HETEROCLINIC CONNECTIONS FOR NONLOCAL EQUATIONS

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ABSTRACT. We construct heteroclinic orbits for a strongly nonlocal integro-differential equation. Since the energy associated to the equation is infinite in such strongly nonlocal regime, the proof, based on variational methods, relies on a renormalized energy functional, exploits a perturbation method of viscosity type and develops a free boundary theory for a double obstacle problem of mixed local and nonlocal type.

The description of the stationary positions for the atom dislocation function in a perturbed crystal, as given by the Peierls-Nabarro model, is a particular case of the result presented.

CONTENTS

1. INTRODUCTION

Heteroclinic orbits are a classical topic in the context of dynamical systems. Not only they are trajectories that show an interesting behavior, providing a connection between two different rest positions, but they are often the "building blocks" for constructing complicated orbits, drifting from one equilibrium to another, possibly leading to a chaotic dynamics. On the other hand, the recent literature has studied the case in which the "classical" differential equations are replaced by integro-differential equations.

The study of these nonlocal equations is not only motivated by mathematical curiosity and by the will driving the scientists of facing with new challenging problems, but it also possesses concrete motivations in applied sciences: in particular, our main motivation for the problem treated in this paper comes from the description of the stationary positions for the atom dislocation in crystals, as provided by the

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Peierls-Nabarro model, see e.g. [Nab79] and Section 2 of [DPV15]. In this context, the evolution of the dislocation function on the "slip line" (i.e., the intersection between the "slip plane", along which the crystal experiences a plastic deformation, and a transversal reference plane) is described by an equation of fractional type, as a consequence of the balance between the elastic bonds that link the atoms and the internal force of the crystals which tends to place all the atoms into a periodically organized lattice.

Concretely, in the Peierls-Nabarro model for edge dislocations, one considers equations that can be written along the slip line as

$$\sqrt{-\Delta}\, Q(x) + W'(Q(x)) = 0 \qquad \text{for any } x \in \mathbb{R}, \tag{1.1}$$

where W is a multi-well potential and the diffusion operator is the square root of the Laplacian, which (up to normalizing multiplicative constants) is the integro-differential operator

$$\sqrt{-\Delta}\, Q(x) := \text{P.V.} \int_{\mathbb{R}} \frac{Q(x) - Q(y)}{|x-y|^2}\, dy := \lim_{\varrho \to 0} \int_{\mathbb{R}\setminus B_\varrho(x)} \frac{Q(x) - Q(y)}{|x-y|^2}\, dy. \tag{1.2}$$

In the setting of (1.1), the function $Q : \mathbb{R} \to \mathbb{R}$ represents a dislocation function (i.e., roughly speaking, a measure of the atomic disregistry with respect to the ideal rest configuration of a perfect crystal); the diffusion operator in (1.1) and (1.2) takes into account the effect on the slip line of the elastic bonds between different atoms in the crystal and the potential W is induced by the large-scale pattern of the crystal itself (see e.g. [Nab79] and Section 2 of [DPV15] for additional details).

The mathematical framework in which we work here is the following. Given a function $Q : \mathbb{R} \to \mathbb{R}$, the nonlocal operator that we take into account in this paper is given by

$$\mathscr{L} Q(x) := \text{P.V.} \int_{\mathbb{R}} \big(Q(x) - Q(y)\big)\, K(x-y)\, dy := \lim_{\varrho \to 0} \int_{\mathbb{R}\setminus B_\varrho(x)} \big(Q(x) - Q(y)\big)\, K(x-y)\, dy. \tag{1.3}$$

The kernel K is supposed to be even and such that

$$\frac{\theta_0}{|r|^{1+2s}} \chi_{[0,r_0]}(r) \leqslant K(r) \leqslant \frac{\Theta_0}{|r|^{1+2s}}, \tag{1.4}$$

for some $\Theta_0 \geqslant \theta_0 > 0$ and some $r_0 > 0$, with

$$s \in \left(\frac{1}{4}, \frac{1}{2} \right]. \tag{1.5}$$

Of course, the case under consideration comprises in particular the original Peierls-Nabarro model in (1.2), which corresponds to the choice

$$s := \frac{1}{2} \quad \text{and} \quad K(r) := \frac{1}{|r|^2}. \tag{1.6}$$

In the equations that we consider, the diffusive operator $\mathscr{L}$ is balanced by a forcing term of potential type. More precisely, we consider a non-negative multi-well potential $W \in C^2(\mathbb{R}, \mathbb{R})$ with a locally finite set of minima. Namely, we suppose that $W \geqslant 0$ and that there exists $\mathscr{Z} \subset \mathbb{R}$ which is a discrete set (i.e., it has no accumulation points) with

$$0 \in \mathscr{Z} \tag{1.7}$$

and such that

$$W(\zeta) = 0 \text{ for any } \zeta \in \mathscr{Z} \text{ and } W(r) > 0 \text{ for any } r \in \mathbb{R} \setminus \mathscr{Z}. \tag{1.8}$$

We also suppose that W grows quadratically from its minima, that is

$$c_0 |\xi|^2 \leqslant W(\zeta + \xi) \leqslant C_0 |\xi|^2, \tag{1.9}$$

for some $C_0 > c_0 > 0$, for all $\zeta \in \mathscr{Z}$ and $\xi \in B_{\delta_0}$, with $\delta_0 > 0$.

In our framework, the potential is modulated by an oscillatory function a. Such function is supposed to maintain the sign of the potential, namely we assume that

$$a(x) \in [\underline{a}, \overline{a}] \quad \text{for all } x \in \mathbb{R}, \tag{1.10}$$

for some $\overline{a} > \underline{a} > 0$.

We also assume that a is "non-degenerate". More precisely, we suppose that there exist $m_1, m_2 \in \mathbb{R}$ and $\omega, \theta > 0$ such that

$$m_2 - m_1 \geqslant 2\omega + \theta, \tag{1.11}$$

and, for $i \in \{1, 2\}$,

$$a(x) - a(x - \theta) \geqslant \gamma \quad \text{and} \quad a(x) - a(x + \theta) \geqslant \gamma, \quad \text{for all } x \in [m_i - \omega, m_i + \omega], \tag{1.12}$$

for[1] some $\gamma > 0$.

In this setting, the equation that we study here has the form

$$\mathscr{L} Q^\star(x) + a(x)\, W\big(Q^\star(x)\big) = 0 \quad \text{for all } x \in \mathbb{R}. \tag{1.13}$$

Of course, when $\mathscr{L}$ is replaced by the classical second order differential operator, equation (1.13) may be seen as a pendulum-like equation.

The main objective of this paper is to construct heteroclinic solutions of (1.13), i.e. orbits which connect two different equilibria. To this aim, given $\zeta_1, \zeta_2 \in \mathscr{Z}$, we take $Q^\sharp_{\zeta_1,\zeta_2} \in C^\infty(\mathbb{R})$ to be such that $Q^\sharp_{\zeta_1,\zeta_2}(x) = \zeta_1$ for any $x \in (-\infty, -1)$ and $Q^\sharp_{\zeta_1,\zeta_2}(x) = \zeta_2$ for any $x \in (1, +\infty)$.

To deal with the problem of constructing special solutions of (1.13), it is convenient to introduce a variational formulation. To this aim, we consider here the energy functional

$$\begin{aligned} I_0(Q) &:= \int_{\mathbb{R}} a(x) W\big(Q(x)\big)\, dx \\ &\quad + \frac{1}{4} \iint_{\mathbb{R}\times\mathbb{R}} \Big(\big|Q(x) - Q(y)\big|^2 - \big|(Q^\sharp_{\zeta_1,\zeta_2})(x) - (Q^\sharp_{\zeta_1,\zeta_2})(y)\big|^2 \Big) K(x-y)\, dx\, dy. \end{aligned} \tag{1.14}$$

We remark that critical points of I_0 satisfy (1.13).

Also, given $X, Y \subseteq \mathbb{R}$, we use the notation

$$[v]_{K, X\times Y} := \sqrt{\iint_{X\times Y} \big|v(x) - v(y)\big|^2 K(x-y)\, dx\, dy}\,. \tag{1.15}$$

Then, in this setting, our main result on the existence of heteroclinics for equation (1.13) is the following:

[1] For concreteness, we mention that the function

$$a(x) := 2 + \varepsilon \cos(\delta x)$$

with $\varepsilon, \delta \in (0, 1]$ satisfies (1.12) with $m_1 := 0$, $m_2 := \frac{2\pi}{\delta}$, $\omega := \frac{\pi}{4\delta}$, $\theta := \frac{\pi}{\delta}$ and $\gamma := \sqrt{2}\,\varepsilon$. Indeed, in this case,

$$\begin{aligned} & \inf_{x \in [m_1 - \omega, m_1 + \omega] \cup [m_2 - \omega, m_2 + \omega]} a(x) - a(x \pm \theta) \\ = \; & \inf_{x \in [-\frac{\pi}{4\delta}, \frac{\pi}{4\delta}] \cup [\frac{2\pi}{\delta} - \frac{\pi}{4\delta}, \frac{2\pi}{\delta} + \frac{\pi}{4\delta}]} \varepsilon\big(\cos(\delta x) - \cos(\delta x \pm \delta\theta)\big) \\ = \; & \inf_{y \in [-\frac{\pi}{4}, \frac{\pi}{4}]} \varepsilon\big(\cos y - \cos(y \pm \pi)\big) \\ = \; & 2 \inf_{y \in [-\frac{\pi}{4}, \frac{\pi}{4}]} \varepsilon \cos y \\ = \; & \sqrt{2}\,\varepsilon. \end{aligned}$$

This example shows that there exist "small and slow perturbations of constant functions" that satisfy (1.12).

Theorem 1.1. *Let $\zeta_1 \in \mathscr{Z}$. Then, there exist $\zeta_2 \in \mathscr{Z} \setminus \{\zeta_1\}$ and a solution $Q^\star$ of* (1.13) *such that*

$$\lim_{x\to-\infty} Q^\star(x) = \zeta_1 \quad \text{and} \quad \lim_{x\to+\infty} Q^\star(x) = \zeta_2. \tag{1.16}$$

Moreover, $Q^\star$ is an energy minimizer, in the sense that

$$I_0(Q^\star) \leqslant I_0(Q) \text{ for all } Q \text{ s.t. } Q - Q^\sharp_{\zeta_1,\zeta_2} \in C_0^\infty(\mathbb{R}). \tag{1.17}$$

In addition, if $v^\star := Q^\star - Q^\sharp_{\zeta_1,\zeta_2}$, we have that

$$[v^\star]_{K,\mathbb{R}\times\mathbb{R}} + \|v^\star\|_{L^2(\mathbb{R})} \leqslant \kappa, \tag{1.18}$$

$$\text{and} \qquad \|v^\star\|_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa \qquad \text{for all } \alpha \in (0, 2s), \tag{1.19}$$

for some $\kappa > 0$, which possibly depends on $Q^\sharp_{\zeta_1,\zeta_2}$ and on structural constants.

We observe that Theorem 1.1 is new even in the model case of the square root of the Laplacian (as described by (1.2) and (1.6)).

Moreover, in the special case in which W is an even and periodic potential vanishing on the integers, the role of ζ_2 in Theorem 1.1 can be made explicit: as a matter of fact, in this case, given any $\zeta_1 \in \mathbb{Z}$, one can take both $\zeta_2 := \zeta_1 - 1$ and $\zeta_2 := \zeta_1 + 1$ in the statement of Theorem 1.1 (this follows from Theorem 1.1 here and the discussion in (5.3) of [CDV17]). That is, in the case of even and periodic potentials, Theorem 1.1 guarantees a heteroclinic connection from each minimum of the potential to each of its closest neighborhood.

We also point out that, differently from the classical case, the asymptotic expression in (1.16) is not an immediate consequence of the energy estimates in (1.18) since, when $s \in \left(0, \frac{1}{2}\right]$, functions in $H^s(\mathbb{R})$ are not necessarily infinitesimal at infinity (see e.g. Appendix B for a simple example of this important phenomenon).

The construction of heteroclinic orbits for ordinary differential equations is a well-studied topic in the literature and, in this sense, Theorem 1.1 here is a nonlocal counterpart of some of the celebrated results obtained in [Rab89, Rab94, RCZ00, Rab00] for ordinary differential equations and Hamiltonian systems. Of course, the case of nonlocal equations is conceptually quite different from that of ordinary differential equations, since usual "glueing" and "cut-and-paste" methods are not available, due to far-away energy interactions. We refer to [BV16] for a general introduction to nonlocal problems, also motivated from water wave models, phase transitions, material sciences and biology.

A result similar to Theorem 1.1 when the nonlocal parameter s lies in the range $\left(\frac{1}{2}, 1\right)$ has been obtained in [DPV17]. In case of homogeneous media (i.e., when a is constant), heteroclinic connections corresponding to parameter ranges $s \in \left(0, \frac{1}{2}\right]$ have been studied in [PSV13, CS15, CMY17] by energy renormalization methods.

Concerning the nonlocal parameter range considered in this paper, we recall that the case $s \in \left(0, \frac{1}{2}\right)$ can present several technical and conceptual differences with respect to the case $s \in \left(\frac{1}{2}, 1\right)$ (the case $s = \frac{1}{2}$ being typically "in between" the two cases). For instance, as shown in [CS10, SV12], several fractional equations corresponding to the parameter range $s \in \left[\frac{1}{2}, 1\right)$ present a "local behavior" at a large scale, while they preserve a "nonlocal behavior" at any scale when $s \in \left(0, \frac{1}{2}\right)$.

The case $s = \frac{1}{2}$, $K(r) = \frac{1}{|r|^2}$ and $W(r) = 1 - \cos(2\pi r)$ (which is indeed a particular case of our general framework) plays also an important role in the description of the atom dislocations in crystals, according to the so-called Peierls-Nabarro model, see e.g. [Nab79] (and compare with (1.1) here). This model is in turn related, at a microscopic scale, to the Frenkel-Kontorova model, see [FIM12].

Related models appear also in the study of the Benjamin-Ono equation, see [Tol97], in boundary reaction equations, see [CSM05], and in spin systems on lattices, see [ABC06].

In addition, the study of nonlocal equations with a singular kernel is a very intense subject of research in terms of harmonic analysis, see e.g. [Ste70], and of regularity theory, see e.g. [Sil05].

In our setting, to deal with the case $s \in \left(\frac{1}{4}, \frac{1}{2}\right]$ we will adopt a strategy that has been also very recently used in [CMY17] and based on two basic steps:

- We will consider a renormalized energy functional. This device is needed in order to avoid the divergence of the energy due to nonlocal effects in this parameter range. We stress that this energy divergence is unavoidable, since, for instance, one can easily check that the fractional Sobolev (or Aronszajn-Gagliardo-Slobodeckij) seminorm in $H^s(-R, R)$ of a smooth function connecting two constants goes like $\log R$ when $s = \frac{1}{2}$, and like R^{1-2s} when $s \in \left(0, \frac{1}{2}\right)$, thus diverging as $R \to +\infty$.
- We will perturb the original energy functional by a classical Dirichlet energy. This step is very convenient, since it allows to deal with continuous trajectory in a perturbed setting (notice that, when $s \in \left(0, \frac{1}{2}\right]$, functions in $H^s(\mathbb{R})$ are not necessarily continuous, see e.g. Appendix B for a simple example). After dealing with a minimization argument for such perturbed energy functional, we will obtain uniform estimates that will allow us to pass to the limit.

A series of analytical techniques coming from elliptic partial differential equations are also crucially exploited in our proofs:

- We will make use of viscosity solution methods in order to obtain regularity theories that are uniform in the perturbation parameter related to the Dirichlet energy (this is a fundamental step in order to "remove" the "local and elliptic energy perturbation" in the limit).
- We will study a double obstacle problem of mixed local and nonlocal type, which arises from the constrained minimization of the energy functional (this step is crucial in order to estimate "how the orbits separates from the constraints").

