2} - K_2\frac{\delta^2}{\epsilon} \sum_{i=m+1}^{k} \frac{1}{(i_0 - i + \gamma)^2}.$$

Hence $|\varphi_\epsilon^{k,\delta}(s) - \varphi_\epsilon^{m,\delta}(s)| \to 0$ as $m, k \to +\infty$. Similar arguments show that the sequence $(\varphi_\epsilon^{\delta,n})'$ converge uniformly on compact sets of $\mathbb{R}$. This implies that φ_ϵ^δ is of class C^1 with $(\varphi_\epsilon^\delta)'(s) = \lim_{n\to+\infty}(\varphi_\epsilon^{\delta,n})'(s)$.

Now, let us show (2.9). Let $s = i_0\epsilon + \gamma\epsilon$ for some $i_0 \in \mathbb{Z}$ and $\gamma \in [0,1)$. Then

$$\begin{aligned}&\varphi_\epsilon^{n,\delta}(s) - i_0\epsilon\\ =&\epsilon\left[\zeta\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right] + \sum_{i=-n}^{i_0-1} \epsilon\left[\zeta\left(\frac{i_0\epsilon + \gamma\epsilon - \epsilon i}{\delta}\right) - 1\right] + \sum_{i=i_0+1}^{n} \epsilon\zeta\left(\frac{i_0\epsilon + \gamma\epsilon - \epsilon i}{\delta}\right)\\ \le&\epsilon\left[\zeta\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right]\epsilon + \frac{\delta^2}{\epsilon}K_2 \sum_{i=-n}^{i_0-1} \frac{1}{(i_0 - i + \gamma)^2} + \frac{\delta^2}{\epsilon}K_2 \sum_{i=i_0+1}^{n} \frac{1}{(i - i_0 - \gamma)^2}\\ =&\epsilon\left[\zeta\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right] + \frac{\delta^2}{\epsilon}K_2 \sum_{i=1}^{n+i_0} \frac{1}{(i+\gamma)^2} + \frac{\delta^2}{\epsilon}K_2 \sum_{i=1}^{n-i_0} \frac{1}{(i-\gamma)^2}.\end{aligned}$$

Similarly

$$\varphi_\epsilon^{n,\delta}(s) - i_0\epsilon \ge \epsilon\left[\zeta\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right] - \frac{\delta^2}{\epsilon}K_2 \sum_{i=1}^{n+i_0} \frac{1}{(i+\gamma)^2} - \frac{\delta^2}{\epsilon}K_2 \sum_{i=1}^{n-i_0} \frac{1}{(i-\gamma)^2}.$$

Letting $n \to +\infty$, we get

$$\left|\varphi_\epsilon^{\delta}(s) - i_0\epsilon - \epsilon\left[\zeta\left(\frac{\gamma\epsilon}{\delta}\right) - 1\right]\right| \le \frac{\delta^2}{\epsilon}K_2 \sum_{i=1}^{+\infty} \frac{1}{(i+\gamma)^2} + \frac{\delta^2}{\epsilon}K_2 \sum_{i=1}^{+\infty} \frac{1}{(i-\gamma)^2}.$$

If $\gamma > 0$ then $\zeta\left(\frac{\gamma\epsilon}{\delta}\right) - 1 \to 0$ as $\delta \to 0^+$ and $\varphi_\epsilon^\delta(s) \to i_0\epsilon$ if $\delta \to 0^+$. If $\gamma = 0$, then $\varphi_\epsilon^\delta(s) \to (i_0 - 1)\epsilon + \zeta(0)\epsilon$ if $\delta \to 0^+$ and (2.9) is proved. □

Let us define

$$\widetilde{U}^{\epsilon,\delta}(t,x,y) := \varphi_\epsilon^\delta(U^\epsilon(t,x,y)).$$

Since F satisfies the "geometrical" assumption (F4), the function $\widetilde{U}^{\epsilon,\delta}$ is solution of

$$\begin{cases} \widetilde{U}_t^{\epsilon,\delta} + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon + y}{\epsilon}, D_x \widetilde{U}^{\epsilon,\delta}, D_y \widetilde{U}^{\epsilon,\delta}\right) = 0, & (t,x,y) \in (0,T) \times \mathbb{R}^{N+1}, \\ \widetilde{U}^{\epsilon,\delta}(0,x,y) = \varphi_\epsilon^\delta(u_0(x) - y), & (x,y) \in \mathbb{R}^{N+1}. \end{cases} \tag{2.10}$$

By stability of viscosity solutions, see e.g. [10], the limit $\widetilde{U}^\epsilon(t,x,y)$ of $\widetilde{U}^{\epsilon,\delta}(t,x,y)$ as $\delta \to 0^+$ is a discontinuous viscosity solution of (2.10) with initial datum $\varphi_\epsilon(u_0(x) - y)$, where $\varphi_\epsilon(s) = \lim_{\delta\to 0^+} \varphi_\epsilon^\delta(s)$. This means that $(\widetilde{U}^\epsilon)^* = \limsup^*_{\delta\to 0^+} \widetilde{U}^{\epsilon,\delta}$ (resp. $(\widetilde{U}^\epsilon)_* = \liminf_{*\,\delta\to 0^+} \widetilde{U}^{\epsilon,\delta}$) is a viscosity subsolution (resp. supersolution) of (2.10), and $(\widetilde{U}^\epsilon)^*(0,x,y) \leq (\varphi_\epsilon)^*(u_0(x) - y)$ (resp. $(\widetilde{U}^\epsilon)_*(0,x,y) \geq (\varphi_\epsilon)_*(u_0(x) - y)$). Moreover, by (2.9)

$$\widetilde{U}^\epsilon(t,x,y) = \begin{cases} i\epsilon, & \text{if } i\epsilon < U^\epsilon(t,x,y) < (i+1)\epsilon, \\ (i-1)\epsilon + \zeta(0)\epsilon, & \text{if } (t,x,y) \in \operatorname{Int}\{U^\epsilon = i\epsilon\}. \end{cases}$$

At the points $(t,x,y) \in \partial\{U^\epsilon = i\epsilon\}$, the value of $\widetilde{U}^\epsilon$ depends on the lower semi-continuous or the upper semi-continuous envelope that we consider in the definition of discontinuous viscosity solution. In particular, since U^ϵ is continuous, $\widetilde{U}^\epsilon$ has the following properties

$$|(\widetilde{U}^\epsilon)^*(t,x,y) - U^\epsilon(t,x,y)|,\ |(\widetilde{U}^\epsilon)_*(t,x,y) - U^\epsilon(t,x,y)| \leq \epsilon \quad \forall (t,x,y) \in [0,T] \times \mathbb{R}^{N+1} \tag{2.11}$$

and

$$D\widetilde{U}^\epsilon(t,x,y) = 0 \quad \text{if } U^\epsilon(t,x,y) \neq i\epsilon,\ i \in \mathbb{Z}. \tag{2.12}$$

Condition (2.12) implies that $\widetilde{U}^\epsilon$ is actually a solution of

$$\begin{cases} \widetilde{U}_t^\epsilon + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, D_x \widetilde{U}^\epsilon, D_y \widetilde{U}^\epsilon\right) = 0, & (t,x,y) \in (0,T) \times \mathbb{R}^{N+1}, \\ \widetilde{U}^\epsilon(0,x,y) = \varphi_\epsilon(u_0(x) - y), & (x,y) \in \mathbb{R}^{N+1}. \end{cases}$$

Indeed, when $i\epsilon < U^\epsilon(t,x,y) < (i+1)\epsilon$, for some $i \in \mathbb{Z}$, the function $\widetilde{U}^\epsilon$ is constant in a neighborhood of (t,x,y). Then the result follows from the fact that $F(t,x,y,0) = 0$. On the other hand, when $U^\epsilon(t,x,y) = i\epsilon$, by periodicity, $F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{U^\epsilon + y}{\epsilon}, P\right) = F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, P\right)$.

In order to estimate $|U^\epsilon - U^0|$ it is convenient to estimate $|\widetilde{U}^\epsilon - U^0|$; indeed, $\frac{U^\epsilon}{\epsilon}$ does not any longer appear in the equation satisfied by $\widetilde{U}^\epsilon$.

Let us define $V^\epsilon(t,x,y) = e^{-t}\widetilde{U}^\epsilon(t,x,y)$ and $V^0(t,x,y) = e^{-t}U^0(t,x,y)$. The functions V^ϵ and V^0 are respectively solutions of

$$\begin{cases} V_t^\epsilon + V^\epsilon + F\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \frac{y}{\epsilon}, D_x V^\epsilon, D_y V^\epsilon\right) = 0, & (t,x,y) \in (0,T) \times \mathbb{R}^{N+1}, \\ V^\epsilon(0,x,y) = \varphi_\epsilon(u_0(x) - y), & (x,y) \in \mathbb{R}^{N+1}, \end{cases} \tag{2.13}$$

and

$$\begin{cases} V^0_t + V^0 + \overline{F}(D_x V^0, D_y V^0) = 0, & (t,x,y) \in (0,T) \times \mathbb{R}^{N+1}, \\ V^0(0,x,y) = u_0(x) - y, & (x,y) \in \mathbb{R}^{N+1}. \end{cases} \tag{2.14}$$

For alleviating the notations, let us denote a vector of $\mathbb{R}^{N+1}$ by $X = (x, x_{N+1})$, where $x \in \mathbb{R}^N$ and $x_{N+1} \in \mathbb{R}$. We first estimate from above the difference $(V^\epsilon)^* - V^0$: for this, let us introduce the auxiliary function

$$\begin{aligned} \Phi(t,X,s,Y) = (V^\epsilon)^*(t,X) - V^0(s,Y) - \epsilon W^\alpha\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, \frac{X-Y}{\epsilon^\beta}\right) \\ - \frac{|X-Y|^2}{2\epsilon^\beta} - \frac{|t-s|^2}{2\sigma} - \frac{r}{2}|X|^2 - \frac{\eta}{T-t}, \end{aligned} \tag{2.15}$$

where $\alpha = \epsilon^\theta$, θ, β, σ, r, $\eta \in (0,1)$ will be fix later on and β and θ satisfy

$$0 < \theta < 1 - \beta. \tag{2.16}$$

In view of (2.6), (2.11), (i) of Proposition 2.1 and (2.3),

$$\Phi(t,X,s,Y) \le 2C_T + \epsilon + |x_{N+1} - y_{N+1}| + \frac{\epsilon}{\alpha} C_1 \frac{|X-Y|}{\epsilon^\beta} - \frac{|X-Y|^2}{2\epsilon^\beta} - \frac{r}{2}|X|^2$$

for all (t,X), $(s,Y) \in [0,T] \times \mathbb{R}^{N+1}$. Hence, Φ attains a global maximum at some point $(\overline{t}, \overline{X}, \overline{s}, \overline{Y}) \in ([0,T] \times \mathbb{R}^{N+1})^2$. Standard arguments show that $\overline{t}$, $\overline{s} < T$ for σ small enough.

Claim 1: There exists a constant $M_1 > 0$ independent of ϵ such that $\frac{|\overline{t}-\overline{s}|}{\sigma} \le M_1(1 + |\overline{y}_{N+1}|)$.
The inequality $\Phi(\overline{t}, \overline{X}, \overline{t}, \overline{Y}) \le \Phi(\overline{t}, \overline{X}, \overline{s}, \overline{Y})$ and Proposition (2.2) imply

$$\begin{aligned} \frac{|\overline{t}-\overline{s}|^2}{2\sigma} &\le V^0(\overline{t}, \overline{Y}) - V^0(\overline{s}, \overline{Y}) \le |e^{-\overline{t}} - e^{-\overline{s}}||U^0(\overline{t}, \overline{Y})| + e^{-\overline{s}}|U^0(\overline{t}, \overline{Y}) - U^0(\overline{s}, \overline{Y})| \\ &\le |\overline{t}-\overline{s}|(C_T + |\overline{y}_{N+1}|) + L|\overline{t}-\overline{s}| \end{aligned}$$

from which Claim 1 follows.