In general, we believe that a very interesting feature provided by the equations related to the Peierls-Nabarro model lies in the fact that their complete understanding requires a *synergic combination of resources and methods coming from different specific backgrounds*, which include, among the others, mathematical physics, calculus of variations, partial differential equations, free boundary problems, geometric measure theory, harmonic analysis and the theory of pseudodifferential operators.

The parameter range considered in this paper has also a special energy feature. Namely, while the interaction energy of fractional Sobolev type of a heteroclinic connection is divergent, the part coming from the potential is typically finite under assumption (1.5). To check this, we recall formula (12) in [PSV13], according to which a heteroclinic orbit $Q(x)$ converges to the equilibrium in the homogeneous case like $\frac{\text{const}}{1+|x|^{2s}}$. Since, by (1.9), the potential W is quadratic near the equilibria, the potential energy term of such trajectory behaves like

$$\int_{\mathbb{R}} \frac{\text{const}}{(1+|x|^{2s})^2}\, dx,$$

which is finite when s lies above the threshold $1/4$.

For this reason, when s lies below $1/4$, it could be expected that a second energy renormalization is needed in order to apply variational methods (e.g. in the approach given by formula (13) in [PSV13]) and we plan to explore this parameter range in future works.

We also remark that the case considered in this paper is not translation invariant, in view of the modulating function a. This is an important difference with respect to the previous literature on the subject, since the translation invariance implies the monotonicity of the heteroclinic, which in turn implies a series of analytic estimates on the energy functional and allows the use of more direct minimization principles (see [PSV13, CS15, CMY17] for further details).

The rest of the paper is organized as follows. In Section 2, we fix some notation, to be used in the rest of the paper. In Section 3, we give two elementary proofs establishing a uniform bound for a nonlocal equation and a regularity result for a perturbed problem (in our setting, such bound is important to obtain uniform estimates in a perturbed problem, and the regularity result is useful to estimate errors in the "cut-and-paste" procedures).

The proof of Theorem 1.1 is then developed in Sections 4, 5, 6, 7, 8 and 9. More precisely, Section 4 is devoted to an energy estimate from below. In our setting, this bound is important to avoid that large excursions of the orbits may drift the renormalized energy to $-\infty$ and to guarantee the necessary compactness for the direct methods of the calculus of variations.

Then, we exploit these variational methods to construct the heteroclinic connections, by proceeding step by step. First, in Section 5 we consider a constrained and perturbed problem. The additional perturbation provides the technical advantage that all the orbits with finite energy are in fact continuous, and this fact will allow us to make use of geometric arguments in the analysis of such orbits. The constrain is also useful to "force" the orbits close to the equilibria at infinity. As a matter of fact, in Section 6, using a double obstacle problem approach, we show that constrained minimizers are continuous with uniform bounds.

Interestingly, this obstacle problem is also of mixed local and nonlocal type, and this is a class of problems rarely studied in the existing literature. For our goals, the achievement of uniform estimates for this problem is crucial in order to have precise information when the orbit touches the variational constraints.

Also, in Sections 7 and 8 we recall the notions of *clean intervals* and *clean points*, and we prove some stickiness properties of the energy minimizers.

Then, in Section 9, by taking the asymptotic constraints "far enough", we will produce a free, i.e. unconstrained, minimizer. Finally, in Section 10, by using estimates that are uniform with respect to the perturbative parameter, we will be able to remove the perturbation and obtain the solution claimed in Theorem 1.1.

2. Notation

- Given I, $J \subseteq \mathbb{R}$ and f, $g : \mathbb{R} \to \mathbb{R}$, we set

$$\mathscr{B}_{I,J}(f,g) := \iint_{I\times J} \big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\,K(x-y)\,dx\,dy, \tag{2.1}$$

and

$$E_{I\times J}(f) := \iint_{I\times J} \Big(\big|f(x)-f(y)\big|^2 - \big|(Q^\sharp_{\zeta_1,\zeta_2})(x) - (Q^\sharp_{\zeta_1,\zeta_2})(y)\big|^2\Big)\,K(x-y)\,dx\,dy. \tag{2.2}$$

Notice that

$$\begin{aligned}\mathscr{B}_{J,I}(f,g) &= \iint_{J\times I} \big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\,K(x-y)\,dx\,dy\\ &= \iint_{I\times J} \big(f(y)-f(x)\big)\big(g(y)-g(x)\big)\,K(y-x)\,dy\,dx = \mathscr{B}_{I,J}(f,g),\end{aligned} \tag{2.3}$$

since K is even. Similarly,

$$E_{I\times J}(f) = E_{J\times I}(f).$$

We will also use the notation

$$E_{I^2}(f) = E_{I\times I}(f).$$

- The Lebesgue measure of a set A will be denoted by $|A|$.

3. A uniform bound and a regularity result for a nonlocal equation

We provide here a general uniform bound for solutions of nonlocal equations, which will be exploited in this paper in the proof of the forthcoming Corollary 5.2, to obtain estimates that are uniform in the perturbation parameter η. The result will be applied to functions whose domain is one dimensional, but, for the sake of generality, we state and prove the result in $\mathbb{R}^N$ for all $\mathbb{N} \in \mathbb{N}$, $N \geqslant 1$, and $s \in (0,1)$ (for this, the power $1+2s$ in (1.4) gets replaced by $N+2s$). So, in this section, $\mathscr{L}u$ denotes the differential operator defined on smooth bounded functions as follows

$$\mathscr{L}u(x) := \text{P.V.}\int_{\mathbb{R}^N} \big(u(x)-u(y)\big)\,K(x-y)\,dy := \lim_{\varrho\to 0}\int_{\mathbb{R}^N\setminus B_\varrho(x)} \big(u(x)-u(y)\big)\,K(x-y)\,dy, \tag{3.1}$$

where K is an even kernel such that

$$\frac{\theta_0}{|r|^{N+2s}}\chi_{[0,r_0]}(r) \leqslant K(r) \leqslant \frac{\Theta_0}{|r|^{N+2s}},$$

for some $\Theta_0 \geqslant \theta_0 > 0$ and some $r_0 > 0$, with $s\in(0,1)$. Of course, the setting in (1.3) is comprised here with $N := 1$. Then we bound the solution of perturbed nonlocal operators as follows:

Lemma 3.1. *Let $\eta \geqslant 0$. Let $u_0 \in L^\infty(\mathbb{R}^N \setminus B_1)$ and $f \in L^\infty(B_1)$. Let $u:\mathbb{R}^N\to\mathbb{R}$ be a solution of*

$$\begin{cases} -\eta\Delta u + \mathscr{L}u = f & \text{in } B_1,\\ u = u_0 & \text{in } \mathbb{R}^N\setminus B_1.\end{cases}$$

Then $u\in L^\infty(\mathbb{R}^N)$ and

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant \text{const}\,\Big(\|u_0\|_{L^\infty(\mathbb{R}^N\setminus B_1)} + \|f\|_{L^\infty(B_1)}\Big).$$

Here, the positive constant "const" depends on N and on the structural constants of $\mathscr{L}$ but it is independent of η.

Proof. We let $\mu\in(0,1)$, to be taken conveniently small in what follows. We define

$$v_\star(x) := \max\left\{0,\ \frac{1}{\mu^2} - |x|^2\right\}.$$

Notice that

$$v_\star > 0 \text{ in } B_{1/\mu} \supset B_1. \tag{3.2}$$

We claim that

$$\mathscr{L}v_\star \geqslant c \text{ in } B_1, \tag{3.3}$$

for some $c\in(0,1)$, as long as μ is sufficiently small. To check this, for any $\bar x\in B_1$ we define

$$r_{\bar x}(x) := 2\bar x\cdot(\bar x - x)\,\chi_{B_{1/\sqrt{\mu}}(\bar x)}(x) + \frac{1}{\mu^2} - |\bar x|^2 = 2\bar x\cdot(\bar x - x)\,\chi_{B_{1/\sqrt{\mu}}(\bar x)}(x) + v_\star(\bar x).$$

We observe that in $B_{1/\sqrt{\mu}}(\bar x)$ the function $r_{\bar x}$ describes the tangent plane to $v_\star$ at $\bar x$. Hence, since $v_\star$ is concave in its positivity set, it follows from (3.2) that

$$r_{\bar x} \geqslant v_\star \text{ in } B_{1/\sqrt{\mu}}(\bar x). \tag{3.4}$$

Furthermore, in $\mathbb{R}^N\setminus B_{1/\sqrt{\mu}}(\bar x)$, it holds that $r_{\bar x}=v_\star(\bar x)$ and therefore

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_{1/\sqrt{\mu}}(\bar x)} \big(r_{\bar x}(y)-v_\star(y)\big)\,K(\bar x-y)\,dy\\
&= \int_{\mathbb{R}^N\setminus B_{1/\sqrt{\mu}}(\bar x)} \big(v_\star(\bar x)-v_\star(y)\big)\,K(\bar x-y)\,dy\\
&= \int_{B_{1/\mu}\setminus B_{1/\sqrt{\mu}}(\bar x)} \big(|y|^2-|\bar x|^2\big)\,K(\bar x-y)\,dy+\int_{\mathbb{R}^N\setminus B_{1/\mu}}\left(\frac{1}{\mu^2}-|\bar x|^2\right)\,K(\bar x-y)\,dy\\
&\geqslant -\int_{\mathbb{R}^N\setminus B_{1/\sqrt{\mu}}(\bar x)} |\bar x|^2\,K(\bar x-y)\,dy\\
&\geqslant -\text{ const}\int_{\mathbb{R}^N\setminus B_{1/\sqrt{\mu}}}\frac{d\xi}{|\xi|^{N+2s}}\\
&= -\text{ const}\,\mu^s.
\end{aligned}\tag{3.5}$$

Also, since K is even, for any $\varrho>0$ we have that

$$\int_{\mathbb{R}^N\setminus B_\varrho(\bar x)} \bar x\cdot(\bar x-y)\,\chi_{B_{1/\sqrt{\mu}}(\bar x)}(y)\,K(\bar x-y)\,dy=\int_{\mathbb{R}^N\setminus B_\varrho}\bar x\cdot\xi\,\chi_{B_{1/\sqrt{\mu}}}(\xi)\,K(\xi)\,d\xi=0.$$

Accordingly, by (3.1) and (3.5),

$$\begin{aligned}
\mathscr{L}v_\star(\bar x) &= \lim_{\varrho\to0}\int_{\mathbb{R}^N\setminus B_\varrho(\bar x)} \big(2\bar x\cdot(\bar x-y)\,\chi_{B_{1/\sqrt{\mu}}(\bar x)}(y)+v_\star(\bar x)-v_\star(y)\big)\,K(\bar x-y)\,dy\\
&= \lim_{\varrho\to0}\int_{\mathbb{R}^N\setminus B_\varrho(\bar x)} \big(r_{\bar x}(y)-v_\star(y)\big)\,K(\bar x-y)\,dy\\
&\geqslant \lim_{\varrho\to0}\int_{B_{1/\sqrt{\mu}}(\bar x)\setminus B_\varrho(\bar x)} \big(r_{\bar x}(y)-v_\star(y)\big)\,K(\bar x-y)\,dy-\text{ const}\,\mu^s.
\end{aligned}$$

Hence, since $B_{1/\sqrt{\mu}}(\bar x)\supseteq B_{r_0}(\bar x)$ if μ is small enough, using (3.4) we can write that

$$\begin{aligned}
\text{const}\,\mu^s+\mathscr{L}v_\star(\bar x) &\geqslant \int_{B_{r_0}(\bar x)\setminus B_{r_0/2}(\bar x)} \big(r_{\bar x}(y)-v_\star(y)\big)\,K(\bar x-y)\,dy\\
&\geqslant \text{const}\int_{B_{r_0}(\bar x)\setminus B_{r_0/2}(\bar x)} \big(r_{\bar x}(y)-v_\star(y)\big)\,|\bar x-y|^{-N-2s}\,dy\\
&= \text{const}\int_{B_{r_0}(\bar x)\setminus B_{r_0/2}(\bar x)} \big(2\bar x\cdot(\bar x-y)-|\bar x|^2+|y|^2\big)\,|\bar x-y|^{-N-2s}\,dy\\
&= \text{const}\int_{B_{r_0}(\bar x)\setminus B_{r_0/2}(\bar x)} |\bar x-y|^2\,|\bar x-y|^{-N-2s}\,dy\\
&= \text{const}\int_{B_{r_0}\setminus B_{r_0/2}} |\xi|^{2-N-2s}\,d\xi\\
&= \text{const}\,.
\end{aligned}$$

By taking μ conveniently small, this proves (3.3), as desired.

The computations above have fixed the size of μ once and for all. Therefore, it holds that

$$\|v_\star\|_{L^\infty(\mathbb{R}^N)}\leqslant\text{ const}\,.\tag{3.6}$$

Let now $M:=c^{-1}\big(\|u_0\|_{L^\infty(\mathbb{R}^N\setminus B_1)}+\|f\|_{L^\infty(B_1)}\big)$ and $\beta:=M(v_\star+1)$. Notice that, outside B_1, we have that $\beta\geqslant M\geqslant u_0=u$. Moreover, in B_1, it holds that $\mathscr{L}\beta=M\mathscr{L}v_\star\geqslant c\,M\geqslant f$, thanks to (3.3). Also, by concavity, we have that $\Delta\beta=M\Delta v_\star\leqslant0$ in B_1.

All in all, we have that
$$-\eta\Delta\beta + \mathscr{L}\beta \geqslant f = -\eta\Delta u + \mathscr{L}u \quad \text{in } B_1.$$
Consequently, by Comparison Principle, we find that $\beta \geqslant u$ in $\mathbb{R}^N$ and therefore, by (3.6),
$$u \leqslant \|\beta\|_{L^\infty(\mathbb{R}^N)} \leqslant M\big(\|v_\star\|_{L^\infty(\mathbb{R}^N)} + 1\big) \leqslant \text{const}\, M.$$
Similarly, we see that $u \geqslant -\text{const}\, M$. These observations imply the desired result. □

Next is a uniform regularity result dealing with a perturbed problem:

Lemma 3.2. *Let $\eta \in [0,1]$, $s \in (0,1)$ and $f_1,\ f_2 \in \mathbb{R}$. Let $u \in L^\infty(\mathbb{R}^N) \cap C(B_1)$ be a viscosity subsolution of*
$$-\eta\Delta u + \mathscr{L}u + f_1 = 0 \quad \text{in } B_1, \tag{3.7}$$
and a viscosity supersolution of
$$-\eta\Delta u + \mathscr{L}u + f_2 = 0 \quad \text{in } B_1. \tag{3.8}$$
Then, $u \in C^{0,\alpha}(B_{1/2})$ for any $\alpha < \min\{2s, 1\}$ and
$$[u]_{C^{0,\alpha}(B_{1/2})} \leqslant C\left(f_2 - f_1 + \|u\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{\alpha}{2s}} \|u\|^{1-\frac{\alpha}{2s}}_{L^\infty(\mathbb{R}^N)}, \tag{3.9}$$
for some $C > 0$ independent of η.

Proof. We use appropriate techniques from the theory of regularity of viscosity solutions of uniformly elliptic second-order local operators, see [IL90], and recently extended to nonlocal operators, see e.g. [BCI11, MP12], adapted to our context. Let us introduce the following notation: given $r > 0$, for a function ϕ we define
$$\mathscr{L}^{1,r}\phi(x) := \int_{\{|x|\leqslant r\}} (\phi(x) - \phi(x+z) + \chi_{B_r}(z)\nabla u(x)\cdot z)K(z)\, dz$$
and
$$\mathscr{L}^{2,r}\phi(x) := \int_{\{|x|\geqslant r\}} (\phi(x) - \phi(x+z))K(z)\, dz$$
where χ_{B_r} is the indicator function of B_r. Then,
$$\mathscr{L}\phi(x) = \mathscr{L}^{1,r}\phi(x) + \mathscr{L}^{2,r}\phi(x). \tag{3.10}$$
We let $\phi \in C^\infty(\mathbb{R}^N; \mathbb{R}^+_0) \cap W^{2,\infty}(\mathbb{R}^N)$ be such that $\phi(x) = 0$ for all $x \in B_{1/2}$ and $\phi(x) \geqslant 1$ for all $x \in \mathbb{R}^N \setminus B_{3/4}$. We then define
$$\psi(x) := 2\|u\|_{L^\infty(\mathbb{R}^N)}\phi(x). \tag{3.11}$$
Since $\phi \equiv 0$ in $B_{1/2}$, to prove that $u \in C^{0,\alpha}(B_{1/2})$ for any $\alpha < 2s$, it is enough to show that given any $\alpha < 2s$, with $\alpha \in (0,1)$, there exists $L > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^N$,
$$u(x_1) - u(x_2) - L|x_1 - x_2|^\alpha - \psi(x_1) \leqslant 0. \tag{3.12}$$
We argue by contradiction, assuming that (3.12) does not hold true. For $\varepsilon > 0$, let u^ε and u_ε be respectively the sup and inf convolution of u in $\mathbb{R}^N$, i.e.,
$$\begin{aligned} u^\varepsilon(x) &:= \sup_{y\in\mathbb{R}^N} \left(u(y) - \frac{1}{2\varepsilon}|x-y|^2\right) \\ \text{and} \quad u_\varepsilon(x) &:= \inf_{y\in\mathbb{R}^N} \left(u(y) + \frac{1}{2\varepsilon}|x-y|^2\right). \end{aligned}$$
We notice that
$$u^\varepsilon(x) \geqslant u(x) \geqslant u_\varepsilon(x). \tag{3.13}$$
Moreover, u^ε is semiconvex and is a subsolution of (3.7) in $B_{2-\rho}$ and u_ε is semiconcave and is a supersolution of (3.8) in $B_{2-\rho}$, for some $\rho = \rho(\varepsilon) > 0$, see e.g. Proposition III.2 in [Awa91].

Since (3.12) does not hold true, there exists $\alpha \in (0, 2s)$ such that, for any $L > 0$ and $\varepsilon > 0$,

$$\sup_{(x_1,x_2)\in\mathbb{R}^{2N}} u^\varepsilon(x_1) - u_\varepsilon(x_2) - L|x_1 - x_2|^\alpha - \psi(x_1) \geqslant \sup_{(x_1,x_2)\in\mathbb{R}^{2N}} u(x_1) - u(x_2) - L|x_1 - x_2|^\alpha - \psi(x_1) > 0,$$

where we also used (3.13). Then, for any $L > 0$ and $\varepsilon > 0$, the supremum on $\mathbb{R}^{2N}$ of the function

$$u^\varepsilon(x_1) - u_\varepsilon(x_2) - L|x_1 - x_2|^\alpha - \psi(x_1), \tag{3.14}$$

is positive and is attained at some point $(\overline{x}_1, \overline{x}_2) \in \mathbb{R}^{2N}$. Moreover, for ε small enough, we have that $\overline{x}_1 \neq \overline{x}_2$. We remark that

$$|\overline{x}_1 - \overline{x}_2| \leqslant \left(\frac{2\|u\|_{L^\infty(\mathbb{R}^N)}}{L}\right)^{\frac{1}{\alpha}}. \tag{3.15}$$

Using that $\phi \geqslant 1$ in $\mathbb{R}^N \setminus B_{3/4}$ and (3.11), we see that for all $x_1 \in \mathbb{R}^N \setminus B_{3/4}$,

$$u^\varepsilon(x_1) - u_\varepsilon(x_2) - L|x_1 - x_2|^\alpha - \psi(x_1) \leqslant u^\varepsilon(x_1) - u_\varepsilon(x_2) - 2\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant o_\varepsilon(1),$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Thus me must have $x_1 \in B_{3/4}$ for ε small enough, and by (3.15), if

$$L \geqslant 16\|u\|_{L^\infty(\mathbb{R}^N)},$$

we have that

$$\overline{x}_1, \overline{x}_2 \in B_{7/8}. \tag{3.16}$$

The function in (3.14) is semiconvex, hence, by Aleksandrov's Theorem, twice differentiable almost everywhere. Let us now introduce a perturbation of it, for which we can choose maximum points of twice differentiability.

First we transform $(\overline{x}_1, \overline{x}_2)$ into a strict maximum point. In order to do that, we consider a smooth function $h : \mathbb{R}^+ \to \mathbb{R}$, with compact support, such that $h(0) = 0$ and $h(t) > 0$ for $0 < t < 1$, we fix a small $\beta > 0$ and we set

$$\theta(x_1, x_2) := \beta h(|x_1 - \overline{x}_1|^2) + \beta h(|x_2 - \overline{x}_2|^2).$$

Clearly, $(\overline{x}_1, \overline{x}_2)$ is a strict maximum point of $u^\varepsilon(x_1) - u_\varepsilon(x_2) - L|x_1 - x_2|^\alpha - \psi(x_1) - \theta(x_1, x_2)$.

Next we consider a smooth function $\tau : \mathbb{R}^N \to \mathbb{R}$ such that $\tau(x) = 1$ if $|x| \leqslant 1/2$ and $\tau(x) = 0$ for $|x| \geqslant 1$. By Jensen's Lemma, see e.g. Lemma A.3 of [CIL92], for every small and positive δ, there exist q_1^δ, $q_2^\delta \in \mathbb{R}^N$ with $|q_1^\delta|, |q_2^\delta| \leqslant \delta$, such that the function

$$\Phi(x_1, x_2) := u^\varepsilon(x_1) - u_\varepsilon(x_2) - L|x_1 - x_2|^\alpha - \varphi_1(x_1) - \varphi_2(x_2), \tag{3.17}$$

where

$$\begin{aligned} \varphi_1(x_1) &:= \psi(x_1) + \beta h(|x_1 - \overline{x}_1|^2) + \tau(x_1 - \overline{x}_1) q_1^\delta \cdot x_1, \\ \text{and} \quad \varphi_2(x_2) &:= \beta h(|x_2 - \overline{x}_2|^2) + \tau(x_2 - \overline{x}_2) q_2^\delta \cdot x_2, \end{aligned}$$

has a maximum at (x_1^δ, x_2^δ), with

$$|x_1^\delta - \overline{x}_1|,\ |x_2^\delta - \overline{x}_2| \leqslant \delta \tag{3.18}$$

and $u^\varepsilon(x_1) - u_\varepsilon(x_2)$ is twice differentiable at (x_1^δ, x_2^δ). In particular, u^ε is twice differentiable with respect to x_1 at x_1^δ and u_ε is twice differentiable with respect to x_2 at x_2^δ.

We remark that the function τ has been introduced to make $\mathscr{L}^{2,r}\varphi_1$ and $\mathscr{L}^{2,r}\varphi_2$ finite. Also, for δ small enough, by (3.16) and (3.18), we have that

$$x_1^\delta, x_2^\delta \in B_{1-\rho}, \tag{3.19}$$

and that $x_1^\delta \neq x_2^\delta$. In particular, this will allow us to compute the derivatives of the function in (3.17).

Since (x_1^δ, x_2^δ) is a maximum point for Φ, we have

$$\begin{aligned} &\nabla u^\varepsilon(x_1^\delta) = \nabla\varphi_1(x_1^\delta) + \alpha L|x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta) \\ \text{and} \quad &\nabla u_\varepsilon(x_2^\delta) = -\nabla\varphi_2(x_2^\delta) + \alpha L|x_1^\delta - x_2^\delta|^{\alpha-2}(x_1^\delta - x_2^\delta). \end{aligned} \tag{3.20}$$

Moreover the inequalities

$$\begin{aligned} &\Phi(x_1^\delta+z,x_2^\delta)\leqslant\Phi(x_1^\delta,x_2^\delta),\\ &\Phi(x_1^\delta,x_2^\delta+z)\leqslant\Phi(x_1^\delta,x_2^\delta)\\ \text{and}\qquad &\Phi(x_1^\delta+z,x_2^\delta+z)\leqslant\Phi(x_1^\delta,x_2^\delta), \end{aligned}$$

for any $z\in\mathbb{R}^N$, together with (3.20), give respectively:

$$\begin{aligned} &u^\varepsilon(x_1^\delta+z)-u^\varepsilon(x_1^\delta)-\nabla u^\varepsilon(x_1^\delta)\cdot z\\ &\leqslant\varphi_1(x_1^\delta+z)-\varphi_1(x_1^\delta)-\nabla\varphi_1(x_1^\delta)\cdot z\\ &\quad+L|x_1^\delta+z-x_2^\delta|^\alpha-L|x_1^\delta-x_2^\delta|^\alpha-\alpha L|x_1^\delta-x_2^\delta|^{\alpha-2}(x_1^\delta-x_2^\delta)\cdot z, \end{aligned}\tag{3.21}$$

and

$$\begin{aligned} &-(u_\varepsilon(x_2^\delta+z)-u_\varepsilon(x_2^\delta)-\nabla u_\varepsilon(x_2^\delta)\cdot z)\\ &\leqslant\varphi_2(x_2^\delta+z)-\varphi_2(x_2^\delta)-\nabla\varphi_2(x_2^\delta)\cdot z\\ &\quad+L|x_1^\delta-z-x_2^\delta|^\alpha-L|x_1^\delta-x_2^\delta|^\alpha+\alpha L|x_1^\delta-x_2^\delta|^{\alpha-2}(x_1^\delta-x_2^\delta)\cdot z, \end{aligned}\tag{3.22}$$

and, for any $r>0$,

$$\begin{aligned} &u^\varepsilon(x_1^\delta+z)-u^\varepsilon(x_1^\delta)-\chi_{B_r}(z)\nabla u^\varepsilon(x_1^\delta)\cdot z\\ &\leqslant u_\varepsilon(x_2^\delta+z)-u_\varepsilon(x_2^\delta)-\chi_{B_r}(z)\nabla u_\varepsilon(x_2^\delta)\cdot z\\ &\quad+\varphi_1(x_1^\delta+z)-\varphi_1(x_1^\delta)-\chi_{B_r}(z)\nabla\varphi_1(x_1^\delta)\cdot z\\ &\quad+\varphi_2(x_2^\delta+z)-\varphi_2(x_2^\delta)-\chi_{B_r}(z)\nabla\varphi_2(x_2^\delta)\cdot z. \end{aligned}\tag{3.23}$$

The last inequality in particular implies that

$$\mathscr{L}^{2,r}u^\varepsilon(x_1^\delta)\leqslant\mathscr{L}^{2,r}u_\varepsilon(x_2^\delta)+\mathscr{L}^{2,r}\varphi_1(x_1^\delta)+\mathscr{L}^{2,r}\varphi_2(x_2^\delta),\tag{3.24}$$

and

$$D^2u^\varepsilon(x_1^\delta)-D^2u_\varepsilon(x_2^\delta)\leqslant C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})I_N,\tag{3.25}$$

where I_N is the $N\times N$ identity matrix. Here and henceforth C denotes various positive constants independent of the parameters.

Now, using that u^ε and u_ε are respectively subsolution of (3.7) and supersolution of (3.8) in $B_{1-\rho}$, and recalling (3.10) and (3.19), we have that

$$-\eta\Delta u^\varepsilon(x_1^\delta)+\mathscr{L}^{1,r}u^\varepsilon(x_1^\delta)+\mathscr{L}^{2,r}u^\varepsilon(x_1^\delta)+f_1\leqslant 0\tag{3.26}$$

and

$$-\eta\Delta u_\varepsilon(x_2^\delta)+\mathscr{L}^{1,r}u_\varepsilon(x_2^\delta)+\mathscr{L}^{2,r}u_\varepsilon(x_2^\delta)+f_2\geqslant 0.\tag{3.27}$$

Thus, by subtracting (3.27) to (3.26) and using (3.24) and (3.25), we obtain

$$\mathscr{L}^{1,r}u^\varepsilon(x_1^\delta)-\mathscr{L}^{1,r}u_\varepsilon(x_2^\delta)+f_1-f_2-C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})\leqslant 0.\tag{3.28}$$

Next, let us estimate the term $\mathscr{L}^{1,r}u^\varepsilon(x_1^\delta)-\mathscr{L}^{1,r}u_\varepsilon(x_2^\delta)$ and show that it contains a main negative part. For $0<\nu_0<1$, let us denote by A_r the cone

$$A_r:=\left\{z\in B_r\,,\;|z\cdot(x_1^\delta-x_2^\delta)|\geqslant\nu_0|z||x_1^\delta-x_2^\delta|\right\}.$$

Then

(3.29)
$$\begin{aligned}
&\mathscr{L}^{1,r}u^\varepsilon(x_1^\delta)-\mathscr{L}^{1,r}u_\varepsilon(x_2^\delta)\\
&=\;-\int_{A_r}\Big[u^\varepsilon(x_1^\delta+z)-u^\varepsilon(x_1^\delta)-\nabla u^\varepsilon(x_1^\delta)\cdot z-(u_\varepsilon(x_2^\delta+z)-u_\varepsilon(x_2^\delta)-\nabla u_\varepsilon(x_2^\delta)\cdot z)\Big]K(z)\,dz\\
&\quad-\int_{B_r\setminus A_r}\Big[u^\varepsilon(x_1^\delta+z)-u^\varepsilon(x_1^\delta)-\nabla u^\varepsilon(x_1^\delta)\cdot z-(u_\varepsilon(x_2^\delta+z)-u_\varepsilon(x_2^\delta)-\nabla u_\varepsilon(x_2^\delta)\cdot z)\Big]K(z)\,dz\\
&=:\;-T_1-T_2.
\end{aligned}$$

From (3.23) we have

$$T_2\leqslant C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)}). \tag{3.30}$$

Let us estimate T_1. Using (3.21) and (3.22), and successively making the change of variable $z\to -z$, we get the following estimate of T_1:

$$\begin{aligned}
T_1&\leqslant\int_{A_r}\Big[L|x_1^\delta+z-x_2^\delta|^\alpha-L|x_1^\delta-x_2^\delta|^\alpha-\alpha L|x_1^\delta-x_2^\delta|^{\alpha-2}(x_1^\delta-x_2^\delta)\cdot z\Big]K(z)\,dz+C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})\\
&\quad+\int_{A_r}\Big[L|x_1^\delta-z-x_2^\delta|^\alpha-L|x_1^\delta-x_2^\delta|^\alpha+\alpha L|x_1^\delta-x_2^\delta|^{\alpha-2}(x_1^\delta-x_2^\delta)\cdot z\Big]K(z)\,dz\\
&=2\int_{A_r}\Big[L|x_1^\delta+z-x_2^\delta|^\alpha-L|x_1^\delta-x_2^\delta|^\alpha-\alpha L|x_1^\delta-x_2^\delta|^{\alpha-2}(x_1^\delta-x_2^\delta)\cdot z\Big]K(z)\,dz+C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})\\
&\leqslant\alpha L\int_{A_r}\sup_{\{|t|\leqslant1\}}\Big\{|x_1^\delta-x_2^\delta+tz|^{\alpha-4}\big(|x_1^\delta-x_2^\delta+tz|^2|z|^2-(2-\alpha)[(x_1^\delta-x_2^\delta+tz)\cdot z]^2\big)\Big\}K(z)\,dz\\
&\quad+C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)}).
\end{aligned}$$