Claim 2: There exists a constant $M_2 > 0$ independent of ϵ and T, such that $\frac{|\overline{X}-\overline{Y}|}{\epsilon^\beta} \le M_2$.
The inequality $\Phi(\overline{t}, \overline{X}, \overline{s}, \overline{X}) \le \Phi(\overline{t}, \overline{X}, \overline{s}, \overline{Y})$ implies

$$\frac{|\overline{X}-\overline{Y}|^2}{\epsilon^\beta} \le V^0(\overline{s}, \overline{X}) - V^0(\overline{s}, \overline{Y}) + \epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, 0\right) - \epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right).$$

Using (2.7), (i) of Lemma 2.1 and (2.16) we then infer

$$\begin{aligned} \frac{|\overline{X}-\overline{Y}|^2}{\epsilon^\beta} &\le (L+1)|\overline{X}-\overline{Y}| + \frac{\epsilon}{\alpha} C_1 \frac{|\overline{X}-\overline{Y}|}{\epsilon^\beta} = (L+1)|\overline{X}-\overline{Y}| + \epsilon^{1-\theta-\beta} C_1 |\overline{X}-\overline{Y}| \\ &\le M_2 |\overline{X}-\overline{Y}|. \end{aligned}$$

This concludes the proof of Claim 2.

Claim 3: There exists a constant $M_3 > 0$ independent of ϵ such that $r|\overline{X}|^2 \le M_3$.
The inequality $\Phi(\overline{t}, 0, \overline{s}, 0) \le \Phi(\overline{t}, \overline{X}, \overline{s}, \overline{Y})$ implies

$$\begin{aligned}\frac{r}{2}|\overline{X}|^2 \le &(V^\epsilon)^*(\overline{t}, \overline{X}) - V^0(\overline{s}, \overline{Y}) + V^0(\overline{s}, 0) - (V^\epsilon)^*(\overline{t}, 0)\\ &+ \epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon}, 0, 0\right) - \epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right).\end{aligned}$$

Then, using (2.6), (2.11), Claims 1 and 2, (iii) of Lemma 2.1, (i) of Proposition 2.1 and (2.3), we deduce

$$\begin{aligned}\frac{r}{2}|\overline{X}|^2 \le\ & e^{-\overline{t}}[U^\epsilon(\overline{t}, \overline{X}) - U^0(\overline{s}, \overline{Y})] + |e^{-\overline{t}} - e^{-\overline{s}}||U^0(\overline{s}, \overline{Y})| + \epsilon\\ &+ V^0(\overline{s}, 0) - (V^\epsilon)^*(\overline{t}, 0) - \epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon}, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)\\ &\le 4C_T + M_2\epsilon^\beta + |\overline{t} - \overline{s}|(C_T + |\overline{y}_{N+1}|) + 2\epsilon + \frac{\epsilon}{\alpha}C_1\frac{|\overline{X}-\overline{Y}|}{\epsilon^\beta}\\ &\le C + 2\sigma M_1|\overline{y}_{N+1}|^2 \le C + 2\sigma M_1|\overline{X}|^2,\end{aligned}$$

and Claim 3 follows by choosing $\sigma < \frac{r}{8M_1}$.

Now, suppose first that $\overline{t} = 0$, then

$$\begin{aligned}&(V^\epsilon)^*(t, X) - V^0(t, X) - \epsilon W^\alpha\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, 0\right) - \frac{\eta}{T-t} - \frac{r}{2}|X|^2\\ &\le (\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - V^0(\overline{s}, \overline{Y}) - \epsilon W^\alpha\left(0, \frac{\overline{X}}{\epsilon}, \frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)\end{aligned}$$

for any $(t, X) \in [0, T] \times \mathbb{R}^{N+1}$, from which, using (i) of Proposition 2.1, (iii) of Lemma 2.1, (2.3) and Claim 2, we deduce

$$(V^\epsilon)^*(t, X) - V^0(t, X) \le (\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - V^0(\overline{s}, \overline{Y}) + \frac{\eta}{T-t} + \frac{r}{2}|X|^2 + \epsilon^{1-\theta}C_1M_2.$$

Letting σ, η and r go to 0^+ and using (2.11) and Claim 2 we obtain

$$\begin{aligned}&(V^\epsilon)^*(t, X) - V^0(t, X)\\ \le&(\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - (u_0(\overline{y}) - \overline{y}_{N+1}) + C\epsilon^{1-\theta}\\ \le&(\varphi_\epsilon)^*(u_0(\overline{x}) - \overline{x}_{N+1}) - (u_0(\overline{x}) - \overline{x}_{N+1}) + (L+1)|\overline{X} - \overline{Y}| + C\epsilon^{1-\theta}\\ \le&C(\epsilon^\beta + \epsilon^{1-\theta}) + \epsilon,\end{aligned}$$

which implies

$$U^\epsilon(t, X) - U^0(t, X) \le Ce^t(\epsilon^\beta + \epsilon^{1-\theta}). \tag{2.17}$$

The same estimate can be showed if $\overline{s} = 0$.

Next, let us consider the case $\overline{t}, \overline{s} > 0$.

Claim 4: There exists a constant $C>0$ independent of ϵ and T such that

$$\frac{\overline{t}-\overline{s}}{\sigma}+\frac{\eta}{(T-\overline{t})^2}+(V^\epsilon)^*(\overline{t},\overline{X})+\overline{F}\left(\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)\le C(\epsilon^{1-\theta-\beta}+\epsilon^\theta).$$

The function

$$(t,X)\to(V^\epsilon)^*(t,X)-\epsilon W^\alpha\left(\frac{t}{\epsilon},\frac{X}{\epsilon},\frac{X-\overline{Y}}{\epsilon^\beta}\right)-\frac{|X-\overline{Y}|^2}{2\epsilon^\beta}-\frac{r}{2}|X|^2-\frac{|t-\overline{s}|^2}{2\sigma}-\frac{\eta}{T-t} \tag{2.18}$$

has a maximum at $(\overline{t},\overline{X})$. By adding to Φ a smooth function vanishing with its first derivative at $(\overline{t},\overline{X})$, we may assume the maximum is strict.

Next, for $j>0$, let us introduce the function

$$\begin{aligned}\Psi_j(t,s,X,Y,Z):=(V^\epsilon)^*(t,X)-\epsilon W^\alpha\left(s,Y,\frac{Z-\overline{Y}}{\epsilon^\beta}\right)-\frac{|X-\overline{Y}|^2}{2\epsilon^\beta}-\frac{r}{2}|X|^2-\frac{|t-\overline{s}|^2}{2\sigma}\\ -\frac{\eta}{T-t}-\frac{j}{2}(|t-\epsilon s|^2+|X-Z|^2+|X-\epsilon Y|^2).\end{aligned}$$

Let $P_j=(t_j,s_j,X_j,Y_j,Z_j)$ be a maximum point of Ψ_j on the set

$$A:=\overline{B}(\overline{t},1)\times\overline{B}\left(\frac{\overline{t}}{\epsilon},1\right)\times\overline{B}(\overline{X},1)\times\overline{B}\left(\frac{\overline{X}}{\epsilon},1\right)\times\overline{B}(\overline{X},1).$$

Since $(\overline{t},\overline{X})$ is a strict maximum point of (2.18), $t_j\to\overline{t}$, $s_j\to\frac{\overline{t}}{\epsilon}$, X_j, $Z_j\to\overline{X}$ and $Y_j\to\frac{\overline{X}}{\epsilon}$ as $j\to+\infty$. Then, for j large enough, P_j lies in the interior of A. Moreover, standard arguments show that

$$j|t_j-\epsilon s_j|^2,\quad j|X_j-Z_j|^2,\quad j|X_j-\epsilon Y_j|^2\to 0\quad\text{as } j\to+\infty. \tag{2.19}$$

Remark that this implies in addition that

$$2j|t_j-\epsilon s_j||X_j-\epsilon Y_j|\le j|t_j-\epsilon s_j|^2+j|X_j-\epsilon Y_j|^2\to 0\quad\text{as } j\to+\infty. \tag{2.20}$$

Since $(V^\epsilon)^*$ and W^α are respectively viscosity subsolutions of (2.13) and supersolution of (2.5), we obtain

$$\begin{aligned}&\frac{t_j-\overline{s}}{\sigma}+\frac{\eta}{(T-t_j)^2}+j(t_j-\epsilon s_j)+(V^\epsilon)^*(t_j,X_j)\\ &+F\left(\frac{t_j}{\epsilon},\frac{X_j}{\epsilon},\frac{X_j-\overline{Y}}{\epsilon^\beta}+rX_j+j(X_j-Z_j)+j(X_j-\epsilon Y_j)\right)\le 0\end{aligned} \tag{2.21}$$

and

$$j(t_j-\epsilon s_j)+\alpha W^\alpha\left(s_j,Y_j,\frac{Z^j-\overline{Y}}{\epsilon^\beta}\right)+F\left(s_j,Y_j,\frac{Z^j-\overline{Y}}{\epsilon^\beta}+j(X_j-\epsilon Y_j)\right)\ge 0. \tag{2.22}$$

Subtracting (2.21) and (2.22) and using the Lipschitz continuity of F, assumption (F2), we get

$$\begin{aligned}&\frac{t_j-\overline{s}}{\sigma}+\frac{\eta}{(T-t_j)^2}+(V^\epsilon)^*(t_j,X_j)-\alpha W^\alpha\left(s_j,Y_j,\frac{Z^j-\overline{Y}}{\epsilon^\beta}\right)\\ \le&\frac{C_1}{\epsilon}\left(|t_j-\epsilon s_j|+|X_j-\epsilon Y_j|\right)\left(\frac{|Z^j-\overline{Y}|}{\epsilon^\beta}+j|X_j-\epsilon Y_j|\right)\\ &+C_1\left(\frac{|X_j-Z_j|}{\epsilon^\beta}+r|X_j|+j|X_j-Z_j|\right).\end{aligned}\tag{2.23}$$

Let us estimate $j|X_j-Z_j|$. From the inequality $\Psi_j(t_j,s_j,X_j,Y_j,X_j)\le\Psi_j(t_j,s_j,X_j,Y_j,Z_j)$ we deduce that

$$\frac{j}{2}|X_j-Z_j|^2\le\epsilon W^\alpha\left(s_j,Y_j,\frac{X^j-\overline{Y}}{\epsilon^\beta}\right)-\epsilon W^\alpha\left(s_j,Y_j,\frac{Z^j-\overline{Y}}{\epsilon^\beta}\right),$$

and using (i) of Lemma 2.1 we get

$$\frac{j}{2}|X_j-Z_j|^2\le C_1\frac{\epsilon}{\alpha}\frac{|X_j-Z_j|}{\epsilon^\beta}=C_1\epsilon^{1-\theta-\beta}|X_j-Z_j|.$$

Then

$$j|X_j-Z_j|\le 2C_1\epsilon^{1-\theta-\beta}.\tag{2.24}$$

Then, passing to the limsup as $j\to+\infty$ in (2.23) and taking into account Claim 2, (2.19) and (2.20), we obtain

$$\frac{\overline{t}-\overline{s}}{\sigma}+\frac{\eta}{(T-\overline{t})^2}+(V^\epsilon)^*(\overline{t},\overline{X})-\alpha W^\alpha\left(\frac{\overline{t}}{\epsilon},\frac{\overline{X}}{\epsilon},\frac{\overline{X}-\overline{Y}}{\epsilon}\right)\le C(\epsilon^{1-\theta-\beta}+r|\overline{X}|).\tag{2.25}$$

By Claim 3, $r|\overline{X}|\le r^{\frac12}M_3^{\frac12}$, hence choosing $r>0$ such that $r^{\frac12}M_3^{\frac12}\le\epsilon^{1-\theta-\beta}$, we have $r|\overline{X}|\le\epsilon^{1-\theta-\beta}$.

Finally, Claim 4 easily follows from (2.25), Claim 2 and the following inequality

$$-\alpha W^\alpha\left(\frac{\overline{t}}{\epsilon},\frac{\overline{X}}{\epsilon},\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)\ge\overline{F}\left(\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)-\alpha K_1\ge\overline{F}\left(\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)-K_1\epsilon^\theta$$

which comes from (ii) of Lemma 2.1 .