Let us fix $r:=\sigma|x_1^\delta-x_2^\delta|$, for some $\sigma>0$. Then, for $z\in A_r$,

$$\begin{aligned}
&|x_1^\delta-x_2^\delta+tz|\leqslant(1+\sigma)|x_1^\delta-x_2^\delta|\\
\text{and}\qquad&|(x_1^\delta-x_2^\delta+tz)\cdot z|\geqslant|(x_1^\delta-x_2^\delta)\cdot z|-|z|^2\geqslant(\nu_0-\sigma)\,|x_1^\delta-x_2^\delta||z|.
\end{aligned}$$

Let us choose $0<\sigma<\nu_0<1$ such that

$$C_0:=-(1+\sigma)^2+(2-\alpha)(\nu_0-\sigma)^2>0,$$

then by (1.4),

$$\begin{aligned}
T_1&\leqslant-CC_0L|x_1^\delta-x_2^\delta|^{\alpha-2}\int_{A_r}|z|^2K(z)\,dz+C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})\\
&\leqslant-CC_0L|x_1^\delta-x_2^\delta|^{\alpha-2}r^{2-2s}+C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})\\
&\leqslant-CC_0L|x_1^\delta-x_2^\delta|^{\alpha-2s}+C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)}).
\end{aligned}\tag{3.31}$$

From (3.28), (3.29), (3.30) and (3.31), we obtain

$$CC_0L|x_1^\delta-x_2^\delta|^{\alpha-2s}-C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})+f_1-f_2\leqslant0.$$

Letting δ go to 0, the last inequality and (3.18) yield

$$CC_0L|\overline{x}_1-\overline{x}_2|^{\alpha-2s}\leqslant C(\beta+\|u\|_{L^\infty(\mathbb{R}^N)})+f_2-f_1.$$

Thus, since $\alpha-2s<0$, using (3.15) and letting β go to 0, we finally obtain

$$L\leqslant C\left(f_2-f_1+\|u\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{\alpha}{2s}}\|u\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{\alpha}{2s}}.$$

Since L was chosen as big as we wish, we get a contradiction for $L>C\left(f_2-f_1+\|u\|_{L^\infty(\mathbb{R}^N)}\right)^{\frac{\alpha}{2s}}\|u\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{\alpha}{2s}}$. This proves (3.9). □

With the aid of Lemma 3.2, we can prove the following regularity result (with uniform bounds):

Lemma 3.3. *Let $T>1$, $\eta\in(0,1)$, $\rho>0$, $\zeta\in\mathscr{Z}$. Let $Q\in L^\infty(\mathbb{R})$ be a solution of*

$$-\eta\ddot{Q}(x)+\mathscr{L}(Q)(x)+a(x)W'(Q(x))=0,\qquad \textit{for any } x\in(-4T,4T).$$

Suppose that

$$Q(x)\in\overline{B_\rho(\zeta)} \textit{ for any } x\in(-4T,4T). \tag{3.32}$$

Then, for any $\alpha<\min\{1,2s\}$,

$$[Q]_{C^{0,\alpha}(-T,T)}\leqslant CT^{-\alpha}\left(\|Q\|_{L^\infty(\mathbb{R})}+T^{2s}\rho\right)^{\frac{\alpha}{2s}}\rho^{1-\frac{\alpha}{2s}}, \tag{3.33}$$

for some $C>0$ independent of η and depending on structural constants.

Proof. Up to a translation, we assume that $\zeta=0$, hence (3.32) becomes

$$|Q(x)|\leqslant\rho,\text{ for any } x\in(-4T,4T). \tag{3.34}$$

We let $\tau_o\in C_0^\infty([-4,4],[0,1])$ be such that $\tau_o(x)=1$ for any $x\in[-3,3]$. We define $\tau(x):=\tau_o(x/T)$ and $u(x):=\tau(x)\,Q(x)$. Notice that, by (3.34),

$$|u(x)|\leqslant\rho \text{ for any } x\in\mathbb{R}. \tag{3.35}$$

Arguing as in Lemma 4.1 in [DPV17], we see that u is solution of

$$-\eta\ddot{u}+\mathscr{L}(u)=f\quad\text{in } (-2T,2T),$$

for some function f satisfying

$$\|f\|_{L^\infty(-2T,2T)}\leqslant\frac{C\|Q\|_{L^\infty(\mathbb{R})}}{T^{2s}}+C\rho,$$

with $C>0$ independent of η. Let $v(x):=u(Tx)$, then v is a solution of

$$-\eta T^{2(s-1)}\ddot{v}+\mathscr{L}(v)=T^{2s}f\quad\text{in } (-2,2).$$

Therefore, by Lemma 3.2 and (3.35), we have that, for any $\alpha<\min\{1,2s\}$,

$$[v]_{C^{0,\alpha}(-1,1)}\leqslant C\left(\|Q\|_{L^\infty(\mathbb{R})}+T^{2s}\rho\right)^{\frac{\alpha}{2s}}\rho^{1-\frac{\alpha}{2s}}.$$

Scaling back we get (3.33). □

4. Energy estimates

Goal of this section is to provide suitable integral estimates, with the aim of bounding the energy from below (this bound is crucial to apply minimization methods in the variational arguments). To start with, we provide a bound on the "mixed term" of the energy functional, as defined in (2.1).

Lemma 4.1. *Let $v\in L^\infty(\mathbb{R})$,*

$$\begin{aligned} &S_-(v):=\left\{x\in\mathbb{R}\setminus[-1,1] \textit{ s.t. } \operatorname{dist}(v(x),\mathscr{Z})\leqslant\delta_0\right\}\\ \textit{and}\qquad &S_+(v):=\left\{x\in\mathbb{R}\setminus[-1,1] \textit{ s.t. } \operatorname{dist}(v(x),\mathscr{Z})>\delta_0\right\}.\end{aligned}$$

Then

$$\left|\mathscr{B}_{\mathbb{R},\mathbb{R}}(v,Q^\sharp_{\zeta_1,\zeta_2})\right|\leqslant\text{ const }\|Q^\sharp_{\zeta_1,\zeta_2}\|_{C^1(\mathbb{R})}\left(\left(|S_+(v)|^{\frac{1-2s}{2}}+1\right)[v]_{K,\mathbb{R}\times\mathbb{R}}+\sqrt{\int_{S_-(v)}|v(x)|^2\,dx}\right).$$

Proof. We fix $L \geqslant 2$, to be chosen conveniently at the end of the proof and we set $I_- := (-\infty, -L)$, $I_+ := (L, +\infty)$ and $J := [-L, L]$, and we notice that $\mathscr{B}_{I_-,I_-}(v, Q^\sharp_{\zeta_1,\zeta_2}) = \mathscr{B}_{I_+,I_+}(v, Q^\sharp_{\zeta_1,\zeta_2}) = 0$, since $Q^\sharp_{\zeta_1,\zeta_2}$ is constant on $I_- \cup I_+$. Using this and (2.3), we see that

$$\mathscr{B}_{\mathbb{R},\mathbb{R}}(v, Q^\sharp_{\zeta_1,\zeta_2}) = \mathscr{B}_{J,J}(v, Q^\sharp_{\zeta_1,\zeta_2}) + 2\mathscr{B}_{J,I_-}(v, Q^\sharp_{\zeta_1,\zeta_2}) + 2\mathscr{B}_{I_-,I_+}(v, Q^\sharp_{\zeta_1,\zeta_2}) + 2\mathscr{B}_{J,I_+}(v, Q^\sharp_{\zeta_1,\zeta_2}). \tag{4.1}$$

Moreover, if $x \in [-L, 1]$ and $y \in (L, +\infty)$ we have that

$$|x - y| = y - x \geqslant \frac{y}{2} + \frac{L}{2} - 1 \geqslant \frac{y}{2},$$

and so, recalling (1.4), we have that

$$\begin{aligned} &\iint_{[-L,1]\times(L,+\infty)} \big|(Q^\sharp_{\zeta_1,\zeta_2})(x) - \zeta_2\big|^2 K(x-y)\,dx\,dy \\ \leqslant\;& \text{const}\, \|Q^\sharp_{\zeta_1,\zeta_2}\|^2_{L^\infty(\mathbb{R})} \iint_{[-L,1]\times(L,+\infty)} y^{-1-2s}\,dx\,dy \\ \leqslant\;& \text{const}\, \|Q^\sharp_{\zeta_1,\zeta_2}\|^2_{L^\infty(\mathbb{R})}\, L^{1-2s}. \end{aligned}$$

Therefore, by the Cauchy-Schwarz Inequality we find that

$$\begin{aligned} &\big|\mathscr{B}_{J,I_+}(v, Q^\sharp_{\zeta_1,\zeta_2})\big| \\ =\;& \left|\iint_{[-L,L]\times(L,+\infty)} \big(v(x) - v(y)\big)\big((Q^\sharp_{\zeta_1,\zeta_2})(x) - \zeta_2\big)\, K(x-y)\,dx\,dy\right| \\ \leqslant\;& \sqrt{\iint_{[-L,1]\times(L,+\infty)} \big|v(x) - v(y)\big|^2 K(x-y)\,dx\,dy} \\ &\times \sqrt{\iint_{[-L,1]\times(L,+\infty)} \big|(Q^\sharp_{\zeta_1,\zeta_2})(x) - \zeta_2\big|^2 K_(x-y)\,dx\,dy} \\ \leqslant\;& \text{const}\, \|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})}\, L^{\frac{1-2s}{2}}\, [v]_{K,\mathbb{R}\times\mathbb{R}}. \end{aligned} \tag{4.2}$$

Similarly, it holds that

$$\big|\mathscr{B}_{J,I_-}(v, Q^\sharp_{\zeta_1,\zeta_2})\big| \leqslant \text{const}\, \|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})}\, L^{\frac{1-2s}{2}}\, [v]_{K,\mathbb{R}\times\mathbb{R}}. \tag{4.3}$$

Also, we have that

$$\begin{aligned} \big|\mathscr{B}_{I_-,I_+}(v, Q^\sharp_{\zeta_1,\zeta_2})\big| &= \left|\iint_{(-\infty,-L)\times(L,+\infty)} \big(v(x) - v(y)\big)\big(\zeta_1 - \zeta_2\big)\, K(x-y)\,dx\,dy\right| \\ &\leqslant \text{const}\, \|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})} \iint_{(-\infty,-L)\times(L,+\infty)} \big(|v(x)| + |v(y)|\big)\,(y-x)^{-1-2s}\,dx\,dy \\ &\leqslant \text{const}\, \|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})} \left[\int_{(-\infty,-L)} \frac{|v(x)|}{(L-x)^{2s}}\,dx + \int_{(L,+\infty)} \frac{|v(y)|}{(y+L)^{2s}}\,dy\right]. \end{aligned} \tag{4.4}$$

In addition, using the Cauchy-Schwarz Inequality we see that

$$\begin{aligned} \int_{S_-(v)\cap(L,+\infty)} \frac{|v(y)|}{(y+L)^{2s}}\,dy &\leqslant \sqrt{\int_{S_-(v)} |v(y)|^2\,dy \int_{(L,+\infty)} \frac{dy}{(y+L)^{4s}}} \\ &\leqslant \frac{\text{const}}{L^{\frac{4s-1}{2}}} \sqrt{\int_{S_-(v)} |v(x)|^2\,dx}. \end{aligned} \tag{4.5}$$

We stress that we have used condition (1.5) here.

Also, by using the Hölder inequality with exponents $2^*_s := \frac{2}{1-2s}$ and $\frac{2}{1+2s}$, and then the fractional Sobolev Inequality (see e.g. Appendix A), we have

$$\begin{aligned}\int_{S_+(v)\cap(L,+\infty)} \frac{|v(y)|}{(y+L)^{2s}}\,dy \;&\leqslant\; \left(\int_{S_+(v)\cap(L,+\infty)} |v(y)|^{2^*_s}\,dy\right)^{\frac{1}{2^*_s}} \left(\int_{S_+(v)\cap(L,+\infty)} \frac{dy}{(y+L)^{\frac{4s}{1+2s}}}\right)^{\frac{1+2s}{2}}\\
&\leqslant\; \frac{\text{const}}{L^{2s}}\,\|v\|_{L^{2^*_s}(\mathbb{R})}\,|S_+(v)\cap(L,+\infty)|^{\frac{1+2s}{2}}\\
&\leqslant\; \frac{\text{const}}{L^{2s}}\,[v]_{H^s(\mathbb{R})}\,|S_+(v)\cap(L,+\infty)|^{\frac{1+2s}{2}}\\
&\leqslant\; \frac{\text{const}}{L^{2s}}\,[v]_{K,\mathbb{R}\times\mathbb{R}}\,|S_+(v)|^{\frac{1+2s}{2}}.\end{aligned}$$

This and (4.5) imply that

$$\int_{(L,+\infty)} \frac{|v(y)|}{(y+L)^{2s}}\,dy \leqslant \frac{\text{const}}{L^{\frac{4s-1}{2}}}\sqrt{\int_{S_-(v)} |v(x)|^2\,dx} + \text{const}\;[v]_{K,\mathbb{R}\times\mathbb{R}}\,\frac{|S_+(v)|^{\frac{1+2s}{2}}}{L^{2s}}. \tag{4.6}$$

Similarly, it holds that

$$\int_{(-\infty,-L)} \frac{|v(x)|}{(x+L)^{2s}}\,dx \leqslant \frac{\text{const}}{L^{\frac{4s-1}{2}}}\sqrt{\int_{S_-(v)} |v(x)|^2\,dx} + \text{const}\;[v]_{K,\mathbb{R}\times\mathbb{R}}\,\frac{|S_+(v)|^{\frac{1+2s}{2}}}{L^{2s}}. \tag{4.7}$$

Thus, we plug (4.6) and (4.7) into (4.4), and we conclude that

$$\left|\mathscr{B}_{I_-,I_+}(v,Q^\sharp_{\zeta_1,\zeta_2})\right| \leqslant \text{const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})}\left(\frac{1}{L^{\frac{4s-1}{2}}}\sqrt{\int_{S_-(v)} |v(x)|^2\,dx} + \text{const}\;[v]_{K,\mathbb{R}\times\mathbb{R}}\,\frac{|S_+(v)|^{\frac{1+2s}{2}}}{L^{2s}}\right). \tag{4.8}$$

Furthermore, by the Cauchy-Schwarz Inequality and (1.15), we have that

$$\begin{aligned}&\left|\mathscr{B}_{J,J}(v,Q^\sharp_{\zeta_1,\zeta_2})\right|\\
&\leqslant \sqrt{\iint_{J\times J} \left|v(x)-v(y)\right|^2 K(x-y)\,dx\,dy\;\iint_{J\times J}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy}\\
&\leqslant \text{const}\,[v]_{K,\mathbb{R}\times\mathbb{R}}\sqrt{\iint_{J\times J}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy}.\end{aligned} \tag{4.9}$$

Now, using that $Q^\sharp_{\zeta_1,\zeta_2}(x)=\zeta_1$ for any $x\in(-\infty,-1)$ and $Q^\sharp_{\zeta_1,\zeta_2}(x)=\zeta_2$ for any $x\in(1,+\infty)$, we have that

$$\begin{aligned}&\iint_{J\times J}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\\
&= \int_{-2}^{2}\int_{-2}^{2}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\\
&+2\int_{-L}^{-2}\int_{-2}^{2}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\\
&+2\int_{-L}^{-2}\int_{2}^{L}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\\
&+2\int_{-2}^{2}\int_{2}^{L}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy.\end{aligned}$$

We estimates the integrals in the right-hand side of the previous equality as follows.

$$\begin{aligned}&\int_{-L}^{-2}\int_{-2}^{2}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\\&=\int_{-L}^{-2}\int_{-1}^{2}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\\&\leqslant \text{ const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|^2_{L^\infty(\mathbb{R})}L^{-2s}.\end{aligned}$$

Similarly,

$$\int_{-2}^{2}\int_{2}^{L}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\leqslant \text{ const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|^2_{L^\infty(\mathbb{R})}L^{-2s},$$

and

$$\int_{-L}^{-2}\int_{2}^{L}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\leqslant \text{ const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|^2_{L^\infty(\mathbb{R})}L^{1-2s}.$$

Therefore, using in addition that

$$\int_{-2}^{2}\int_{-2}^{2}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\leqslant \text{ const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|^2_{C^1(\mathbb{R})},$$

we infer that

$$\iint_{J\times J}\left|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\right|^2 K(x-y)\,dx\,dy\leqslant \text{ const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|^2_{C^1(\mathbb{R})}L^{1-2s}. \tag{4.10}$$

From (4.9) and (4.10) we obtain

$$\left|\mathscr{B}_{J,J}(v,Q^\sharp_{\zeta_1,\zeta_2})\right|\leqslant \text{ const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|_{C^1(\mathbb{R})}[v]_{K,\mathbb{R}\times\mathbb{R}}L^{\frac{1-2s}{2}}. \tag{4.11}$$

Now, we insert (4.2), (4.3), (4.8) and (4.11) into (4.1) and we obtain

$$\begin{aligned}&\left|\mathscr{B}_{\mathbb{R},\mathbb{R}}(v,Q^\sharp_{\zeta_1,\zeta_2})\right|\\&\leqslant \text{ const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|_{C^1(\mathbb{R})}\left(L^{\frac{1-2s}{2}}\,[v]_{K,\mathbb{R}\times\mathbb{R}}+\frac{1}{L^{\frac{4s-1}{2}}}\sqrt{\int_{S_-(v)}|v(x)|^2\,dx}+\,[v]_{K,\mathbb{R}\times\mathbb{R}}\,\frac{|S_+(v)|^{\frac{1+2s}{2}}}{L^{2s}}\right).\end{aligned}$$

Therefore, choosing $L:=2+|S_+(v)|$ we obtain the desired result. □

Now, we provide a lower bound for the potential energy.

Lemma 4.2. *Let* $v\in L^\infty(\mathbb{R})$,

$$\begin{aligned}&S_-(v):=\left\{x\in\mathbb{R}\setminus[-1,1]\ s.t.\ \operatorname{dist}(v(x),\mathscr{Z})\leqslant\delta_0\right\}\\ \textit{and}\qquad &S_+(v):=\left\{x\in\mathbb{R}\setminus[-1,1]\ s.t.\ \operatorname{dist}(v(x),\mathscr{Z})>\delta_0\right\}.\end{aligned}$$

Then

$$\int_{(-\infty,-1)\cup(1,+\infty)}W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v(x)\big)\,dx\geqslant \text{ const}\int_{S_-(v)}\left|v(x)\right|^2dx+\inf_{\operatorname{dist}(r,\mathscr{Z})\geqslant\delta_0}W(r)\,\left|S_+(v)\right|.$$

Proof. Notice that if $x \in (-\infty,-1)\cup(1,+\infty)$ then $Q^\sharp_{\zeta_1,\zeta_2}(x) \in \mathscr{Z}$ and consequently $W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v(x)\big) = W\big(v(x)\big)$. Therefore, recalling (1.9) we compute that

$$\begin{aligned}\int_{(-\infty,-1)\cup(1,+\infty)} W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v(x)\big)\,dx &= \int_{(-\infty,-1)\cup(1,+\infty)} W\big(v(x)\big)\,dx\\ &= \int_{S_-(v)} W\big(v(x)\big)\,dx + \int_{S_+(v)} W\big(v(x)\big)\,dx\\ &\geqslant \text{const}\int_{S_-(v)} \big|v(x)\big|^2\,dx + \inf_{\text{dist}(r,\mathscr{Z})\geqslant\delta_0} W(r)\,\big|S_+(v)\big|,\end{aligned}$$

as desired. □

Combining Lemmata 4.1 and 4.2 we obtain:

Corollary 4.3. *Let $v\in L^\infty(\mathbb{R})$,*

$$\begin{aligned} & S_-(v) := \big\{x\in\mathbb{R}\setminus[-1,1] \text{ s.t. } \text{dist}(v(x),\mathscr{Z})\leqslant\delta_0\big\}\\ \text{and}\qquad & S_+(v) := \big\{x\in\mathbb{R}\setminus[-1,1] \text{ s.t. } \text{dist}(v(x),\mathscr{Z})>\delta_0\big\}.\end{aligned}$$

Assume that

$$|S_+(v)| < +\infty. \tag{4.12}$$

Then, there exist κ_1, $\kappa_2>0$, possibly depending on $\|Q^\sharp_{\zeta_1,\zeta_2}\|_{C^1(\mathbb{R})}$ and on structural constants, such that

$$\begin{aligned}&\mathscr{B}_{\mathbb{R},\mathbb{R}}(v,Q^\sharp_{\zeta_1,\zeta_2}) + \frac12\int_{\mathbb{R}} a(x)\,W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v(x)\big)\,dx\\ &\qquad\geqslant \kappa_1\left(\int_{S_-(v)}\big|v(x)\big|^2\,dx + \big|S_+(v)\big|\right) - \kappa_2\left([v]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}}+1\right).\end{aligned}$$

Proof. We fix $\varepsilon>0$, to be chosen conveniently small and we denote by C_ε positive quantities, possibly varying from line to line and possibly depending on ε, on $\|Q^\sharp_{\zeta_1,\zeta_2}\|_{C^1(\mathbb{R})}$ and on structural constants.

By the Cauchy-Schwarz Inequality we have that

$$\text{const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|_{C^1(\mathbb{R})}\sqrt{\int_{S_-(v)}|v(x)|^2\,dx} \leqslant C_\varepsilon + \varepsilon\int_{S_-(v)}|v(x)|^2\,dx. \tag{4.13}$$

Also, by Young's Inequality with exponents $\frac{2}{1-2s}$ and $\frac{2}{1+2s}$, we see that

$$\text{const}\,\|Q^\sharp_{\zeta_1,\zeta_2}\|_{C^1(\mathbb{R})}\,|S_+(v)|^{\frac{1-2s}{2}}\,[v]_{K,\mathbb{R}\times\mathbb{R}} \leqslant \varepsilon\,|S_+(v)| + C_\varepsilon\,[v]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}}. \tag{4.14}$$

As a consequence, exploiting Lemma 4.1 and the estimates in (4.13) and (4.14), we obtain that

$$\begin{aligned}\big|\mathscr{B}_{\mathbb{R},\mathbb{R}}(v,Q^\sharp_{\zeta_1,\zeta_2})\big| &\leqslant \varepsilon\left(\int_{S_-(v)}|v(x)|^2\,dx + |S_+(v)|\right) + C_\varepsilon\left([v]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}} + [v]_{K,\mathbb{R}\times\mathbb{R}}+1\right)\\ &\leqslant \varepsilon\left(\int_{S_-(v)}|v(x)|^2\,dx + |S_+(v)|\right) + C_\varepsilon\left([v]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}}+1\right).\end{aligned}$$

From this and Lemma 4.2 we deduce that

$$\begin{aligned}&\mathscr{B}_{\mathbb{R},\mathbb{R}}(v,Q^\sharp_{\zeta_1,\zeta_2}) + \int_{\mathbb{R}} a(x)\,W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v(x)\big)\,dx\\ \geqslant\;& (\text{const}-\varepsilon)\int_{S_-(v)}\big|v(x)\big|^2\,dx + \left(\text{const}\inf_{\text{dist}(r,\mathscr{Z})\geqslant\delta_0} W(r)-\varepsilon\right)\big|S_+(v)\big| - C_\varepsilon\left([v]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}}+1\right)\\ \geqslant\;& \frac12\left(\text{const}\int_{S_-(v)}\big|v(x)\big|^2\,dx + \text{const}\inf_{\text{dist}(r,\mathscr{Z})\geqslant\delta_0} W(r)\,\big|S_+(v)\big|\right) - C_\varepsilon\left([v]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}}+1\right),\end{aligned}$$

as long as ε is taken suitably small. $\square$

5. Variational methods and constrained minimization for a perturbed problem

Fixed ζ_1, $\zeta_2 \in \mathscr{Z}$ and $r \in (0, \min\{\delta_0, r_0\}]$ (where δ_0 and r_0 are given by (1.4) and (1.9), respectively), we construct constrained minimizers for our variational problems. To this aim, we take $b_1 \leqslant -1$ and $b_2 \geqslant 1$ and consider ϕ and ψ solutions to

$$\begin{cases} -\eta\ddot{\phi} + \mathscr{L}\phi = C_0 & \text{in } (b_1-\tau, b_2+\tau), \\ \phi = \zeta_1 + r & \text{in } (-\infty, b_1-\tau], \\ \phi = \zeta_2 + r & \text{in } [b_2+\tau, +\infty), \end{cases} \tag{5.1}$$

and

$$\begin{cases} -\eta\ddot{\psi} + \mathscr{L}\psi = -C_0 & \text{in } (b_1-\tau, b_2+\tau), \\ \psi = \zeta_1 - r & \text{in } (-\infty, b_1-\tau], \\ \psi = \zeta_2 - r & \text{in } [b_2+\tau, +\infty), \end{cases} \tag{5.2}$$

where $C_0 := \|aW'\|_{L^\infty(\mathbb{R})}$ and $\tau \in (0,1)$. It is known that solutions to (5.1) and (5.2) with $\eta = 0$ grow like $d^s(x)$ plus the boundary data away from the boundary of $(b_1-\tau, b_2+\tau)$, where $d(x)$ is the distance function to the boundary of $(b_1-\tau, b_2+\tau)$, see [ROS14]. Thus, by stability of viscosity solutions, there exist c, $C > 0$ such that, for τ small enough,

$$\begin{cases} c(x-b_1+\tau)^s + o_\eta(1) \leqslant \phi(x) - \zeta_1 - r \leqslant C(x-b_1+\tau)^s + o_\eta(1) & \text{for } x \in [b_1-\tau, b_1], \\ c(b_2+\tau-x)^s + o_\eta(1) \leqslant \phi(x) - \zeta_2 - r \leqslant C(b_2+\tau-x)^s + o_\eta(1) & \text{for } x \in [b_2, b_2+\tau], \\ -C(x-b_1+\tau)^s + o_\eta(1) \leqslant \psi(x) - \zeta_1 + r \leqslant -c(x-b_1+\tau)^s + o_\eta(1) & \text{for } x \in [b_1-\tau, b_1], \\ -C(b_2+\tau-x)^s + o_\eta(1) \leqslant \psi(x) - \zeta_2 + r \leqslant -c(b_2+\tau-x)^s + o_\eta(1) & \text{for } x \in [b_2, b_2+\tau], \end{cases}$$

where $o_\eta(1) \to 0$ as $\eta \to 0$. In particular, for small τ,

$$\begin{cases} |\phi(x) - \zeta_1 - r| \leqslant \dfrac{r}{4} & \text{for } x \in [b_1-\tau, b_1], \\[2ex] |\phi(x) - \zeta_2 - r| \leqslant \dfrac{r}{4} & \text{for } x \in [b_2, b_2+\tau], \\[2ex] |\psi(x) - \zeta_1 + r| \leqslant \dfrac{r}{4} & \text{for } x \in [b_1-\tau, b_1], \\[2ex] |\psi(x) - \zeta_2 + r| \leqslant \dfrac{r}{4} & \text{for } x \in [b_2, b_2+\tau]. \end{cases} \tag{5.3}$$

Next, consider smooth functions $\Phi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \Phi(x) = \phi(x) & \text{for } x \in (-\infty, b_1-2\tau] \cup [b_2+2\tau, +\infty), \\ \zeta_1 + \dfrac{3}{4}r \leqslant \Phi(x) \leqslant \phi(x) \leqslant \zeta_1 + \dfrac{5}{4}r & \text{for } x \in (b_1-2\tau, b_1], \\ \Phi(x) \geqslant \phi(x) & \text{for } x \in (b_1, b_2), \\ \zeta_2 + \dfrac{3}{4}r \leqslant \Phi(x) \leqslant \phi(x) \leqslant \zeta_2 + \dfrac{5}{4}r & \text{for } [b_2, b_2+2\tau) \end{cases} \tag{5.4}$$

and

$$\begin{cases} \Psi(x) = \psi(x) & \text{for all } x \in (-\infty, b_1-2\tau] \cup [b_2+2\tau, +\infty), \\ \zeta_1 - \dfrac{5}{4}r \leqslant \psi(x) \leqslant \Psi(x) \leqslant \zeta_1 - \dfrac{3}{4}r & \text{for all } x \in (b_1-2\tau, b_1], \\ \Psi(x) \leqslant \psi(x) & \text{for all } x \in (b_1, b_2), \\ \zeta_2 - \dfrac{5}{4}r \leqslant \psi(x) \leqslant \Psi(x) \leqslant \zeta_2 - \dfrac{3}{4}r & \text{for all } [b_2, b_2+2\tau). \end{cases} \tag{5.5}$$

With this, we can define the set

$$\begin{aligned}\Gamma(b_1,b_2) := \Big\{ & Q:\mathbb{R}\to\mathbb{R} \text{ s.t. } Q-Q^\sharp_{\zeta_1,\zeta_2}\in H^1(\mathbb{R}),\\ & \Psi(x)\leqslant Q(x)\leqslant \Phi(x) \text{ for all } x\in(-\infty,b_1]\cup[b_2,+\infty)\Big\}.\end{aligned}\tag{5.6}$$

Given $\eta\in(0,1]$, we also consider the energy functional

$$\begin{aligned}I_\eta(Q) \;:=\; & \frac{\eta}{2}\int_{\mathbb{R}}|\dot Q(x)|^2\,dx+\int_{\mathbb{R}}a(x)W\big(Q(x)\big)\,dx\\ &+\frac14\iint_{\mathbb{R}\times\mathbb{R}}\Big(\big|Q(x)-Q(y)\big|^2-\big|(Q^\sharp_{\zeta_1,\zeta_2})(x)-(Q^\sharp_{\zeta_1,\zeta_2})(y)\big|^2\Big)\,K(x-y)\,dx\,dy.\end{aligned}\tag{5.7}$$

Then, we can construct a constrained minimizer for I_η in $\Gamma(b_1,b_2)$ (later on, in Proposition 9.2, we will take b_1 and b_2 conveniently separated, in order to employ condition (1.12), so to obtain an unconstrained minimizer, and then, in Section 10, we will send $\eta\to0$ in order to obtain a true solution, as claimed in Theorem 1.1).

Lemma 5.1. *There exists $Q_\eta\in\Gamma(b_1,b_2)$ such that*

$$I_\eta(Q_\eta)\leqslant I_\eta(Q)\ \textit{for all}\ Q\in\Gamma(b_1,b_2).\tag{5.8}$$

Also, letting $v_\eta:=Q_\eta-Q^\sharp_{\zeta_1,\zeta_2}$, it holds that

$$[v_\eta]^2_{H^1(\mathbb{R})}\leqslant\frac{\kappa}{\eta},\tag{5.9}$$

$$[v_\eta]_{K,\mathbb{R}\times\mathbb{R}}\leqslant\kappa,\tag{5.10}$$

$$\|v_\eta\|_{L^\infty(\mathbb{R})}\leqslant\frac{\kappa}{\eta},\tag{5.11}$$

$$\|v_\eta\|_{L^2(\mathbb{R})}\leqslant\kappa\big(1+\|v_\eta\|^2_{L^\infty(\mathbb{R})}\big),\tag{5.12}$$

$$\textit{and}\qquad E_{\mathbb{R}^2}(Q_\eta)\geqslant-\kappa,\tag{5.13}$$

for some $\kappa>0$, which possibly depends on $Q^\sharp_{\zeta_1,\zeta_2}$ and on structural constants.