Claim 5: There exists a constant $C>0$ independent of ϵ and T such that

$$\frac{\overline{t}-\overline{s}}{\sigma}+V^0(\overline{s},\overline{Y})+\overline{F}\left(\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)\ge-C\epsilon^{1-\theta-\beta}.$$

The function

$$(s,Y)\to\phi(s,Y):=V^0(s,Y)+\epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon},\frac{\overline{X}}{\epsilon},\frac{\overline{X}-Y}{\epsilon^\beta}\right)+\frac{|\overline{X}-Y|^2}{2\epsilon^\beta}+\frac{|\overline{t}-s|^2}{2\sigma}$$

has a minimum at $(\overline{s},\overline{Y})$, consequently $(0,0)\in D^-\phi(\overline{s},\overline{Y})$. If we set

$$\widetilde{V}(s,Y):=V^0(s,Y)+\frac{|\overline{X}-Y|^2}{2\epsilon^\beta}+\frac{|\overline{t}-s|^2}{2\sigma},\quad \widetilde{W}(Y):=\epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon},\frac{\overline{X}}{\epsilon},\frac{\overline{X}-Y}{\epsilon^\beta}\right),$$

by properties of semijets of Lipschitz functions, see e.g. Lemma 2.4 in [7], there exists $Q\in\mathbb{R}^{N+1}$ such that

$$(0,Q)\in D^-\widetilde{V}(\overline{s},\overline{Y})=D^-V^0(\overline{s},\overline{Y})-\left(\frac{\overline{t}-\overline{s}}{\sigma},\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)\quad -Q\in D^-\widetilde{W}(\overline{Y}).$$

Since V^0 is a supersolution of (2.14), we have

$$\frac{\overline{t}-\overline{s}}{\sigma}+V^0(\overline{s},\overline{Y})+\overline{F}\left(\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}+Q\right)\geq 0. \tag{2.26}$$

By (i) of Lemma 2.1,

$$\left|\epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon},\frac{\overline{X}}{\epsilon},\frac{\overline{X}-Y}{\epsilon^\beta}\right)-\epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon},\frac{\overline{X}}{\epsilon},\frac{\overline{X}-Z}{\epsilon^\beta}\right)\right|\leq\frac{\epsilon}{\alpha}C_1\frac{|Y-Z|}{\epsilon^\beta}=C_1\epsilon^{1-\theta-\beta}|Y-Z|,$$

from which we get the following estimate of Q:

$$|Q|\leq C_1\epsilon^{1-\theta-\beta}. \tag{2.27}$$

Then, Claim 5 follows from (2.26) using estimate (2.27) and the Lipschitz continuity of $\overline{F}$ assured by (iv) of Lemma 2.1.

Claims 4 and 5 imply

$$(V^\epsilon)^*(\overline{t},\overline{X})-V^0(\overline{s},\overline{Y})\leq C(\epsilon^{1-\theta-\beta}+\epsilon^\theta),$$

for some constant C independent of ϵ and T. Since $(\overline{t},\overline{X},\overline{s},\overline{Y})$ is a maximum point of Φ, we have

$$(V^\epsilon)^*(t,X)-V^0(t,X)\leq\Phi(\overline{t},\overline{X},\overline{s},\overline{Y})+\epsilon W^\alpha\left(\frac{t}{\epsilon},\frac{X}{\epsilon},0\right)+\frac{r}{2}|X|^2+\frac{\eta}{T-t},$$

for all $(t,X)\in[0,T]\times\mathbb{R}^{N+1}$. Then, by (iii) of Lemma 2.1

$$\begin{aligned}&(V^\epsilon)^*(t,X)-V^0(t,X)\\&\leq(V^\epsilon)^*(\overline{t},\overline{X})-V^0(\overline{s},\overline{Y})-\epsilon W^\alpha\left(\frac{\overline{t}}{\epsilon},\frac{\overline{X}}{\epsilon},\frac{\overline{X}-\overline{Y}}{\epsilon^\beta}\right)+\frac{r}{2}|X|^2+\frac{\eta}{T-t}\\&\leq C(\epsilon^{1-\theta-\beta}+\epsilon^\theta)+\frac{\epsilon}{\alpha}C_1\frac{|\overline{X}-\overline{Y}|}{\epsilon^\beta}+\frac{r}{2}|X|^2+\frac{\eta}{T-t}\\&\leq C(\epsilon^{1-\theta-\beta}+\epsilon^\theta)+\frac{r}{2}|X|^2+\frac{\eta}{T-t},\end{aligned}$$

for some positive constant C. Hence, sending r, $\eta,\to 0^+$ and taking into account (2.11), we get

$$U^\epsilon(t,X)-U^0(t,X)\leq Ce^t(\epsilon^{1-\theta-\beta}+\epsilon^\theta).$$

Then, from the previous estimate and (2.17), we can conclude that for all β, $\theta \in (0,1)$ satisfying (2.16) we have

$$U^\epsilon(t,X) - U^0(t,X) \le Ce^t(\epsilon^{1-\theta-\beta} + \epsilon^\theta + \epsilon^\beta),$$

for all $(t,X) \in [0,T] \times \mathbb{R}^{N+1}$. The optimal choice of the parameters is $\theta = \beta = \frac{1}{3}$, which gives

$$\sup_{[0,T]\times\mathbb{R}^{N+1}} (U^\epsilon(t,X) - U^0(t,X)) \le C\epsilon^{\frac{1}{3}}.$$

The opposite inequality follows by similar arguments, replacing $(V^\epsilon)^*$ with V^0 and V^0 with $(V^\epsilon)_*$ in (2.15), and the proof of Theorem 2.1 in the general case is complete.

Now, let us consider the case when u_0 is affine. Let us suppose that $u_0(x) = p \cdot x + c_0$ for some $p \in \mathbb{R}^N$ and $c_0 \in \mathbb{R}$. In this case, the solution of (1.3) is $u^0(t,x) = p \cdot x + c_0 - \overline{H}(p)t$. Let $\overline{V}$ be a bounded viscosity supersolution of (1.8) with $p_x = p$ and $p_y = -1$. Let us define

$$V^\epsilon(t,X) = U^0(t,X) + \epsilon\overline{V}\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}\right).$$

Since $u_0(x) - y \ge \varphi_\epsilon(u_0(x) - y) - \epsilon$ then $V^\epsilon(0,X) \ge \varphi_\epsilon(u_0(x) - y) - (M+1)\epsilon$ where $M = \|\overline{V}\|_\infty$. Hence, it is easy to check that V^ϵ is a supersolution of

$$\begin{cases} V^\epsilon_t + F\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}, D_X V^\epsilon\right) = 0, & (t,X) \in (0,T) \times \mathbb{R}^{N+1}, \\ V^\epsilon(0,X) = \varphi_\epsilon(u_0(x) - y) - (M+1)\epsilon, & (x,y) \in \mathbb{R}^{N+1}. \end{cases}$$

By comparison we get $V^\epsilon(t,X) \ge (\widetilde{U}^\epsilon)^*(t,X) - (M+1)\epsilon$ and this implies that $U^0(t,X) - U^\epsilon(t,X) \ge -C\epsilon$. A similar argument shows that $U^0(t,X) - U^\epsilon(t,X) \le C\epsilon$ and this concludes the proof of the theorem.

3. Approximation of the effective Hamiltonian by Eulerian schemes

In this section we give an approximation of the effective Hamiltonian $\overline{F}(P)$. To this end, we introduce an approximation scheme for the equation (2.5) and for simplicity we only discuss the case $N = 1$. Given N_X and N_t positive integers, we introduce $\Delta t = 1/N_t$, $h = 1/N_X$ and

$$\mathbb{R}^2_h := \{X_{i,j} = (x_i, y_j) \,|\, x_i = ih,\ y_j = jh,\ i,j \in \mathbb{Z}\},$$

$$\mathbb{R}_{\Delta t} := \{t_n = n\Delta t \,|\, n \in \mathbb{Z}\}.$$

An anisotropic mesh with steps h_1 and h_2 is possible too; we take $h_1 = h_2$ only for simplicity. We denote by $W^{n,P,\alpha}_{i,j}$ our numerical approximation of $W^{P,\alpha}$ at $(t_n, x_i, y_j) \in \mathbb{R}_{\Delta t} \times \mathbb{R}^2_h$. For (2.5) we consider the implicit Eulerian scheme of the form

$$\frac{W^{n+1,P,\alpha}_{i,j} - W^{n,P,\alpha}_{i,j}}{\Delta t} + \alpha W^{n+1,P,\alpha}_{i,j} + S(t_n, x_i, y_j, h, [W^{n+1,P,\alpha}]_{i,j}) = 0, \qquad (3.1)$$

where

$$\begin{aligned}&S(t_n, x_i, y_j, h, [W]_{i,j})\\&= g(t_n, x_i, y_j, (\Delta_1^+ W)_{i,j} + p_x, (\Delta_1^+ W)_{i-1,j} + p_x, (\Delta_2^+ W)_{i,j} + p_y, (\Delta_2^+ W)_{i,j-1} + p_y)\end{aligned} \tag{3.2}$$

and

$$(\Delta_1^+ W)_{i,j} = \frac{W_{i+1,j} - W_{i,j}}{h}, \quad (\Delta_2^+ W)_{i,j} = \frac{W_{i,j+1} - W_{i,j}}{h}.$$

We make the following assumptions on g:

(g1) Monotonicity: g is nonincreasing with respect to its fourth and sixth arguments, and nondecreasing with respect to its fifth and seventh arguments;

(g2) Consistency: for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $(q_x, q_y) \in \mathbb{R}^2$

$$g(t, x, y, q_x, q_x, q_y, q_y) = F(t, x, y, q_x, q_y).$$

(g3) Periodicity: for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $Q \in \mathbb{R}^4$

$$g(t + 1, x + 1, y + 1, Q) = g(t, x, y, Q);$$

(g4) Regularity: g is locally Lipschitz continuous and there exists $\widetilde{C}_1 > 0$ such that for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $Q \in \mathbb{R}^4$

$$|D_Q g(t, x, y, Q)| \le \widetilde{C}_1;$$

(g5) Coercivity: there exist $\widetilde{C}_2$, $\widetilde{C}_3 > 0$ such that for any $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$, $(q_1, q_2) \in \mathbb{R}^2$

$$g(t, x, y, q_1, q_2, 0, 0) \ge \widetilde{C}_2(|q_1^-|^2 + |q_2^+|^2)^{\frac{1}{2}} - \widetilde{C}_3;$$

(g6) For any $t \in \mathbb{R}$, $(x, y_1), (x, y_2) \in \mathbb{R}^2$, $q_1, q_2 \in \mathbb{R}$

$$g(t, x, y_1, q_1, q_2, 0, 0) = g(t, x, y_2, q_1, q_2, 0, 0).$$

The points (g1)-(g4) are standard assumptions in the study of numerical schemes for Hamilton-Jacobi equations. The coercivity hypothesis (g5) can be substituted by the weaker condition

$$\lim_{q_1^+ + q_2^- \to +\infty} g(x, y, q_1, q_2, q_3, q_4) = +\infty$$

if g (and hence F) does not depend on time. If g is homogeneous of degree 1 w.r.t. Q, then the two coercivity conditions are equivalent.

As an example, we suppose that the Hamiltonian F is of the form $F(t, x, y, p_x, p_y) = a(t, x)|p_x| + b(t, x, y)|p_y|$, with a and b Lipschitz continuous and periodic functions and $a(t, x) \ge \widetilde{C}_2 > 0$; we consider a generalization of the Godunov scheme proposed in [21]:

$$\begin{aligned}&g(t, x, y, q_1, q_2, q_3, q_4)\\&= a(t, x)[(q_1^-)^2 + (q_2^+)^2]^{\frac{1}{2}} + b^+(t, x, y)[(q_3^-)^2 + (q_4^+)^2]^{\frac{1}{2}} - b^-(t, x, y)[(q_3^+)^2 + (q_4^-)^2]^{\frac{1}{2}}.\end{aligned}$$

where $q^+ = \max(q,0)$ and $q^- = (-q)^+$. Then hypothesis (g1)-(g6) are satisfied.

The following theorem is the discrete version of the analogous result in [3] for the exact solution $W^{P,\alpha}$ of (2.5).

Theorem 3.1. *Assume (g1)-(g6). Then we have*

(i) For any $P = (p_x, p_y) \in \mathbb{R}^2$, α, h, $\Delta t > 0$ *there exists a unique* $(W^{n,P,\alpha}_{i,j})$ *periodic solution of* (3.1);

(ii) There exists a constant $\widetilde{K}_1$ *depending on* $\|F(\cdot,\cdot,\cdot,P)\|_\infty$, $\widetilde{C}_1$ *in (g4)*, $\widetilde{C}_2$, $\widetilde{C}_3$ *in (g5)*, p_x *and* p_y, *but independent of* α, h *and* Δt *such that*

$$\max_{i,j,n} W^{n,P,\alpha}_{i,j} - \min_{i,j,n} W^{n,P,\alpha}_{i,j} \le \widetilde{K}_1;$$

(iii) There exists a constant $\overline{F}^{\Delta t}_h(P)$ *such that*

$$\lim_{\alpha\to 0^+} \alpha W^{n,P,\alpha}_{i,j} = -\overline{F}^{\Delta t}_h(P) \quad \forall i,j,n; \tag{3.3}$$

(iv) $\overline{F}^{\Delta t}_h(P)$ *is the unique number* $\overline{\lambda}^{\Delta t}_h \in \mathbb{R}$ *such that the equation*

$$\frac{W^{n+1,P}_{i,j} - W^{n,P}_{i,j}}{\Delta t} + S(t_n, x_i, y_j, h, [W^{n+1,P}]_{i,j}) = \overline{\lambda}^{\Delta t}_h \tag{3.4}$$

admits a bounded solution.