Proof. We take a minimizing sequence $Q_j\in\Gamma(b_1,b_2)$ for the functional I_η, and we let $v_j:=Q_j-Q^\sharp_{\zeta_1,\zeta_2}\in H^1(\mathbb{R})$. In particular, we have that

$$\lim_{x\to\pm\infty}v_j(x)=0,$$

for any $j\in\mathbb{N}$. From this and (1.7), we have that the set

$$S_+(v_j):=\big\{x\in\mathbb{R}\setminus[-1,1]\text{ s.t. }\operatorname{dist}(v_j(x),\mathscr{Z})>\delta_0\big\}$$

is bounded. This means that condition (4.12) is satisfied for any fixed $j\in\mathbb{N}$ and, as a consequence, by exploiting Corollary 4.3 we obtain that

$$\begin{aligned}&\mathscr{B}_{\mathbb{R},\mathbb{R}}(v_j,Q^\sharp_{\zeta_1,\zeta_2})+\frac12\int_{\mathbb{R}}a(x)\,W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v_j(x)\big)\,dx\\ &\qquad\geqslant\kappa\left(\int_{S_-(v_j)}\big|v_j(x)\big|^2\,dx+\big|S_+(v_j)\big|\right)-\kappa\left([v_j]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}}+1\right),\end{aligned}\tag{5.14}$$

for some κ, possibly varying from line to line and possibly depending on $Q^\sharp_{\zeta_1,\zeta_2}$ and on structural constants, where

$$S_-(v_j):=\big\{x\in\mathbb{R}\setminus[-1,1]\text{ s.t. }\operatorname{dist}(v_j(x),\mathscr{Z})\leqslant\delta_0\big\}.$$

We also define $J_\eta(v) := I_\eta(Q^\sharp_{\zeta_1,\zeta_2}+v)$. In this way, the sequence v_j is minimizing for J_η, and

$$\begin{aligned} J_\eta(v) &= \frac{\eta}{2}\int_{\mathbb{R}} \big(|\dot Q^\sharp_{\zeta_1,\zeta_2}(x)|^2+|\dot v(x)|^2+2\dot Q^\sharp_{\zeta_1,\zeta_2}(x)\dot v(x)\big)\,dx+\int_{\mathbb{R}} a(x)W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v(x)\big)\,dx \\ &\quad+\frac14\iint_{\mathbb{R}\times\mathbb{R}} \big|v(x)-v(y)\big|^2 K(x-y)\,dx\,dy+\frac12\mathscr{B}_{\mathbb{R},\mathbb{R}}(v,Q^\sharp_{\zeta_1,\zeta_2}). \end{aligned} \tag{5.15}$$

Since v_j is minimizing and the zero function is an admissible competitor for J_η, we can also suppose that

$$J_\eta(v_j)\leqslant J_\eta(0)+1\leqslant \frac12\int_{\mathbb{R}}|\dot Q^\sharp_{\zeta_1,\zeta_2}(x)|^2\,dx+\int_{\mathbb{R}}\overline{a}\,W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)\big)\,dx+1\leqslant\kappa. \tag{5.16}$$

In addition, by Cauchy-Schwarz Inequality,

$$2\left|\dot Q^\sharp_{\zeta_1,\zeta_2}(x)\cdot\dot v_j(x)\right|\leqslant 4\left|\dot Q^\sharp_{\zeta_1,\zeta_2}(x)\right|^2+\frac14\left|\dot v_j(x)\right|^2.$$

Combining this estimate with formulas (5.14), (5.15) and (5.16), we conclude that

$$\begin{aligned} \kappa &\geqslant \frac{3\eta}{8}\int_{\mathbb{R}}|\dot v_j(x)|^2\,dx+\frac34\int_{\mathbb{R}} a(x)W\big(Q^\sharp_{\zeta_1,\zeta_2}(x)+v_j(x)\big)\,dx \\ &\quad+\frac14\,[v_j]^2_{K,\mathbb{R}\times\mathbb{R}}-\kappa\,[v_j]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}} \\ &\quad+\kappa\left(\int_{S_-(v_j)}\big|v_j(x)\big|^2\,dx+\big|S_+(v_j)\big|\right). \end{aligned} \tag{5.17}$$

In particular,

$$\kappa\geqslant [v_j]^2_{K,\mathbb{R}\times\mathbb{R}}-\kappa[v_j]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}},$$

which implies that

$$[v_j]_{K,\mathbb{R}\times\mathbb{R}}\leqslant\kappa. \tag{5.18}$$

This and the Sobolev Inequality (see e.g. Appendix A) imply that

$$\|v_j\|_{L^{2^*_s}(\mathbb{R})}\leqslant\kappa,$$

with $2^*_s:=\frac{2}{1-2s}>2$. Therefore, for any interval $\mathscr{I}\subset\mathbb{R}$ of length 1, we have that

$$\|v_j\|_{L^{2^*_s}(\mathscr{I})}\leqslant\kappa. \tag{5.19}$$

Furthermore, exploiting (5.17) once more, we see that

$$\eta\int_{\mathbb{R}}|\dot v_j(x)|^2\,dx\leqslant\kappa-[v_j]^2_{K,\mathbb{R}\times\mathbb{R}}+\kappa[v_j]^{\frac{2}{1+2s}}_{K,\mathbb{R}\times\mathbb{R}}\leqslant\kappa. \tag{5.20}$$

Now we prove that

$$\sup_{x\in\mathbb{R}}|Q_j(x)-\zeta_1|\leqslant\frac{\kappa}{\eta}. \tag{5.21}$$

For this, we suppose that, for some $\bar x\in\mathbb{R}$, it holds that

$$Q_j(\bar x)\geqslant\zeta_1+\nu, \tag{5.22}$$

with $\nu>1$. Our goal is to bound ν. To this end, we use formulas (5.19) and (5.20) to see that

$$\|v_j\|^2_{H^1(\mathscr{I})}\leqslant\frac{\kappa}{\eta},$$

for any interval $\mathscr{I}$ of length 1, and, consequently, by the classical Sobolev Embedding Theorem,

$$[v_j]^2_{C^{0,\frac12}(\mathscr{I})}\leqslant\frac{\kappa}{\eta}.$$

and so

$$[Q_j]^2_{C^{0,\frac12}(\mathscr{I})} \leqslant \frac{\kappa}{\eta}. \tag{5.23}$$

Moreover, from (5.22), we know that there exist $\nu' \in \mathbb{N}$ with $\nu' \geqslant \text{const}\,\nu$ and points $x_1, \dots, x_{\nu'} \in \mathbb{R}$ for which $\text{dist}(Q_j(x_m), \mathscr{Z}) \geqslant \frac14$, for all $m \in \{1, \dots, \nu'\}$. This and (5.23) imply that $\text{dist}(Q_j(x), \mathscr{Z}) \geqslant \frac18$ for all $m \in \{1, \dots, \nu'\}$ and all $x \in (x_m - \kappa\eta, x_m + \kappa\eta)$. Thereupon, we obtain that

$$\int_{\mathbb{R}} a(x)W\big(Q^\sharp_{\zeta_1,\zeta_2}(x) + v_j(x)\big)\,dx = \int_{\mathbb{R}} a(x)W\big(Q_j(x)\big)\,dx \geqslant \kappa\nu'\eta \geqslant \kappa\nu\eta.$$

This and (5.17) imply that $\nu \leqslant \kappa/\eta$.

This argument shows that $Q_j(x) \leqslant \zeta_1 + \frac{\kappa}{\eta}$. Other estimates can be obtained in a similar way, thus proving (5.21).

From (5.21), it follows that

$$\|v_j\|_{L^\infty(\mathbb{R})} \leqslant \frac{\kappa}{\eta}. \tag{5.24}$$

Now, we observe that

$$\begin{aligned}\int_{\mathbb{R}} |v_j(x)|^2\,dx &= \int_{S_-(v_j)} |v_j(x)|^2\,dx + \int_{S_+(v_j)} |v_j(x)|^2\,dx \\ &\leqslant \int_{S_-(v_j)} |v_j(x)|^2\,dx + \|v_j\|^2_{L^\infty(\mathbb{R})}\,|S_+(v_j)| \\ &\leqslant \big(1 + \|v_j\|^2_{L^\infty(\mathbb{R})}\big)\left(\int_{S_-(v_j)} |v_j(x)|^2\,dx + |S_+(v_j)|\right).\end{aligned}$$

This and (5.17) give that

$$\int_{\mathbb{R}} |v_j(x)|^2\,dx \leqslant \kappa\big(1 + \|v_j\|^2_{L^\infty(\mathbb{R})}\big). \tag{5.25}$$

From (5.18) and (5.25), we obtain that, up to a subsequence, v_j converges locally uniformly in $\mathbb{R}$ and weakly in the Hilbert space induced by $[\cdot]_{K,\mathbb{R}\times\mathbb{R}}$ to a minimizer v_η. We then set $Q_\eta := v_\eta + Q^\sharp_{\zeta_1,\zeta_2}$ and we obtain (5.8). Also, the claim in (5.9) follows by taking the limit in (5.20), as well as the claim in (5.10) follows by taking the limit in (5.18). Similarly, the claim in (5.11) follows from (5.24) and the claim in (5.12) follows from (5.25). Finally, (5.13) follows by taking the limit in (5.14) and by (5.10). □

Now, by virtue of the uniform bound in Lemma 3.1, we are in the position of improving (5.11) and (5.12), obtaining uniform estimates in the perturbative parameter η.

Corollary 5.2. *In the setting of Lemma 5.1, it holds that*

$$\|v_\eta\|_{L^\infty(\mathbb{R})} \leqslant \kappa, \tag{5.26}$$

$$\|v_\eta\|_{L^2(\mathbb{R})} \leqslant \kappa, \tag{5.27}$$

$$\text{and} \qquad [Q_\eta]^2_{C^{0,\frac12}(\mathbb{R})} \leqslant \frac{\kappa}{\eta}, \tag{5.28}$$

for some $\kappa > 0$, which possibly depends on $Q^\sharp_{\zeta_1,\zeta_2}$ and on structural constants.

Proof. If $x \in (b_1, b_2)$, the minimizer Q_η is unconstrained and we can therefore differentiate the energy functional I_η, thus obtaining that

$$-\eta\ddot{Q}_\eta + aW'(Q_\eta) + \mathscr{L}Q_\eta = 0 \qquad \text{in } (b_1, b_2).$$

Moreover, by (5.4) and (5.5), we see that

$$|Q_\eta(x) - \zeta_1| \leqslant \frac54 r \text{ for all } x \leqslant b_1 \quad \text{and} \quad |Q_\eta(x) - \zeta_2| \leqslant \frac54 r \text{ for all } x \geqslant b_2.$$

In this way, we are in position of using Lemma 3.1 with $f(x) := -a(x)W'\big(Q_\eta(x)\big)$: thus we deduce that there exists κ, independent of C_0, such that $\|Q_\eta\|_{L^\infty(\mathbb{R})} \leqslant \kappa$, and therefore

$$\|v_\eta\|_{L^\infty(\mathbb{R})} \leqslant \|Q_\eta\|_{L^\infty(\mathbb{R})} + \|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})} \leqslant \kappa.$$

This proves (5.26).

The claim in (5.27) follows from (5.12) and (5.26).

Finally, (5.9), (5.26) and the classical Sobolev Theorem yield (5.28). □

Now we define

$$J_* := (b_1, b_2)$$

and

$$L := \big\{x \in (-\infty, b_1] \cup [b_2, +\infty) \text{ s.t. } \Psi(x) < Q_\eta(x) < \Phi(x)\big\}.$$

Let also

$$F := J_* \cup L. \tag{5.29}$$

As usual, by taking inner variations, one sees that in the set F the minimization problem is "free" and so it satisfies an Euler-Lagrange equation, as stated explicitly in the next result:

Lemma 5.3. *Let Q_η be as in Lemma 5.1. For any $x \in F$, we have that*

$$-\eta\, \ddot{Q}_\eta(x) + \mathscr{L} Q_\eta(x) + a(x)\, W'(Q_\eta(x)) = 0. \tag{5.30}$$

Now we define the set

$$\Sigma := \Big\{Q : \mathbb{R} \to \mathbb{R} \text{ s.t. } Q - Q^\sharp_{\zeta_1,\zeta_2} \in H^1(\mathbb{R}) \text{ and } \Psi(x) \leqslant Q(x) \leqslant \Phi(x) \text{ for all } x \in \mathbb{R}\Big\}. \tag{5.31}$$

We notice that, differently from the definition of $\Gamma(b_1, b_2)$ given in (5.6), we require here that a function Q belongs to Σ if it satisfies $\Psi \leqslant Q \leqslant \Phi$ in the whole of $\mathbb{R}$, and not only in $(-\infty, b_1] \cup [b_2, +\infty)$.

As a matter of fact, we prove that the minimizer $Q_\eta \in \Gamma(b_1, b_2)$, given by Lemma 5.1, is actually a minimizer of I_η in Σ:

Lemma 5.4. *Let Q_η be as in Lemma 5.1. Then, we have that $Q_\eta \in \Sigma$. In particular,*

$$\inf_{Q\in\Sigma} I_\eta(Q) = \inf_{Q\in\Gamma(b_1,b_2)} I_\eta(Q) = I_\eta(Q_\eta). \tag{5.32}$$

Proof. We first prove that Q_η belongs to Σ. For this, it is enough to show that

$$\Psi(x) \leqslant Q_\eta(x) \leqslant \Phi(x) \quad \text{for any } x \in (b_1, b_2). \tag{5.33}$$

To check this, we observe that, by Lemma 5.3, Q_η is solution of

$$-\eta\, \ddot{Q}_\eta(x) + \mathscr{L} Q_\eta(x) + a(x)\, W'(Q_\eta(x)) = 0 \quad \text{for any } x \in (b_1, b_2).$$

In addition, since $Q_\eta \in \Gamma(b_1, b_2)$, recalling (5.4) and (5.6), we see that

$$Q_\eta(x) \leqslant \Phi(x) \leqslant \phi(x) \quad \text{for any } x \in (-\infty, b_1] \cup [b_2, +\infty). \tag{5.34}$$

Thus, using also (5.1) and the Comparison Principle, we conclude that

$$Q_\eta(x) \leqslant \phi(x) \quad \text{for any } x \in (b_1, b_2).$$

Consequently, making again use of (5.4),

$$Q_\eta(x) \leqslant \phi(x) \leqslant \Phi(x) \quad \text{for any } x \in (b_1, b_2),$$

which proves the second inequality in (5.33). Similarly, one can check that

$$Q_\eta(x) \geqslant \Psi(x) \quad \text{for any } x \in (b_1, b_2),$$

which completes the proof of (5.33).