Proof. A proof of the existence of a unique solution of (3.1) in the uniform grid on the torus with step h is given in [9].

Let us prove (ii). First, remark that by comparison with constants we have

$$\max_{i,j,n} |\alpha W^{n,P,\alpha}_{i,j}| \le C_0, \tag{3.5}$$

where $C_0 := \|F(\cdot,\cdot,\cdot,P)\|_\infty$. Next, let us define

$$\overline{W}^n_i := \max_j W^{n,P,\alpha}_{i,j}.$$

We claim that $\overline{W}^n_i$ satisfies

$$\frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t} + \alpha \overline{W}^{n+1}_i + \overline{S}(t_n, x_i, h, [\overline{W}^{n+1}]_i) \le 0,$$

where

$$\overline{S}(t_n, x_i, h, [W]_i) := \min_j g(t_n, x_i, y_j, (\Delta^+_1 W)_i + p_x, (\Delta^+_1 W)_{i-1} + p_x, p_y, p_y).$$

Indeed, for any i and n, denote by $\overline{j}_{(i,n)}$ the index j such that $\overline{W}^n_i = \max_j W^{n,P,\alpha}_{i,j} = W^{n,P,\alpha}_{i,\overline{j}_{(i,n)}}$, then

$$\frac{W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}} - W^{n,P,\alpha}_{i,\overline{j}_{(i,n+1)}}}{\Delta t} \ge \frac{W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}} - W^{n,P,\alpha}_{i,\overline{j}_{(i,n)}}}{\Delta t} = \frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t},$$

$$\begin{aligned}(\Delta_1^+ W^{n+1,P,\alpha})_{i,\overline{j}_{(i,n+1)}} &= \frac{W^{n+1,P,\alpha}_{i+1,\overline{j}_{(i,n+1)}} - W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}}}{h}\\ &\le \frac{W^{n+1,P,\alpha}_{i+1,\overline{j}_{i+1,n+1}} - W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}}}{h} = (\Delta_1^+ \overline{W}^{n+1})_i,\end{aligned}$$

$$\begin{aligned}(\Delta_1^+ W^{n+1,P,\alpha})_{i-1,\overline{j}_{(i,n+1)}} &= \frac{W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}} - W^{n+1,P,\alpha}_{i-1,\overline{j}_{(i,n+1)}}}{h}\\ &\ge \frac{W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}} - W^{n+1,P,\alpha}_{i-1,\overline{j}_{(i-1,n+1)}}}{h} = (\Delta_1^+ \overline{W}^{n+1})_{i-1},\end{aligned}$$

and

$$(\Delta_2^+ W^{n+1,P,\alpha})_{i,\overline{j}_{(i,n+1)}} = \frac{W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}+1} - W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}}}{h} \le 0,$$

$$(\Delta_2^+ W^{n+1,P,\alpha})_{i,\overline{j}_{(i,n+1)}-1} = \frac{W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}} - W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}-1}}{h} \ge 0.$$

Since $(W^{n,P,\alpha}_{i,j})$ satisfies (3.1), using the monotonicity assumption (g1), we get

$$\begin{aligned}&\frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t} + \alpha \overline{W}^{n+1}_i + \overline{S}(t_n, x_i, h, [\overline{W}^{n+1}]_i)\\ &\le \frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t} + \alpha W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}}\\ &+ g(t_n, x_i, y_{\overline{j}_{(i,n+1)}}, (\Delta_1^+ \overline{W}^{n+1})_i + p_x, (\Delta_1^+ \overline{W}^{n+1})_{i-1} + p_x, p_y, p_y)\\ &\le \frac{W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}} - W^{n,P,\alpha}_{i,\overline{j}_{(i,n+1)}}}{\Delta t} + \alpha W^{n+1,P,\alpha}_{i,\overline{j}_{(i,n+1)}}\\ &+ g(t_n, x_i, y_{\overline{j}_{(i,n+1)}}, (\Delta_1^+ W^{n+1,P,\alpha})_{i,\overline{j}_{(i,n+1)}} + p_x, (\Delta_1^+ W^{n+1,P,\alpha})_{i-1,\overline{j}_{(i,n+1)}} + p_x,\\ &(\Delta_2^+ W^{n+1,P,\alpha})_{i,\overline{j}_{(i,n+1)}} + p_y, (\Delta_2^+ W^{n+1,P,\alpha})_{i,\overline{j}_{(i,n+1)}-1} + p_y)\\ &\le 0,\end{aligned}$$

as desired. Then, by (g4), (g5) and (3.5), we see that $\overline{W}^n_i$ satisfies

$$\frac{\overline{W}^{n+1}_i - \overline{W}^n_i}{\Delta t} + \widetilde{C}_2 \left(|[(\Delta_1^+ \overline{W}^{n+1})_i + p_x]^-|^2 + |[(\Delta_1^+ \overline{W}^{n+1})_{i-1} + p_x]^+|^2 \right)^{\frac{1}{2}} - \le 0,$$

where $K_1 = C_0 + \widetilde{C}_3 + 2\widetilde{C}_1 |p_y|$. In particular we infer that

$$\overline{W}^{n+1}_i - \overline{W}^n_i \le K_1 \Delta t,$$

which implies that if $n \ge m$ then

$$\overline{W}^n_i - \overline{W}^m_i \le K_1 (n - m)\Delta t = K_1 (t_n - t_m). \tag{3.6}$$

Next, let us consider

$$\overline{\overline{W}}_i = \max_n \overline{W}_i^n.$$

Similar arguments as before show that $\overline{\overline{W}}_i$ satisfies

$$\widetilde{C}_2 \left(|[(\Delta_1^+ \overline{\overline{W}})_i + p_x]^-|^2 + |[(\Delta_1^+ \overline{\overline{W}})_{i-1} + p_x]^+|^2 \right)^{\frac{1}{2}} \le K_1,$$

which implies the existence of a constant $K_2 > 0$ depending on $C_0, \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, p_x$ and p_y such that

$$\max_i |(\Delta_1^+ \overline{\overline{W}})_i| \le K_2. \tag{3.7}$$

Now, let (i_1, n_1) and (i_2, n_2) be such that $\max_{i,n} \overline{W}_i^n = \overline{W}_{i_1}^{n_1}$ and $\min_{i,n} \overline{W}_i^n = \overline{W}_{i_2}^{n_2}$, and let n_{i_2} be such that $\overline{\overline{W}}_{i_2} = \max_n \overline{W}_{i_2}^n = \overline{W}_{i_2}^{n_{i_2}}$. By periodicity, we may take $|x_{i_1} - x_{i_2}| \le 1$ and $0 \le t_{n_{i_2}} - t_{n_2} \le 1$. Then using (3.7) and (3.6), we get

$$\begin{aligned}
\overline{W}_{i_1}^{n_1} &= \overline{\overline{W}}_{i_1} \\
&\le \overline{\overline{W}}_{i_2} + K_2 |x_{i_1} - x_{i_2}| \\
&\le \overline{W}_{i_2}^{n_{i_2}} + K_2 \\
&\le \overline{W}_{i_2}^{n_2} + K_1 (t_{n_{i_2}} - t_{n_2}) + K_2 \\
&\le \overline{W}_{i_2}^{n_2} + K_0.
\end{aligned}$$

Then we have proved that

$$\max_{i,n} \overline{W}_i^n - \min_{i,n} \overline{W}_i^n \le K_0, \tag{3.8}$$

where K_0 depends only on $C_0, \widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3, p_x$ and p_y.

Next, we consider the behavior of $W_{i,j}^{n,P,\alpha}$ in j. We claim that

$$\begin{gathered}
W_{i,j_1}^{n,P,\alpha} + p_y y_{j_1} \le W_{i,j_2}^{n,P,\alpha} + p_y y_{j_2} \quad \text{if } j_1 \ge j_2 \text{ and } p_y < 0, \\
W_{i,j_1}^{n,P,\alpha} = W_{i,j_2}^{n,P,\alpha} \quad \text{for any } j_1, j_2 \text{ if } p_y = 0, \\
W_{i,j_1}^{n,P,\alpha} + p_y y_{j_1} \ge W_{i,j_2}^{n,P,\alpha} + p_y y_{j_2} \quad \text{if } j_1 \ge j_2 \text{ and } p_y > 0.
\end{gathered} \tag{3.9}$$

Let us consider the case $p_y < 0$. Suppose by contradiction that

$$M := \max_{i,n,j_1 \ge j_2} (W_{i,j_1}^{n,P,\alpha} - W_{i,j_2}^{n,P,\alpha} + p_y (y_{j_1} - y_{j_2})) = W_{\bar{i},\bar{j}_1}^{\overline{n},P,\alpha} - W_{\bar{i},\bar{j}_2}^{\overline{n},P,\alpha} + p_y (y_{\bar{j}_1} - y_{\bar{j}_2}) > 0.$$

Then $\bar{j}_1 \ge \bar{j}_2 + 1$. We have the following estimate

$$\begin{aligned}
(\Delta_1^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_1} - (\Delta_1^+ W^{\overline{n},P,\alpha})_{\bar{i},\bar{j}_2} &= \frac{W_{\bar{i}+1,\bar{j}_1}^{\overline{n},P,\alpha} - W_{\bar{i},\bar{j}_1}^{\overline{n},P,\alpha}}{h} - \frac{W_{\bar{i}+1,\bar{j}_2}^{\overline{n},P,\alpha} - W_{\bar{i},\bar{j}_2}^{\overline{n},P,\alpha}}{h} \\
&= \frac{W_{\bar{i}+1,\bar{j}_1}^{\overline{n},P,\alpha} - W_{\bar{i}+1,\bar{j}_2}^{\overline{n},P,\alpha}}{h} - \frac{W_{\bar{i},\bar{j}_1}^{\overline{n},P,\alpha} - W_{\bar{i},\bar{j}_2}^{\overline{n},P,\alpha}}{h} \le 0.
\end{aligned}$$

Similarly

$$(\Delta_1^+ W^{\overline{n},P,\alpha})_{\overline{i}-1,\overline{j}_1} \geq (\Delta_1^+ W^{\overline{n},P,\alpha})_{\overline{i}-1,\overline{j}_2},$$

and

$$\frac{W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_1} - W^{\overline{n}-1,P,\alpha}_{\overline{i},\overline{j}_1}}{\Delta t} \geq \frac{W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_2} - W^{\overline{n}-1,P,\alpha}_{\overline{i},\overline{j}_2}}{\Delta t}.$$

Moreover, we have

$$\begin{aligned}
&(\Delta_2^+ W^{\overline{n},P,\alpha})_{\overline{i},\overline{j}_1} + p_y = \frac{W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_1+1} - W^{\overline{n}P,\alpha}_{\overline{i},\overline{j}_1}}{h} + p_y \\
=&\frac{W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_1+1} - W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_2}}{h} + p_y \frac{y_{\overline{j}_1+1} - y_{\overline{j}_2}}{h} - \frac{W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_1} - W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_2}}{h} - p_y \frac{y_{\overline{j}_1} - y_{\overline{j}_2}}{h} \leq 0,
\end{aligned}$$

similarly

$$(\Delta_2^+ W^{\overline{n},P,\alpha})_{\overline{i},\overline{j}_1-1} + p_y \geq 0,\ (\Delta_2^+ W^{\overline{n},P,\alpha})_{\overline{i},\overline{j}_2} + p_y \geq 0,\ (\Delta_2^+ W^{\overline{n},P,\alpha})_{\overline{i},\overline{j}_2-1} + p_y \leq 0.$$

Then, since $W^{\overline{n},P,\alpha}_{i,j}$ satisfies (3.1), using assumptions (g1) and (g6), we get

$$\begin{aligned}
&\alpha(W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_1} - W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_2}) \\
\leq& - g(t_{\overline{n}}, x_{\overline{i}}, y_{\overline{j}_1}, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\overline{i},\overline{j}_1} + p_x, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\overline{i}-1,\overline{j}_1} + p_x, 0, 0) \\
&+ g(t_{\overline{n}}, x_{\overline{i}}, y_{\overline{j}_2}, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\overline{i},\overline{j}_1} + p_x, (\Delta_1^+ W^{\overline{n},P,\alpha})_{\overline{i}-1,\overline{j}_1} + p_x, 0, 0) = 0.
\end{aligned}$$

This implies that

$$0 < \alpha M = \alpha(W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_1} - W^{\overline{n},P,\alpha}_{\overline{i},\overline{j}_2} + p_y(y_{\overline{j}_1} - y_{\overline{j}_2})) \leq \alpha p_y (y_{\overline{j}_1} - y_{\overline{j}_2}) < 0,$$

which is a contradiction and this concludes the proof of (3.9) for $p_y < 0$. The case $p_y \geq 0$ can be treated in an analogous way.

Now, to prove (ii), we use the properties (3.8) and (3.9) of $W^{n,P,\alpha}_{i,j}$ and again we only consider the case $p_y < 0$. Let (i_1, j_1, n_1) and (i_2, j_2, n_2) be such that $W^{n_1,P,\alpha}_{i_1,j_1} = \max_{i,j,n} W^{n,P,\alpha}_{i,j}$ and $W^{n_2,P,\alpha}_{i_2,j_2} = \min_{i,j,n} W^{n,P,\alpha}_{i,j}$. Let $\overline{j}$ be such that $\overline{W}^{n_2}_{i_2} = W^{n_2,P,\alpha}_{i_2,\overline{j}}$. By periodicity, we can take $0 \leq y_{\overline{j}} - y_{j_2} \leq 1$ and $|x_{i_1} - x_{i_2}| \leq 1$. Then

$$\begin{aligned}
W^{n_1,P,\alpha}_{i_1,j_1} &= \overline{W}^{n_1}_{i_1} \\
&\leq \overline{W}^{n_2}_{i_2} + K_0 \\
&= W^{n_2,P,\alpha}_{i_2,\overline{j}} + K_0 \\
&\leq W^{n_2,P,\alpha}_{i_2,j_2} + p_y(y_{j_2} - y_{\overline{j}}) + K_0 \\
&\leq W^{n_2,P,\alpha}_{i_2,j_2} - p_y + K_0,
\end{aligned}$$

and this concludes the proof of (ii).

The property (iii) easily follows from (ii) and (3.5). Indeed, from (3.5), up to subsequence, $\alpha \min_{i,j,n} W_{i,j}^{n,P,\alpha}$ converges to a constant $-\overline{F}_h^{\Delta t}(P)$ as $\alpha \to 0^+$. Then from (ii), for any i, j, n, we get

$$\begin{aligned} |\alpha W_{i,j}^{n,P,\alpha} + \overline{F}_h^{\Delta t}(P)| &\le |\alpha \min_{i,j,n} W_{i,j}^{n,P,\alpha} + \overline{F}_h^{\Delta t}(P)| + \alpha |W_{i,j}^{n,P,\alpha} - \min_{i,j,n} W_{i,j}^{n,P,\alpha}| \\ &\le |\alpha \min_{i,j,n} W_{i,j}^{n,P,\alpha} + \overline{F}_h^{\Delta t}| + \alpha \widetilde{K}_1 \to 0 \quad \text{as } \alpha \to 0^+, \end{aligned}$$

and (iii) is proved.

Let us turn to (iv). Let us define $Z_{i,j}^{n,P,\alpha} = W_{i,j}^{n,P,\alpha} - \min_{i,j,n} W_{i,j}^{n,P,\alpha}$. By (ii), up to subsequence, $(Z_{i,j}^{n,P,\alpha})$ converges to a grid function $(Z_{i,j}^{n,P})$ as $\alpha \to 0^+$. The grid function $(Z_{i,j}^{n,P,\alpha})$ satisfies

$$\frac{Z_{i,j}^{n+1,P,\alpha} - Z_{i,j}^{n,P,\alpha}}{\Delta t} + \alpha Z_{i,j}^{n+1,P,\alpha} + S(t_n, x_i, y_j, h, [Z^{n+1,P,\alpha}]_{i,j}) = -\alpha \min_{i,j,n} W_{i,j}^{n,P,\alpha}.$$

Letting $\alpha \to 0^+$, since by (ii) $(Z_{i,j}^{n,P,\alpha})$ is bounded and $\alpha \min_{i,j,n} W_{i,j}^{n,P,\alpha} \to -\overline{F}_h^{\Delta t}$, we see that $(Z_{i,j}^{n,P})$ is a solution of (3.4) with $\overline{\lambda}_h^{\Delta t} = \overline{F}_h^{\Delta t}$.

To prove the uniqueness of a solution $(\overline{\lambda}_h^{\Delta t}, (W_{i,j}^{n,P}))$ of (3.4), we show that if there exists a subsolution $(U_{i,j}^{n,P})$ of (3.4) with $\overline{\lambda}_h^{\Delta t} = \lambda_1$ and a supersolution $(V_{i,j}^{n,P})$ of (3.4) with $\overline{\lambda}_h^{\Delta t} = \lambda_2$, then $\lambda_2 \le \lambda_1$.

Let $M = \max_{i,j,n}(U_{i,j}^{n,P} - V_{i,j}^{n,P}) = U_{i_0,j_0}^{n_0,P} - V_{i_0,j_0}^{n_0,P}$. Then

$$\frac{U_{i_0,j_0}^{n_0,P} - U_{i_0,j_0}^{n_0-1,P}}{\Delta t} \ge \frac{V_{i_0,j_0}^{n_0,P} - V_{i_0,j_0}^{n_0-1,P}}{\Delta t},$$

$$(\Delta_1^+ U^{n_0,P})_{i_0,j_0} \le (\Delta_1^+ V^{n_0,P})_{i_0,j_0}, \quad (\Delta_1^+ U^{n_0,P})_{i_0-1,j_0} \ge (\Delta_1^+ V^{n_0,P})_{i_0-1,j_0},$$

$$(\Delta_2^+ U^{n_0,P})_{i_0,j_0} \le (\Delta_2^+ V^{n_0,P})_{i_0,j_0}, \quad (\Delta_2^+ U^{n_0,P})_{i_0,j_0-1} \ge (\Delta_2^+ V^{n_0,P})_{i_0,j_0-1}.$$

From the monotonicity of g,

$$\begin{aligned} \lambda_1 \ge\ & \frac{U_{i_0,j_0}^{n_0,P} - U_{i_0,j_0}^{n_0-1,P}}{\Delta t} + g\big(t_{n_0}, x_{i_0}, y_{j_0}, (\Delta_1^+ U^{n_0,P})_{i_0,j_0} + p_x, (\Delta_1^+ U^{n_0,P})_{i_0-1,j_0} + p_x, \\ & (\Delta_2^+ U^{n_0,P})_{i_0,j_0} + p_y, (\Delta_2^+ U^{n_0,P})_{i_0,j_0-1} + p_y\big) \\ \ge\ & \frac{V_{i_0,j_0}^{n_0,P} - V_{i_0,j_0}^{n_0-1,P}}{\Delta t} + g\big(t_{n_0}, x_{i_0}, y_{j_0}, (\Delta_1^+ V^{n_0,P})_{i_0,j_0} + p_x, (\Delta_1^+ V^{n_0,P})_{i_0-1,j_0} + p_x, \\ & (\Delta_2^+ V^{n_0,P})_{i_0,j_0} + p_y, (\Delta_2^+ V^{n_0,P})_{i_0,j_0-1} + p_y\big) \\ \ge\ & \lambda_2. \end{aligned}$$

This concludes the proof of (iv). □

We need a more precise estimate on the rate of convergence of $\alpha W_{i,j}^{n,\alpha,P}$ to $\overline{F}_h^{\Delta t}(P)$:

Proposition 3.1. *Assume (g1)-(g6). Then for any* i, j, n

$$|\alpha W_{i,j}^{n,\alpha,P} + \overline{F}_h^{\Delta t}(P)| \le \widetilde{K}_1 \alpha,$$

where $\widetilde{K}_1 = \widetilde{K}_1(P)$ *is the constant in (ii) of Theorem 3.1.*

Proof. As in the proof of (ii) of Lemma 2.1, the result follows from the comparison principle for (3.1) and (ii) of Theorem 3.1. □

Now, we are ready to show that the function $\overline{F}_h^{\Delta t}$ is actually an approximation of the effective Hamiltonian $\overline{F}$.

Proposition 3.2. *Assume (g1)-(g6). Let* $\overline{F}_h^{\Delta t}$ *be defined by* (3.3) *and let* $\overline{F}$ *be the effective Hamiltonian. Then, for any* $P \in \mathbb{R}^2$

$$\lim_{(\Delta t, h)\to(0,0)} \overline{F}_h^{\Delta t}(P) = \overline{F}(P)$$

uniformly on compact sets of $\mathbb{R}^2$.

Proof. To show the result we estimate $W^{P,\alpha}(t_n, x_i, y_j) - W_{i,j}^{n,P,\alpha}$. To this end, following the same proof as in [11] and [1], we assume that

$$\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n, x_i, y_j) - \alpha W_{i,j}^{n,P,\alpha}| = \sup_{i,j,n} (\alpha W^{P,\alpha}(t_n, x_i, y_j) - \alpha W_{i,j}^{n,P,\alpha}) = m \ge 0.$$

The case when $\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n, x_i, y_j) - \alpha W_{i,j}^{n,P,\alpha}| = \sup_{i,j,n}(\alpha W_{i,j}^{n,P,\alpha} - \alpha W^{P,\alpha}(t_n, x_i, y_j))$ is handled in a similar manner.

For simplicity of notations we omit the index P. Let us denote $W_{h,\Delta t}^{\alpha}(t_n, X_{i,j}) := W_{i,j}^{n,\alpha}$, $(t_n, X_{i,j}) \in \mathbb{R}_{\Delta t} \times \mathbb{R}_h^2$. For $(X, Y) \in \mathbb{R}^2 \times \mathbb{R}_h^2$ and $(t, s) \in \mathbb{R} \times \mathbb{R}_{\Delta t}$, consider the function

$$\Psi(t, X, s, Y) = \alpha W^{\alpha}(t, X) - \alpha W_{h,\Delta t}^{\alpha}(s, Y) + \left(5C_0 + \frac{m}{2}\right) \beta_\epsilon(t - s, X - Y),$$

where, as before, $C_0 = \|F(\cdot,\cdot,\cdot,P)\|_\infty$ and $\beta_\epsilon = \beta\left(\frac{t}{\epsilon}, \frac{X}{\epsilon}\right)$ with β a non-negative smooth function such that

$$\begin{cases} \beta(t, X) = 1 - |X|^2 - |t|^2, & \text{if } |X|^2 + |t|^2 \le \frac{1}{2}, \\ \beta \le \frac{1}{2}, & \text{if } \frac{1}{2} \le |X|^2 + |t|^2 \le 1, \\ \beta = 0, & \text{if } |X|^2 + |t|^2 > 1. \end{cases}$$

We have the following lemma:

Lemma 3.1. *The function* Ψ *attains its maximum at a point* (t_0, X_0, s_0, Y_0) *such that*

(i) $\Psi(t_0, X_0, s_0, Y_0) \ge 5C_0 + \frac{3}{2}m$;
(ii) $\beta_\epsilon(t_0 - s_0, X_0 - Y_0) \ge \frac{3}{5}$.

For the proof, see Lemma 4.1 in [11].