Now, since $Q_\eta \in \Sigma \subset \Gamma(b_1, b_2)$, it holds that

$$\inf_{Q\in\Sigma} I_\eta(Q) \geqslant \inf_{Q\in\Gamma(b_1,b_2)} I_\eta(Q) = I_\eta(Q_\eta) \geqslant \inf_{Q\in\Sigma} I_\eta(Q),$$

which proves (5.32). The proof of Lemma 5.4 is thus complete. ☐

6. Lewy-Stampacchia estimates and continuity results for a double obstacle problem

In this section, we prove that constrained minimizers of the perturbed problem are continuous, with uniform bounds. This estimate is based on a double obstacle problem approach. We follow a technique introduced by Lewy and Stampacchia in [LS70] and suitably modified in [SV13] to deal with nonlocal problems. In our situation, differently from the previous literature, we need to take into account the fact that the problem is constrained by two obstacles. Moreover, our problem is a superposition of a local and a nonlocal operators and we aim at estimates which are uniform with respect to the local contribution. The result that suits for our purposes is the following:

Proposition 6.1. *Let I be a bounded interval and $f \in L^\infty(I)$. Let $u \in \Sigma$, with Σ defined as in* (5.31), *and assume that*

$$\begin{aligned}\eta \int_{\mathbb{R}} \dot u(x)\big(\dot u(x) - \dot v(x)\big)\,dx + \frac{1}{2}\iint_{\mathbb{R}^2} \big(u(x)-u(y)\big)\big((u-v)(x)-(u-v)(y)\big)K(x-y)\,dx\,dy \\ \leqslant \int_{\mathbb{R}} f(x)\,(u-v)(x)\,dx,\end{aligned} \tag{6.1}$$

for every $v \in \Sigma$ with $v = u$ in $\mathbb{R}\setminus I$. Then,

$$\min\left\{\inf_{x\in I} -|\ddot\Phi(x)| + \mathscr{L}\Phi(x),\ \inf_{x\in I} f(x)\right\} \leqslant -\eta\ddot u(x) + \mathscr{L}u(x) \leqslant \max\left\{\sup_{x\in I}|\ddot\Psi(x)| + \mathscr{L}\Psi(x),\ \sup_{x\in I} f(x)\right\} \tag{6.2}$$

in the sense of distributions.

Proof. Let

$$M^* := \max\left\{\sup_{x\in I}|\ddot\Psi(x)| + \mathscr{L}\Psi(x),\ \sup_{x\in I} f(x)\right\} \tag{6.3}$$

and

$$I^*(v) := \frac{\eta}{2}\int_I |\dot v(x)|^2\,dx + \frac14 \iint_{Q_I} \big|v(x)-v(y)\big|^2 K(x-y)\,dx\,dy - M^*\int_I v(x)\,dx,$$

where $Q_I := (I\times I)\cup\big(I\times(\mathbb{R}\setminus I)\big)\cup\big((\mathbb{R}\setminus I)\times I\big)$. We take z^* to be a minimizer of I^* in the class of functions $v:\mathbb{R}\to\mathbb{R}$ with $v(x)\leqslant u(x)$ for any $x\in\mathbb{R}$ and $v(x)=u(x)$ for any $x\in\mathbb{R}\setminus I$. The existence of such minimizer follows by compactness, along the lines given in the proof of Lemma 5.1. In particular,

$$z^*(x)\leqslant u(x) \text{ for any } x\in\mathbb{R} \text{ and } z^*(x)=u(x) \text{ for any } x\in\mathbb{R}\setminus I. \tag{6.4}$$

Moreover, for any $\varepsilon\in[0,1]$ and any $w:\mathbb{R}\to\mathbb{R}$ with $w(x)\leqslant u(x)$ for any $x\in\mathbb{R}$ and $w(x)=u(x)$ for any $x\in\mathbb{R}\setminus I$, we have that $z_\varepsilon(x) := \varepsilon w(x) + (1-\varepsilon)z^*(x)$ is an admissible competitor for z^* and consequently $I^*(z_\varepsilon)\geqslant I^*(z^*)$, which gives that

$$\begin{aligned}0 &\leqslant \frac{d}{d\varepsilon} I^*(z_\varepsilon)\Big|_{\varepsilon=0} \\ &= \eta\int_I \dot z^*(x)\big(\dot w(x)-\dot z^*(x)\big)\,dx + \frac12\iint_{Q_I}\big(z^*(x)-z^*(y)\big)\big((w-z^*)(x)-(w-z^*)(y)\big)K(x-y)\,dx\,dy \\ &\qquad - M^*\int_I (w-z^*)(x)\,dx.\end{aligned} \tag{6.5}$$

We claim that

$$z^*\in\Sigma. \tag{6.6}$$

To check this, we first use (6.4) to observe that

$$z^*(x) \leqslant u(x) \leqslant \Phi(x). \tag{6.7}$$

Then, we take

$$w^*(x) := \max\{z^*(x), \Psi(x)\} = z^*(x) + \big(\Psi(x) - z^*(x)\big)_+ .$$

By (6.4), we know that $w^*(x) \leqslant u(x)$ for any $x \in \mathbb{R}$. Also, if $x \in \mathbb{R} \setminus I$, we have that $w^*(x) = \max\{u(x), \Psi(x)\} = u(x)$. Therefore, we can make use of (6.5) with $w := w^*$, and so we find that

$$\begin{aligned} 0 \,&\leqslant \eta \int_{I\cap\{\Psi>z^*\}} \dot z^*(x)\big(\dot\Psi(x) - \dot z^*(x)\big)\,dx \\ &\quad + \frac12 \iint_{Q_I} \big(z^*(x) - z^*(y)\big)\Big(\big(\Psi(x) - z^*(x)\big)_+ - \big(\Psi(y) - z^*(y)\big)_+\Big) K(x-y)\,dx\,dy \\ &\quad - M^* \int_I \big(\Psi(x) - z^*(x)\big)_+\,dx. \end{aligned} \tag{6.8}$$

Furthermore, on ∂I we have that $z^* = u \geqslant \Psi$, hence, from (6.3) and integrating by parts, we see that

$$\begin{aligned} &\eta \int_{I\cap\{\Psi>z^*\}} \dot\Psi(x)\big(\dot\Psi(x) - \dot z^*(x)\big)\,dx \\ &\qquad + \frac12 \iint_{Q_I} \big(\Psi(x) - \Psi(y)\big)\big(\big(\Psi(x) - z^*(x)\big)_+ - \big(\Psi(y) - z^*(y)\big)_+\big) K(x-y)\,dx\,dy \\ &\qquad - M^* \int_I \big(\Psi(x) - z^*(x)\big)_+\,dx \\ &= -\eta \int_{\mathbb{R}} \ddot\Psi(x)\big(\Psi(x) - z^*(x)\big)_+\,dx \\ &\qquad + \iint_{\mathbb{R}^2} \big(\Psi(x) - \Psi(y)\big)\big(\Psi(x) - z^*(x)\big)_+ K(x-y)\,dx\,dy \\ &\qquad - M^* \int_{\mathbb{R}} \big(\Psi(x) - z^*(x)\big)_+\,dx \\ &= \int_{\mathbb{R}} \Big(-\eta\ddot\Psi(x) + \mathscr{L}\Psi(x) - M^*\Big)\big(\Psi(x) - z^*(x)\big)_+\,dx \\ &\leqslant 0. \end{aligned} \tag{6.9}$$

Thus, subtracting (6.8) to (6.9), we conclude that

$$\begin{aligned} 0 \,&\geqslant \eta \int_I \big(\dot\Psi(x) - \dot z^*(x)\big)\big(\dot\Psi(x) - \dot z^*(x)\big)_+\,dx \\ &\quad + \frac12 \iint_{Q_I} \Big(\big(\Psi(x) - z^*(x)\big) - \big(\Psi(y) - z^*(y)\big)\Big)\Big(\big(\Psi(x) - z^*(x)\big)_+ - \big(\Psi(y) - z^*(y)\big)_+\Big) K(x-y)\,dx\,dy. \end{aligned} \tag{6.10}$$

The last term in (6.10) is nonnegative (see e.g. page 1115 in [SV13]), therefore we get that

$$0 \geqslant \int_I \big(\dot\Psi(x) - \dot z^*(x)\big)_+^2\,dx.$$

This says that $\Psi(x) \leqslant z^*(x)$ for any $x \in I$ (and so for any $x \in \mathbb{R}$, due to (6.4)). This and (6.7) imply (6.6), as desired.

Then, from (6.6) we deduce that both the minimum and the maximum between u and z^* belong to Σ, that is

$$v^\sharp(x) := \min\{u(x), z^*(x)\} = u(x) - \big(u(x) - z^*(x)\big)_+ \in \Sigma$$

$$\text{and} \qquad w^\sharp(x) := \max\{u(x), z^*(x)\} = z^*(x) + \big(u(x) - z^*(x)\big)_+ \in \Sigma .$$

In particular, we can take $v := v^\sharp$ in (6.1) and $w := w^\sharp$ in (6.5). This gives that

$$\begin{aligned}
&\eta \int_{\mathbb{R}} \dot u(x)\,\big(\dot u(x) - \dot z^*(x)\big)_+\,dx \\
&\qquad + \frac12 \iint_{\mathbb{R}^2} \big(u(x)-u(y)\big)\Big(\big(u(x)-z^*(x)\big)_+ - \big(u(y)-z^*(y)\big)_+\Big)\,K(x-y)\,dx\,dy \\
&\qquad \leqslant \int_{\mathbb{R}} f(x)\,\big(u(x)-z^*(x)\big)_+\,dx
\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}
M^* \int_I \big(u(x)-z^*(x)\big)_+\,dx &\leqslant \eta \int_I \dot z^*(x)\big(\dot u(x)-\dot z^*(x)\big)_+\,dx \\
&\quad + \frac12 \iint_{Q_I} \big(z^*(x)-z^*(y)\big)\Big(\big(u(x)-z^*(x)\big)_+ - \big(u(y)-z^*(y)\big)_+\Big)\,K(x-y)\,dx\,dy.
\end{aligned} \tag{6.12}$$

Hence, subtracting (6.12) to (6.11) and recalling (6.3), we obtain

$$\begin{aligned}
0 &\geqslant \eta \int_I \big(\dot u(x)-\dot z^*(x)\big)\big(\dot u(x)-\dot z^*(x)\big)_+\,dx \\
&\quad + \frac12 \iint_{Q_I} \Big(\big(u(x)-z^*(x)\big)-\big(u(y)-z^*(y)\big)\Big)\Big(\big(u(x)-z^*(x)\big)_+ - \big(u(y)-z^*(y)\big)_+\Big)\,K(x-y)\,dx\,dy.
\end{aligned}$$

As above, this implies that $u \leqslant z^*$. Combining this with (6.4), we obtain that z^* coincides with u. As a consequence, taking any function $v \geqslant 0$, supported in I, and defining $w := u - v$ in (6.5),

$$\eta \int_I \dot u(x)\dot v(x)\,dx + \frac12 \iint_{Q_I} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,K(x-y)\,dx\,dy \leqslant M^* \int_I v(x)\,dx.$$

Integrating by parts the latter inequality, we obtain that

$$\int_{\mathbb{R}} \Big(-\eta \ddot u(x) + \mathscr{L} u(x)\Big)\,v(x)\,dx \leqslant M^* \int_{\mathbb{R}} v(x)\,dx.$$

By duality, we thus obtain that

$$-\eta \ddot u(x) + \mathscr{L} u(x) \leqslant M^*,$$

in the sense of distributions, which is one of the inequalities in (6.2). The other inequality in (6.2) follows by similar arguments. □

Using Lemma 3.2, Proposition 6.1 and a convolution method (see e.g. formula (3.2) in [SV14]), we obtain a useful uniform continuity result for a perturbed problem. The statement that we need for our purposes is the following:

Corollary 6.2. *Let Q_η be as in Lemma 5.1 and $\alpha \in (0, 2s)$. Then $Q_\eta \in C^{0,\alpha}(\mathbb{R})$ and*

$$\|Q_\eta\|_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa, \tag{6.13}$$

for some $\kappa > 0$, which possibly depends on $Q^\sharp_{\zeta_1,\zeta_2}$ and on structural constants.

Proof. We take $v_\eta := Q_\eta - Q^\sharp_{\zeta_1,\zeta_2}$, as in Lemma 5.1. By Lemma 5.4, we know that $Q_\eta \in \Sigma$. We fix an interval $I \subset \mathbb{R}$ and take any $\xi \in \Sigma$. For any $\varepsilon \in (0,1)$, let $\xi_\varepsilon := \varepsilon\xi + (1-\varepsilon)Q_\eta = Q_\eta + \varepsilon(\xi - Q_\eta)$.

Then $\xi_\varepsilon \in \Sigma$ and therefore, by (5.32), we know that

$$\begin{aligned}
0 &\leqslant I_\eta(\xi_\varepsilon) - I_\eta(Q_\eta)\\
&= \frac{\eta}{2}\int_I \left(|\dot\xi_\varepsilon(x)|^2 - |\dot Q_\eta(x)|^2\right)dx + \int_I a(x)\Big(W\big(\xi_\varepsilon(x)\big) - W\big(Q_\eta(x)\big)\Big)\,dx\\
&\quad+\frac{1}{4}\iint_{\mathbb{R}\times\mathbb{R}}\Big(\big|\xi_\varepsilon(x)-\xi_\varepsilon(y)\big|^2 - \big|Q_\eta(x)-Q_\eta(y)\big|^2\Big)K(x-y)\,dx\,dy\\
&= \varepsilon\eta\int_I \dot Q_\eta(x)\cdot\big(\dot\xi(x)-\dot Q_\eta(x)\big)\,dx + \varepsilon\int_I a(x)\,W'\big(Q_\eta(x)\big)\,\big(\xi(x)-Q_\eta(x)\big)dx\\
&\quad+\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\Big(\big(Q_\eta(x)-Q_\eta(y)\big)\,\big((\xi-Q_\eta)(x)-(\xi-Q_\eta)(y)\big)\Big)K(x-y)\,dx\,dy + o(\varepsilon).
\end{aligned}$$

Thus, dividing this inequality by ε and sending $\varepsilon \searrow 0$, we conclude that Q_η satisfies (6.1) with $f := -aW'(Q_\eta)$. Accordingly, by formula (6.2) in Proposition 6.1, we know that

$$-\operatorname{const} \leqslant -\eta\ddot Q_\eta + \mathscr{L}Q_\eta \leqslant \operatorname{const}$$

and therefore

$$-\kappa \leqslant -\eta\ddot v_\eta + \mathscr{L}v_\eta \leqslant \kappa \tag{6.14}$$

in the sense of distributions, for some $\kappa > 0$, which possibly depends on $Q^\sharp_{\zeta_1,\zeta_2}$.