Lemma 3.1 (ii) implies that

$$\beta_\epsilon(t_0-s_0,X_0-Y_0)=1-\left|\frac{X_0-Y_0}{\epsilon}\right|^2-\left|\frac{t_0-s_0}{\epsilon}\right|^2.$$

Then, from the inequality $\Psi(s_0,Y_0,s_0,Y_0)\le\Psi(t_0,X_0,s_0,Y_0)$ we deduce that

$$\left(5C_0+\frac{m}{2}\right)\left(\left|\frac{X_0-Y_0}{\epsilon}\right|^2+\left|\frac{t_0-s_0}{\epsilon}\right|^2\right)\le\alpha W^\alpha(t_0,X_0)-\alpha W^\alpha(s_0,Y_0)\le 2C_0. \tag{3.10}$$

This implies that $|t_0-s_0|\to 0$ and $|X_0-Y_0|\to 0$ as $\epsilon\to 0$. Moreover, since W^α and $W^\alpha_{h,\Delta t}$ are periodic, we can assume that (t_0,X_0,s_0,Y_0) lies in a compact set of $(\mathbb{R}\times\mathbb{R}^2)^2$. Hence, from (3.10) and the continuity of W^α we get that

$$\left|\frac{X_0-Y_0}{\epsilon}\right|^2+\left|\frac{t_0-s_0}{\epsilon}\right|^2\to 0\quad\text{as }\epsilon\to 0. \tag{3.11}$$

Since (t_0,X_0) is a maximum point of $(t,X)\to\alpha W^\alpha(t,X)+\left(5C_0+\frac{m}{2}\right)\beta_\epsilon(t-s_0,X-Y_0)$, we have

$$\begin{aligned}&-\frac{5C_0+\frac{m}{2}}{\alpha}\partial_t\beta_\epsilon(t_0-s_0,X_0-Y_0)+\alpha W^\alpha(t_0,X_0)\\&+F\left(t_0,X_0,-\frac{5C_0+m/2}{\alpha}D_X\beta_\epsilon(t_0-s_0,X_0-Y_0)+P\right)\le 0.\end{aligned} \tag{3.12}$$

Let i_0,j_0 and n_0 be such that $X_{i_0,j_0}=Y_0$ and $s_0=t_{n_0}$. Since (s_0,Y_0) is a minimum point of $(s,Y)\to\alpha W^\alpha_{h,\Delta t}(s,Y)-(5C_0+m/2)\,\beta_\epsilon(t_0-s,X_0-Y)$, we obtain

$$W^{n_0,\alpha}_{i_0+1,j_0}-W^{n_0,\alpha}_{i_0,j_0}\ge\frac{5C_0+m/2}{\alpha}[\beta_\epsilon(t_0-s_0,X_0-Y_0-he_1)-\beta_\epsilon(t_0-s_0,X_0-Y_0)],$$

where $e_1=(1,0)^T$. From the monotonicity of g,

$$\begin{aligned}&\frac{W^{n_0,\alpha}_{i_0,j_0}-W^{n_0-1,\alpha}_{i_0,j_0}}{\Delta t}+\alpha W^{n_0,\alpha}_{i_0,j_0}+g\left(s_0,Y_0,\frac{5C_0+m/2}{\alpha}(\Delta_1^+\beta_\epsilon(t_0-s_0,X_0-\cdot))_{i_0,j_0}+p_x,\right.\\&\left.(\Delta_1^+W^{n_0,\alpha})_{i_0-1,j_0}+p_x,(\Delta_2^+W^{n_0,\alpha})_{i_0,j_0}+p_y,(\Delta_2^+W^{n_0,\alpha})_{i_0,j_0-1}+p_y\right)\ge 0.\end{aligned} \tag{3.13}$$

But

$$|(\Delta_1^+\beta_\epsilon(t_0-s_0,X_0-\cdot))_{i_0,j_0}-e_1\cdot D_Y\beta_\epsilon(t_0-s_0,X_0-Y_0)|=\frac{h}{2}|e_1^TD^2_{YY}\beta_\epsilon(t_0-s_0,X_0-\overline{Y})e_1|,$$

for some $\overline{Y}$ belonging to the segment (Y_0,Y_0+he_1). Assuming h small enough, so that Lemma 3.1 (ii) implies that $|t_0-s_0|^2+|X_0-Y_0|^2+h^2\le\frac{\epsilon^2}{2}$, we obtain that $D^2_{YY}\beta_\epsilon(t_0-s_0,X_0-\overline{Y})=\frac{2}{\epsilon^2}I$, then

$$|(\Delta_1^+\beta_\epsilon(t_0-s_0,X_0-\cdot))_{i_0,j_0}-e_1\cdot D_Y\beta_\epsilon(t_0-s_0,X_0-Y_0)|=\frac{h}{\epsilon^2}. \tag{3.14}$$

Now, (3.13), (3.14) and the monotonicity of g yield

$$\begin{aligned}
&\frac{W^{n_0,\alpha}_{i_0,j_0} - W^{n_0-1,\alpha}_{i_0,j_0}}{\Delta t} + \alpha W^{n_0,\alpha}_{i_0,j_0}\\
&+ g\left(\begin{array}{l} s_0, Y_0, \dfrac{5C_0+m/2}{\alpha} e_1 \cdot D_Y \beta_\epsilon(t_0-s_0, X_0-Y_0) + p_x,\\ (\Delta_1^+ W^{n_0,\alpha})_{i_0-1,j_0} + p_x, (\Delta_2^+ W^{n_0,\alpha})_{i_0,j_0} + p_y, (\Delta_2^+ W^{n_0,\alpha})_{i_0,j_0-1} + p_y \end{array}\right)\\
&+ \widetilde{C}_1 h \frac{5C_0+m/2}{\epsilon^2 \alpha} \geq 0.
\end{aligned}$$

Repeating similar estimates for the other arguments in g and for the derivative with respect to time, we finally find that

$$\begin{aligned}
&\frac{5C_0+m/2}{\alpha} \partial_s \beta_\epsilon(t_0-s_0, X_0-Y_0) + \alpha W^{n_0,\alpha}_{i_0,j_0} +\\
&F\left(s_0, Y_0, \frac{5C_0+m/2}{\alpha} D_Y \beta_\epsilon(t_0-s_0, X_0-Y_0) + P\right) + C\frac{h+\Delta t}{\epsilon^2 \alpha} \geq 0,
\end{aligned} \tag{3.15}$$

where C is independent of $h, \Delta t, \epsilon$ and α.

Subtracting (3.12) and (3.15) and using (F2) we get

$$\alpha W^\alpha(t_0, X_0) - \alpha W^\alpha_{h,\Delta t}(s_0, Y_0) \leq C\frac{h+\Delta t}{\epsilon^2\alpha} + \frac{C}{\alpha}\left|\frac{X_0-Y_0}{\epsilon}\right|^2 + \frac{C}{\alpha}\left|\frac{t_0-s_0}{\epsilon}\right|^2, \tag{3.16}$$

where C is independent of $h, \Delta t, \epsilon$ and α.

Choose $\epsilon = \epsilon(\Delta t, h)$ such that $\epsilon \to 0$ as $(\Delta t, h) \to (0,0)$ and $\frac{h+\Delta t}{\epsilon^2} \to 0$ as $(\Delta t, h) \to (0,0)$. From (i) of Lemma 3.1

$$\begin{aligned}
\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n, x_i, y_j) - \alpha W^{n,P,\alpha}_{i,j}| = m &\leq \sup \Psi - \left(5C_0 + \frac{m}{2}\right) \beta_\epsilon(t_0-s_0, X_0-Y_0)\\
&= \alpha W^\alpha(t_0, X_0) - \alpha W^\alpha_{h,\Delta t}(s_0, Y_0).
\end{aligned}$$

Then from (3.16) and (3.11), we obtain

$$\sup_{i,j,n} |\alpha W^{P,\alpha}(t_n, x_i, y_j) - \alpha W^{n,P,\alpha}_{i,j}| \leq \frac{C}{\alpha} o(1) \quad \text{as } (\Delta t, h) \to (0,0).$$

From the previous estimate, (ii) of Lemma 2.1 and Proposition 3.1 we finally obtain

$$|\overline{F}(P) - \overline{F}^{\Delta t}_h(P)| \leq \widetilde{K}_1 \alpha + K_1 \alpha + \frac{C}{\alpha} o(1),$$

and letting $(h, \Delta t) \to (0,0)$, we find that

$$\limsup_{(\Delta t, h)\to(0,0)} |\overline{F}(P) - \overline{F}^{\Delta t}_h(P)| \leq \widetilde{K}_1 \alpha + K_1 \alpha,$$

for any fixed $\alpha > 0$. This implies that $\lim_{(\Delta t,h)\to(0,0)} \overline{F}^{\Delta t}_h(P) = \overline{F}(P)$. Since $K_1 = K_1(P)$ and $\widetilde{K}_1 = \widetilde{K}_1(P)$ are bounded for P lying on compact subsets of $\mathbb{R}^2$, the convergence is uniform on compact sets. $\square$

Remark 3.1. If F is coercive, then we can get an estimate of the rate of convergence of $\overline{F}_h^{\Delta t}$ to $\overline{F}$. Indeed, we have:

$$|\overline{F}_h^{\Delta t} - \overline{F}| \leq (h + \Delta t)^{\frac{1}{2}},$$

see Proposition A.3 in [1].

We conclude this subsection by recalling the principal properties of $\overline{F}_h^{\Delta t}$.

Proposition 3.3. *Assume (g1)-(g6), (H1)-(H4). Then the approximate effective Hamiltonian* $\overline{F}_h^{\Delta t}$ *is Lipschitz continuous with a Lipschitz constant independent of* h *and* Δt *and for any* $p_x \in \mathbb{R}$

$$\overline{F}_h^{\Delta t}(p_x, 0) \geq C_2|p_x|.$$

Proof. For the proof of the Lipschitz continuity of $\overline{F}$, see the proof of Proposition A.2 in [1].

Let us show the coercivity property. Let $(W_{i,j}^{n,P,\alpha})$ be a solution of (3.4) for $P = (p_x, 0)$. Let (i_0, j_0, n_0) be a maximum point of $(W_{i,j}^{n,P,\alpha})$, then

$$\frac{W_{i_0,j_0}^{n_0,P,\alpha} - W_{i_0,j_0}^{n_0-1,P,\alpha}}{\Delta t} \geq 0,\ (\Delta_1^+ W^{n_0,P,\alpha})_{i_0,j_0} \leq 0,\ (\Delta_1^+ W^{n_0,P,\alpha})_{i_0-1,j_0} \geq 0,$$

$$(\Delta_2^+ W^{n_0,P,\alpha})_{i_0,j_0} \leq 0,\ (\Delta_2^+ W^{n_0,P,\alpha})_{i_0,j_0-1} \geq 0.$$

By the monotonicity assumption (g1) and (2.4), we have

$$\overline{F}_h^{\Delta t}(p_x, 0) \geq g(t_{n_0}, x_{i_0}, y_{i_0}, p_x, p_x, 0, 0) = F(t_{n_0}, x_{i_0}, y_{i_0}, p_x, 0) \geq C_2|p_x|. \qquad \square$$

3.1. *Long time approximation*

A different way to approximate the effective Hamiltonian is given by the evolutive Hamilton-Jacobi equation

$$\begin{cases} V_t + F(t, x, y, p_x + D_xV, p_y + D_yV) = 0, & (t, x, y) \in (0, +\infty) \times \mathbb{R}^{N+1}, \\ V(0, x, y) = V_0(x, y), & (x, y) \in \mathbb{R}^{N+1}, \end{cases} \tag{3.17}$$

where V_0 is bounded and uniformly continuous on $\mathbb{R}^{N+1}$. Indeed, it is proved in [3] that (3.17) admits a unique solution V which is bounded and uniformly continuous on $[0, T] \times \mathbb{R}^{N+1}$ for any $T > 0$, and satisfies

$$\lim_{t \to +\infty} \frac{V(t, x, y)}{t} = -\overline{F}(P).$$

We approximate (3.17) by the implicit Eulerian scheme

$$\begin{aligned} &\tfrac{V_{i,j}^{n+1,P} - V_{i,j}^{n,P}}{\Delta t} + S(t_n, x_i, y_j, h, [V^{n+1,P}]_{i,j}) = 0 \\ &V_{i,j}^{0,P} = V_0(x_i, y_j), \end{aligned} \tag{3.18}$$

where S is defined as in (3.2). A proof of the existence of a solution $V = (V_{i,j}^{n,P})$ of (3.18) is given in [9] under assumptions (g1)-(g5).

Let $W = (W_{i,j}^{n,P,\alpha})$ be a solution of (3.4), then by comparison, there exist constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c} + W_{i,j}^{n,P,\alpha} - n\overline{F}_h^{\Delta t}(P)\Delta t \le V_{i,j}^{n,P} \le \overline{c} + W_{i,j}^{n,P,\alpha} - n\overline{F}_h^{\Delta t}(P)\Delta t.$$

Since W is bounded, this proves that

$$\lim_{n\to+\infty} \frac{V_{i,j}^{n,P}}{n\Delta t} = -\overline{F}_h^{\Delta t}(P).$$

3.2. *Approximation of the homogenized problem*

We now come back to the N-dimensional homogenized problem (1.3). From Theorem 1.2 we know that if $\overline{H}$ is the effective Hamiltonian in (1.3), then $\overline{H}(p) = \overline{F}(p,-1)$ for any $p \in \mathbb{R}^N$. Hence, from Proposition 3.2, the discrete Hamiltonian

$$\overline{H}_h^{\Delta t}(p) := \overline{F}_h^{\Delta t}(p,-1),$$

is an approximation of $\overline{H}(p)$ for any $p \in \mathbb{R}^N$.

As in [1], we approximate (1.3) by the problem

$$\begin{cases} \partial_t u_{\Delta t,h} + \overline{H}_h^{\Delta t}(Du_{\Delta t,h}) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}^N, \\ u_{\Delta t,h}(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.19}$$

where h and Δt are fixed, and u_0 is the same initial datum as in (1.3).

By Proposition 3.3 $\overline{H}_h^{\Delta t}$ is Lipschitz continuous and coercive, so (3.19) has a unique viscosity solution $u_{\Delta t,h}$ which is an approximation of the solution u^0 of (1.3):

Proposition 3.4. *Let* u^0 *and* $u_{\Delta t,h}$ *be respectively the viscosity solutions of* (1.3) *and* (3.19)*. Then for any* $T > 0$

$$\sup_{[0,T]\times\mathbb{R}^N} |u_{\Delta t,h} - u^0| \to 0 \quad \text{as } (\Delta t, h) \to (0,0). \tag{3.20}$$

Proof. If L_0 is the Lipschitz constant of the initial datum u_0, then, by Proposition 2.2, the functions $u^0(t,\cdot)$ and $u_{\Delta t,h}(t,\cdot)$ are Lipschitz continuous with same Lipschitz constant L_0. By Proposition 3.2 the approximate Hamiltonian $\overline{H}_h^{\Delta t}$ converges to $\overline{H}$ uniformly for $|p| \le L$. Hence (3.20) follows by the following proposition, which is a standard estimate in the regular perturbation theory of Hamilton-Jacobi equations (see Theorem VI.22.1 in [2])

Proposition 3.5. *If there exists* $\eta > 0$ *such that if* H_i*,* $i = 1,2$*, satisfy (H1)-(H3) with*

$$\|H_1 - H_2\|_\infty \le \eta,$$

and if u_i*,* $i = 1, 2$*, are viscosity solutions of*

$$\begin{cases} u_t + H_i(Du) = 0, \ (t, x) \in (0, T) \times \mathbb{R}^N \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases}$$

where u_0 *is bounded and uniformly continuous on* $\mathbb{R}^N$*, then, for some constant* C*,*

$$\|u_1 - u_2\|_\infty \le C\eta. \qquad \square$$

Remark 3.2. *In order to compute numerically the approximation of* u^0*, we need further discretizations. Indeed, we have approximated* $\overline{H}(p)$ *by* $\overline{H}_h^{\Delta t}(p)$ *for any fixed* $p \in \mathbb{R}^N$*. Since it is not possible to compute* $\overline{H}_h^{\Delta t}(p)$ *for any* p*, one possibility is to introduce a triangulation of a bounded region of* $\mathbb{R}^N$ *and compute* $\overline{H}_h^{\Delta t}(p_i)$*, where* p_i *are the vertices of the simplices and to approximate all the other values* $\overline{H}_h^{\Delta t}(p)$ *by* $\overline{H}_{h,k}^{\Delta t}(p)$*, where* $\overline{H}_{h,k}^{\Delta t}$ *is the linear interpolation of* $\overline{H}_h^{\Delta t}$ *and we denote by* k *the maximal diameter of the simplices. The solution* $u^k_{\Delta t,h}$ *of*

$$\begin{cases} \partial_t u^k_{\Delta t,h} + \overline{H}_{h,k}^{\Delta t}(Du^k_{\Delta t,h}) = 0, \ (t, x) \in (0, +\infty) \times \mathbb{R}^N, \\ u^k_{\Delta t,h}(0, x) = u_0(x), \qquad x \in \mathbb{R}^N, \end{cases} \tag{3.21}$$

is an approximation of $u_{\Delta t,h}$ *as* $k \to 0$ *and hence, by Proposition 3.4, of* u^0 *as* $(\Delta t, h, k) \to (0, 0, 0)$*. Finally, discretizing* (3.21) *by means a monotone, consistent and stable approximation scheme, we can compute numerically an approximation of the solution* u^0 *of 1.3. See* [1] *for details.*

4. Numerical Tests

The present paragraph is devoted to the description of numerical approximations of the effective Hamiltonian.

4.1. ***Results***

4.1.1. *First case*

We discuss a one dimensional case where the Hamiltonian is

$$H(x, u, p) = 2\cos(2\pi x) + \sin(8\pi u) + (1 - \cos(6\pi x)/2)|p|.$$

We have used two approaches for computing the effective Hamiltonian.

(g1) Barles cell problem: the first approach consists of increasing the dimension and considering the long time behavior of the continuous viscosity solution w of

$$\begin{aligned} w_t + F(x, y, p + D_x w, -1 + D_y w) = 0, &\quad (t, x, y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}, \\ w(0, x, y) = 0, &\quad (x, y) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{4.1}$$

where F is given by (1.5). In the present case, from the periodicity of H with respect to x and u, w is 1-periodic with respect to x and 1/4-periodic with respect to y. We know that when $t \to \infty$, $w(t, \cdot, \cdot)/t$ tends to a real number λ

and that $\overline{H}(p) = -\lambda$.
For approximating (4.1) on a uniform grid, we have used an explicit Euler time marching method with a Godunov monotone scheme (see [12,22]). A semi-implicit time marching scheme which allows for large time steps may be used as well, see [1], but very large time steps cannot be taken because of the periodic in time asymptotic behaviour of w.
Alternatively, we have also used the higher order method described in [19], see also [20]. It is a third order TVD explicit Runge-Kutta time marching method with a weighted ENO scheme in the spatial variables. This weighted ENO scheme is constructed upon and has the same stencil nodes as the third order ENO scheme but can be as as high as fifth order accurate in the smooth part of the solution.

(g2) Imbert-Monneau cell problem: when p is a rational number ($p = \frac{n}{q}$), instead of considering a problem posed in two space dimensions, one possible way of approximating the effective Hamiltonian $\overline{H}(p)$ is to consider the cell problem

$$\begin{aligned} v_t + H(x, v + p \cdot x, p + Dv) = 0, \quad & (t, x) \in (0, \infty) \times \mathbb{R}, \\ v(0, x) = 0 \quad & x \in \mathbb{R}. \end{aligned} \tag{4.2}$$

This problem has a unique continuous solution which is periodic of period q with respect to x (in fact, the smallest period of v may be a divisor of q). From [16] (Theorem 1), we know that there exists a unique real number λ such that $\frac{v(\tau,x)}{\tau}$ converges to λ as $\tau \to \infty$ uniformly in x, and that $\overline{H}(p) = -\lambda$. Moreover, when t is large, the function $v(t, x) - \lambda t$ becomes close to a periodic function of time. In what follows, (4.2) will be referred to as Imbert-Monneau cell problem. Note that the size of the period varies with p and may be arbitrary large. This is clearly a drawback of this approach which is yet the fastest one for one dimensional problems and moderate values of q.
For approximating (4.2) on a uniform grid, we have used either the above-mentioned explicit Euler time marching method with a Godunov monotone scheme or the third order TVD explicit Runge-Kutta time marching method with a weighted ENO scheme in the spatial variable.

In Figure 1, we plot the graph of the effective Hamiltonian computed with the high order methods and both Imbert-Monneau and Barles cell problems. For Barles cell problems, the grid of the square $[0, 1] \times [0, 1/4]$ has 400×100 nodes and the time step is $1/1000$. For Imbert-Monneau cell problems, the grids in the x variable are uniform with a step of $1/400$ and the time step is $1/1000$. The two graphs are undistinguishable. It can be seen that the effective Hamiltonian is symmetric with respect to p and constant for small values of p, i.e. $|p| \lesssim 1.3$. The points where we have computed the effective Hamiltonian are concentrated near 1.3 where the slope of the graph changes. Our computations clearly indicate that the effective Hamiltonian is piecewise linear.
In order to show the convergence of $\frac{v(\tau,x)}{\tau}$ and $\frac{w(\tau,x)}{\tau}$, we take $p = 1.3$ so the space

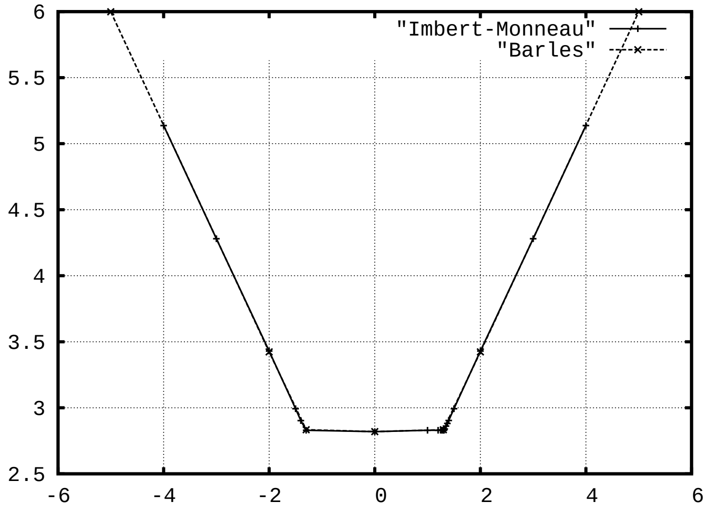

Fig. 1. First case: the effective Hamiltonian as a function of p obtained with both Barles and Imbert-Monneau cell problems.

period of the Imbert-Monneau cell problem is 5. In Figure 2, we plot $\frac{\langle w(\tau)\rangle}{\tau}$ (left) and $\frac{\langle v(\tau)\rangle}{\tau}$ (right) as a function of τ, where $\langle v(\tau)\rangle$ is the median value of $v(\tau,\cdot)$ on a spatial period. Both functions converge to constants when $\tau \to \infty$ and the limit are close to each other (the error between the two scaled median values is smaller than 10^{-3} at $\tau \sim 60$ and we did not consider much longer times). In Figure 3, we plot the graphs of the functions $w(\tau,0,0)-\langle w(\tau)\rangle$ (left) and $v(\tau,0)-\langle v(\tau)\rangle$ (right). We see that these functions become close to time-periodic. In Figure 4 (top), we plot the contour lines of the function $w(\tau,x,y)/\tau$ as a function of (x,y) for $\tau = 60$. In the bottom part of the figure we plot the graph of $y \to w(\tau, 0.13, y)/\tau$ for the same value of τ. We see that w has internal layers. In Figure 5, we plot the graph $x \to v(\tau,x)/\tau$ for $\tau = 60$. We first see that the function takes all its values in a small interval and has very rapid variations with respect to x (is nearly discontinuous). This does not contradict the theory, because there are no uniform estimates on the modulus of continuity of $v(\tau,\cdot)/\tau$.

4.1.2. *Second case*

We consider a two dimensional problem, where the Hamiltonian is

$$\begin{aligned} H(x,u,p) = {} & \cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi(x_1 - x_2)) + \sin(2\pi u) \\ & + \left(1 - \frac{\cos(2\pi x_1)}{2} - \frac{\sin(2\pi x_2)}{4}\right)|p|. \end{aligned}$$

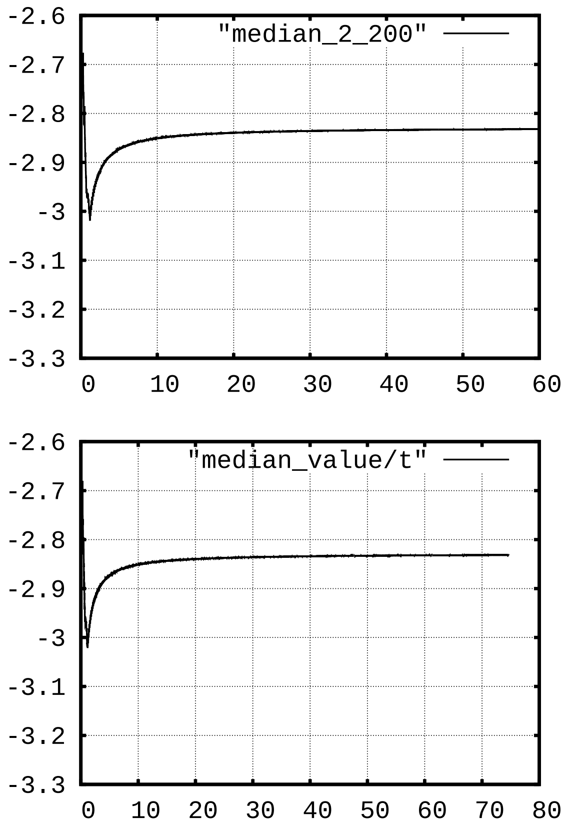

Fig. 2. First case, $p = 1.3$. Top: the median value of $w(\tau, \cdot)/\tau$ on a period as a function of τ. Bottom: the median value of $v(\tau, \cdot)/\tau$ on a period as a function of τ

For this case, only the Imbert-Monneau cell problems have been approximated on uniform grids with step 1/200. The time step is 0.005. In Figure 6, we plot the contours and the graph of the effective Hamiltonian computed with the high order method. We can see that the effective Hamiltonian is symmetric with respect to $p = (0, 0)$, constant for small vectors p. In Figure 7, we plot $\frac{\langle v(\tau)\rangle}{\tau}$ as a function of τ. We see that this function converges when $\tau \to \infty$. In Figure 8, we plot the contours of $v(\tau, \cdot)/\tau$ for $\tau = 59.935$ and $p = (1, 1)$. We see that for large values of τ, v is close to discontinuous.

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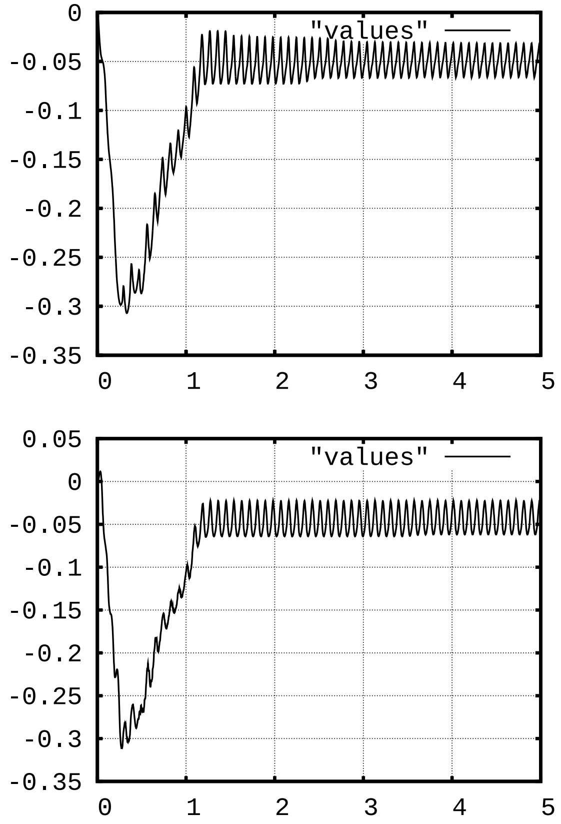

Fig. 3. First case, $p = 1.3$: $w(\tau, 0, 0) - \langle w(\tau)\rangle$ (top) and $v(\tau, 0) - \langle v(\tau)\rangle$ (bottom) as a function of τ

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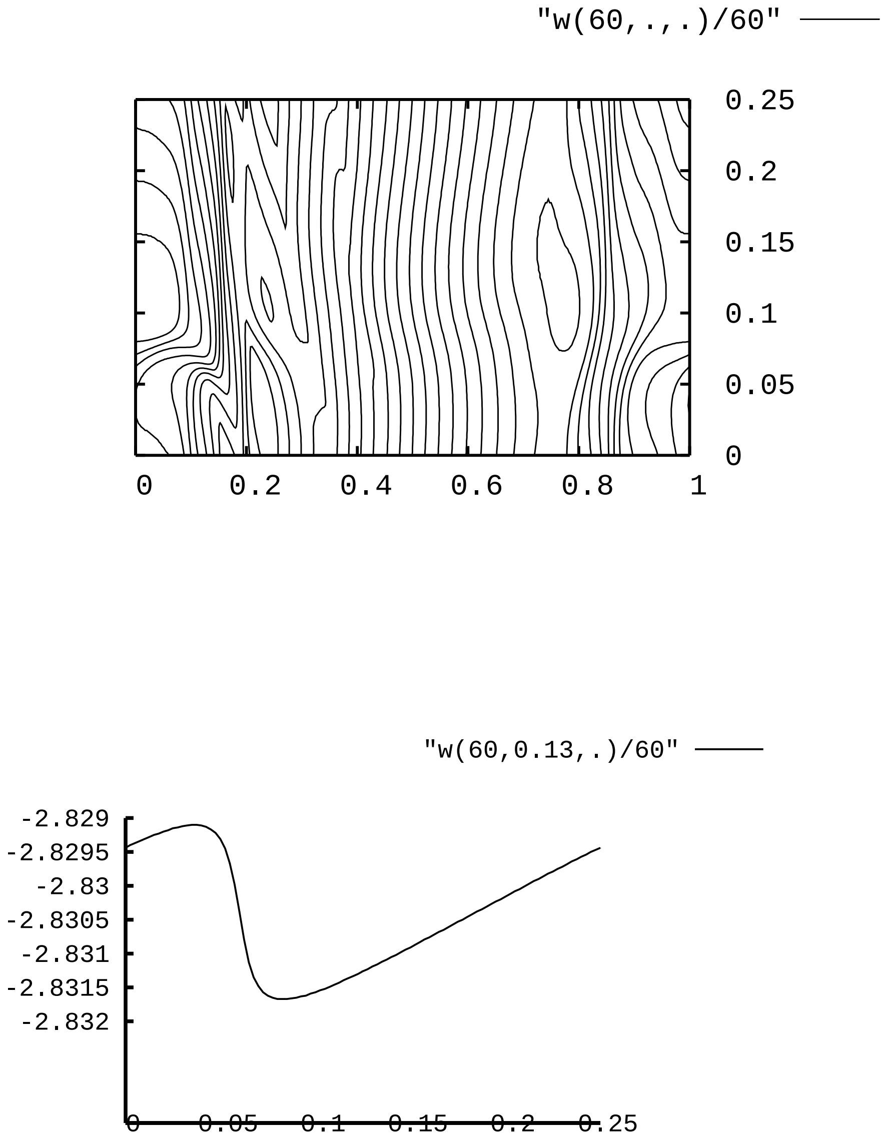

Fig. 4. First case, Barles cell problem, $p = 1.3$. Top: contour lines of $w(\tau, \cdot)/\tau$ on a period as a function of (x, y). Bottom: the cross-section $x = 0.13$.

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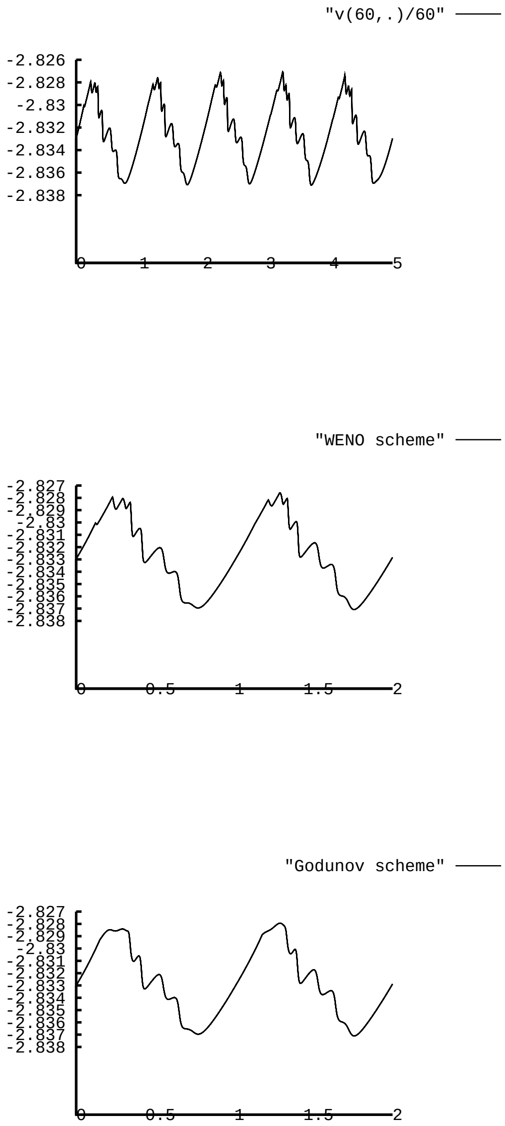

Fig. 5. First case, Imbert-Monneau cell problem, $p = 1.3$: Top: Third order Runge Kutta/WENO scheme: $v(\tau, x)/\tau$ as a function of x for $\tau = 60$; Middle part: a zoom. Bottom: same computation with Euler/Godunov scheme with the same grid parameters: some oscillations are smeared out, but the average value of the solution is well computed.

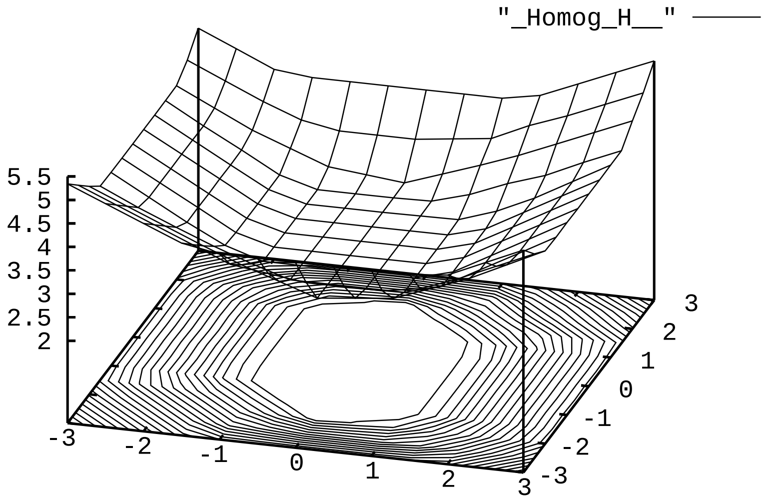

Fig. 6. Second case, the effective Hamiltonian computed by solving Imbert-Monneau cell problems.

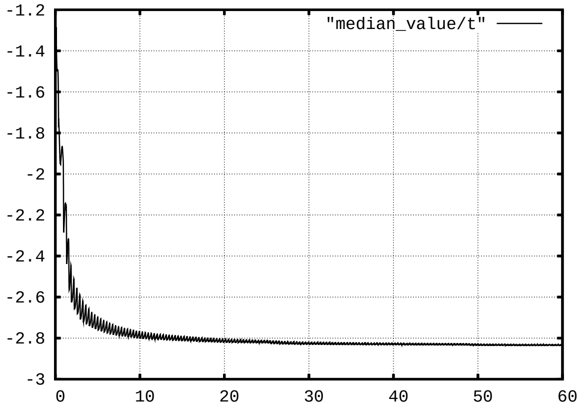

Fig. 7. Second case, $p = (1, 1)$. The median value of $v(\tau, \cdot)/\tau$ on a period as a function of τ.

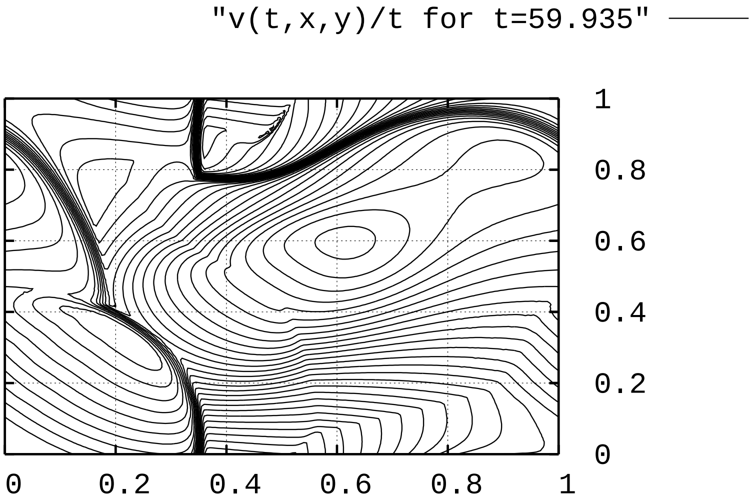

Fig. 8. Second case, the contours of the solution of Imbert-Monneau cell problem for $p = (1, 1)$ at time $\tau = 59.935$.