Now we take an even function $\mu \in C^\infty_0([-1,1])$ and for any $\varepsilon \in (0,1)$ we set $\mu_\varepsilon(x) := \varepsilon^{-1}\mu(x/\varepsilon)$. We consider the mollification $v_{\eta,\varepsilon} := v_\eta * \mu_\varepsilon$. Notice that, as $\varepsilon \searrow 0$, we have that

$$v_{\eta,\varepsilon} \text{ converges locally uniformly to } v_\eta, \tag{6.15}$$

thanks to (5.28). Moreover, we observe that, for any $\varphi \in C^\infty_0(\mathbb{R})$,

$$\begin{aligned}
&\Big|\big(v_\eta(x)-v_\eta(y)\big)\big(\varphi(x)-\varphi(y)\big)\,\mu_\varepsilon(z)\,K(x-y)\Big|\\
&\qquad\leqslant \Big(\big|v_\eta(x)-v_\eta(y)\big|^2 K(x-y) + \big|\varphi(x)-\varphi(y)\big|^2 K(x-y)\Big)\,\chi_{[-1,1]}(z),
\end{aligned}$$

which, as a function of $(x,y,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, belongs to $L^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, thanks to (5.10). This implies that we can exploit the Dominated Convergence Theorem and obtain that

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}}\big(v_{\eta,\varepsilon}(x)-v_{\eta,\varepsilon}(y)\big)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\\
&= \iint_{\mathbb{R}\times\mathbb{R}}\left[\int_{\mathbb{R}}\big(v_\eta(x-z)-v_\eta(y-z)\big)\,\mu_\varepsilon(z)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dz\right]dx\,dy\\
&= \int_{\mathbb{R}}\left[\iint_{\mathbb{R}\times\mathbb{R}}\big(v_\eta(x-z)-v_\eta(y-z)\big)\,\mu_\varepsilon(z)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\right]dz\\
&= \int_{\mathbb{R}}\left[\iint_{\mathbb{R}\times\mathbb{R}}\big(v_\eta(x)-v_\eta(y)\big)\,\mu_\varepsilon(z)\,\big(\varphi(x+z)-\varphi(y+z)\big)\,K(x-y)\,dx\,dy\right]dz\\
&= \iint_{\mathbb{R}\times\mathbb{R}}\left[\int_{\mathbb{R}}\big(v_\eta(x)-v_\eta(y)\big)\,\mu_\varepsilon(z)\,\big(\varphi(x+z)-\varphi(y+z)\big)\,K(x-y)\,dz\right]dx\,dy\\
&= \iint_{\mathbb{R}\times\mathbb{R}}\left[\int_{\mathbb{R}}\big(v_\eta(x)-v_\eta(y)\big)\,\mu_\varepsilon(z)\,\big(\varphi(x-z)-\varphi(y-z)\big)\,K(x-y)\,dz\right]dx\,dy\\
&= \iint_{\mathbb{R}\times\mathbb{R}}\big(v_\eta(x)-v_\eta(y)\big)\big(\varphi_\varepsilon(x)-\varphi_\varepsilon(y)\big)\,K(x-y)\,dx\,dy,
\end{aligned}\tag{6.16}$$

where $\varphi_\varepsilon := \varphi * \mu_\varepsilon$. Similarly, by (5.9), we see that

$$\int_{\mathbb{R}} \dot v_{\eta,\varepsilon}(x)\,\dot\varphi(x)\,dx = \int_{\mathbb{R}} \dot v_\eta(x)\,\dot\varphi_\varepsilon(x)\,dx. \tag{6.17}$$

From (6.14), (6.16) and (6.17) we infer that, for any $\varphi \in C_0^\infty(\mathbb{R}, [0,1])$,

$$\begin{aligned}&\left|\eta \int_{\mathbb{R}} \dot v_{\eta,\varepsilon}(x)\,\dot\varphi(x)\,dx + \frac12 \iint_{\mathbb{R}\times\mathbb{R}} \big(v_{\eta,\varepsilon}(x)-v_{\eta,\varepsilon}(y)\big)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\right|\\ &\qquad = \left|\eta \int_{\mathbb{R}} \dot v_{\eta}(x)\,\dot\varphi_\varepsilon(x)\,dx + \frac12 \iint_{\mathbb{R}\times\mathbb{R}} \big(v_{\eta}(x)-v_{\eta}(y)\big)\big(\varphi_\varepsilon(x)-\varphi_\varepsilon(y)\big)\,K(x-y)\,dx\,dy\right|\\ &\qquad \leqslant \kappa\left|\int_{\mathbb{R}} \varphi_\varepsilon(x)\,dx\right| \leqslant \kappa \int_{\mathbb{R}} \varphi(x)\,dx.\end{aligned}$$

That is,

$$-\kappa \leqslant -\eta\ddot v_{\eta,\varepsilon} + \mathscr{L} v_{\eta,\varepsilon} \leqslant \kappa$$

in the sense of distributions, and also in the classical and viscosity senses, since $v_{\eta,\varepsilon}$ is smooth. Therefore, we are in the position of applying Lemma 3.2 to $v_{\eta,\varepsilon}$ and conclude that

$$[v_{\eta,\varepsilon}]_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa\big(1+\|v_{\eta,\varepsilon}\|_{L^\infty(\mathbb{R})}\big)^{\frac{\alpha}{2s}}\|v_{\eta,\varepsilon}\|_{L^\infty(\mathbb{R})}^{1-\frac{\alpha}{2s}} \leqslant \kappa\big(1+\|v_{\eta}\|_{L^\infty(\mathbb{R})}\big)^{\frac{\alpha}{2s}}\|v_{\eta}\|_{L^\infty(\mathbb{R})}^{1-\frac{\alpha}{2s}},$$

for any $\alpha \in (0,2s)$ (up to freely renaming κ). As a consequence of this and (5.26), we obtain that $[v_{\eta,\varepsilon}]_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa$. This and (6.15) imply that $[v_\eta]_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa$. Using this and (5.26), we obtain that $\|v_\eta\|_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa$, which in turn implies (6.13), as desired. □

7. Clean intervals and clean points

Here we deal with the notions of *clean intervals* and *clean points*, which have been introduced in Section 6 of [DPV17] to perform glueing techniques in the nonlocal setting.

Definition 7.1. *Given $\rho > 0$ and $Q : \mathbb{R} \to \mathbb{R}$, we say that an interval $J \subseteq \mathbb{R}$ is a "clean interval" for (ρ, Q) if $|J| \geqslant |\log \rho|$ and there exists $\zeta \in \mathscr{Z}$ such that*

$$\sup_{x\in J} |Q(x) - \zeta| \leqslant \rho.$$

Definition 7.2. *If J is a bounded clean interval for (ρ, Q), the center of J is called a "clean point" for (ρ, Q).*

Here we show that any sufficiently large interval contains a clean interval.

Lemma 7.3. *Let $J \subseteq \mathbb{R}$ be an interval. Let Q_η be as in Lemma 5.1. Then, there exist $\rho_0 \in (0,1)$ and $\kappa_1 > 0$ depending on $Q^\sharp_{\zeta_1,\zeta_2}$ and on the structural constants such that, if $\rho \in (0, \rho_0)$ and*

$$|J| \geqslant \frac{\kappa_1 [Q_\eta]^{\frac1\alpha}_{C^{0,\alpha}(J)}}{\rho^{2+\frac1\alpha}}|\log\rho|, \tag{7.1}$$

for $\alpha \in (0,2s)$, then there exists a clean interval for (ρ, Q_η) that is contained in J.

Proof. By Corollary 6.2, we know that $Q \in C^{0,\alpha}(J)$ for any $\alpha \in (0,2s)$. Without loss of generality we can assume that $[Q_\eta]_{C^{0,\alpha}(J)} \geqslant 1$. Assume, by contradiction, that

$$J \text{ does not contain any clean subinterval.} \tag{7.2}$$

By (7.1), the interval J contains N disjoint subintervals, say $J_1, \dots, J_N$, each of length $|\log \rho|$, with

$$N \geqslant \frac{\kappa_1 [Q_\eta]^{\frac1\alpha}_{C^{0,\alpha}(J)}}{\rho^{2+\frac1\alpha}} - 1, \tag{7.3}$$

and, by (7.2), none of the subintervals J_i is clean. Hence, for any $i \in \{1, \dots, N\}$, there exists $p_i \in J_i$ such that $Q(p_i)$ stays at distance larger than ρ from $\mathscr{Z}$. Also, letting

$$\ell_\rho := \left(\frac{\rho}{2[Q_\eta]_{C^{0,\alpha}(J)}}\right)^{\frac1\alpha},$$

we have that, for any $x \in J_i' := [p_i - \ell_\rho, p_i + \ell_\rho]$,

$$|Q_\eta(x) - Q_\eta(p_i)| \leqslant [Q_\eta]_{C^{0,\alpha}(J)} |x - p_i|^\alpha \leqslant [Q_\eta]_{C^{0,\alpha}(J)}\, \ell_\rho^\alpha = \frac{\rho}{2}.$$

Accordingly, $Q_\eta(x)$ stays at distance larger than $\frac{\rho}{2}$ from $\mathscr{Z}$ for any $x \in J_i'$ and then, by (1.9),

$$W(Q_\eta(x)) \geqslant \frac{c_0\, \rho^2}{4}.$$

Moreover, for ρ sufficiently small, at least half of the interval J_i' lies in J_i, hence

$$\int_{J_i \cap J_i'} W(Q_\eta(x))\, dx \geqslant \frac{c_0\, \rho^2\, \ell_\rho}{4} = \frac{\kappa \rho^{2+\frac{1}{\alpha}}}{[Q_\eta]^{\frac{1}{\alpha}}_{C^{0,\alpha}(J)}}.$$

Summing up over $i = 1, \dots, N$, using that the intervals J_i are disjoint and recalling (1.10), (5.8) and (5.13), we find that

$$\begin{aligned} I_\eta(Q^\sharp_{\zeta_1,\zeta_2}) &\geqslant I_\eta(Q_\eta) \\ &\geqslant -\kappa + \sum_{i=1}^N \int_{J_i \cap J_i'} a(x) W(Q_\eta(x))\, dx \\ &\geqslant -\kappa + \frac{N \underline{a} \kappa \rho^{2+\frac{1}{\alpha}}}{[Q_\eta]^{\frac{1}{\alpha}}_{C^{0,\alpha}(J)}}, \end{aligned}$$

which gives

$$N \leqslant \frac{\kappa [Q_\eta]^{\frac{1}{\alpha}}_{C^{0,\alpha}(J)}}{\rho^{2+\frac{1}{\alpha}}}.$$

This is a contradiction with (7.3) for $\kappa_1 > \kappa + 1$ and so it proves the desired result. □

Lemma 7.4. *Let Q_η be as in Lemma 5.1. Let $T > 1$ and $J := (x_0 - 4T, x_0 + 4T)$ be a clean interval for (ρ, Q_η). Then, for any $\alpha \in (0, 2s)$,*

$$[Q_\eta]_{C^{0,\alpha}(x_0 - T, x_0 + T)} \leqslant C \left(\frac{\rho^{1-\frac{\alpha}{2s}}}{|\log \rho|^\alpha} + \rho \right),$$

for some $C > 0$, independent of η.

Proof. Let $\zeta \in \mathscr{Z}$ be such that $\sup_{x \in J} |Q_\eta(x) - \zeta| \leqslant \rho$. Then, according to Definition 7.1, we have that

$$T \geqslant \frac{|\log \rho|}{8}, \tag{7.4}$$

and $J \subset F$, where F is defined as in (5.29). Therefore, by Lemma 5.3, Q_η is solution of

$$-\eta\, \ddot{Q}_\eta + \mathscr{L} Q_\eta + a\, W'(Q_\eta) = 0 \quad \text{in } J.$$

Then by Lemma 3.3, (5.26) and (7.4), for $\alpha < 2s$, we have that

$$\begin{aligned} [Q_\eta]_{C^{0,\alpha}(x_0 - T, x_0 + T)} &\leqslant C T^{-\alpha} \left(1 + T^{2s} \rho\right)^{\frac{\alpha}{2s}} \rho^{1-\frac{\alpha}{2s}} \\ &\leqslant C T^{-\alpha} \left(1 + T^\alpha \rho^{\frac{\alpha}{2s}}\right) \rho^{1-\frac{\alpha}{2s}} \\ &\leqslant C \left(T^{-\alpha} \rho^{1-\frac{\alpha}{2s}} + \rho\right) \\ &\leqslant C \left(\frac{\rho^{1-\frac{\alpha}{2s}}}{|\log \rho|^\alpha} + \rho \right), \end{aligned}$$

by possibly renaming C. This proves the desired estimate of Lemma 7.4. □

Remark 7.5. Given $x_0 \in \mathbb{R}$ and $\beta \in (1,+\infty)$, let $P : \mathbb{R} \to \mathbb{R}$ be a function such that

$$v := P - Q^\sharp_{\zeta_1,\zeta_2} \in H^1(\mathbb{R}) \tag{7.5}$$

and P is Hölder continuous in $(x_0-\beta, x_0+\beta)$, with

$$[P]_{C^{0,\alpha}(x_0-\beta,x_0+\beta)} \leqslant \delta, \tag{7.6}$$

for some $\delta > 0$. Given T_1, T_2 such that $-\infty \leqslant T_1 \leqslant x_0-\beta < x_0+\beta \leqslant T_2 \leqslant +\infty$, let us denote

$$I_- := (T_1, x_0), \quad I_+ := (x_0, T_2)$$

and

$$J_- := (T_1, x_0-\beta), \quad D_- := (x_0-\beta, x_0), \quad D_+ := (x_0, x_0+\beta), \quad J_+ := (x_0+\beta, T_2).$$

We want to estimate $E_{(T_1,T_2)^2}(P)$ in terms of $E_{I_-^2}(P)$ and $E_{I_+^2}(P)$. We have that

$$E_{(T_1,T_2)^2}(P) = E_{I_-^2}(P) + E_{I_+^2}(P) + 2E_{I_-\times I_+}(P) \tag{7.7}$$

and

$$E_{I_-\times I_+}(P) = E_{J_-\times I_+}(P) + E_{D_-\times D_+}(P) + E_{D_-\times J_+}(P). \tag{7.8}$$

By (7.6) and (1.4),

$$\begin{aligned} 0 \leqslant E_{D_-\times D_+}(P) + [Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,D_-\times D_+} &= \int_{x_0-\beta}^{x_0}\int_{x_0}^{x_0+\beta} |P(x)-P(y)|^2\, K(x-y)\,dx\,dy \\ &\leqslant \Theta_0\delta^2 \int_{x_0-\beta}^{x_0}\int_{x_0}^{x_0+\beta} |x-y|^{2\alpha-1-2s}\,dx\,dy \\ &\leqslant \kappa\delta^2\beta^{2\alpha+1-2s}. \end{aligned} \tag{7.9}$$

Moreover, recalling (7.5), we have that

$$E_{J_-\times I_+}(P) = [v]^2_{K,J_-\times I_+} + 2\mathscr{B}_{J_-\times I_+}(v, Q^\sharp_{\zeta_1,\zeta_2}). \tag{7.10}$$

Now, by (1.4),

$$\begin{aligned} \left|\mathscr{B}_{J_-\times I_+}(v, Q^\sharp_{\zeta_1,\zeta_2})\right| &= \left|\int_{T_1}^{x_0-\beta}\int_{x_0}^{T_2} \big(v(x)-v(y)\big)\big((Q^\sharp_{\zeta_1,\zeta_2})(x) - (Q^\sharp_{\zeta_1,\zeta_2})(y)\big)\,K(x-y)\,dx\,dy\right| \\ &\leqslant 2\Theta_0 \|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{x_0-\beta}\int_{x_0}^{+\infty} \big(|v(x)|+|v(y)|\big)|x-y|^{-1-2s}\,dx\,dy \\ &= \frac{\Theta_0 \|Q^\sharp_{\zeta_1,\zeta_2}\|_{L^\infty(\mathbb{R})}}{s}\left[\int_{-\infty}^{x_0-\beta} |v(x)|(x_0-x)^{-2s}\,dx + \int_{x_0}^{+\infty} |v(y)|(y-x_0+\beta)^{-2s}\,dy\right]. \end{aligned}$$

In addition, using the Cauchy-Schwarz inequality and (1.5), we see that

$$\begin{aligned} &\int_{-\infty}^{x_0-\beta} |v(x)|(x_0-x)^{-2s}\,dx \leqslant \left(\int_{-\infty}^{x_0-\beta} |v(x)|^2\,dx\right)^{\frac12}\left(\int_{-\infty}^{x_0-\beta} (x_0-x)^{-4s}\,dx\right)^{\frac12} \\ &\leqslant \kappa\|v\|_{L^2(\mathbb{R})}\beta^{-\frac{4s-1}{2}}. \end{aligned}$$

Similarly,

$$\int_{x_0}^{+\infty} |v(y)|(y-x_0+\beta)^{-2s}\,dy \leqslant \kappa\|v\|_{L^2(\mathbb{R})}\beta^{-\frac{4s-1}{2}}.$$

Plugging these pieces of information into (7.10), we have that

$$\left|E_{J_-\times I_+}(P)\right| \leqslant [v]^2_{K,(-\infty,x_0-\beta)\times(x_0,+\infty)} + \kappa\|v\|_{L^2(\mathbb{R})}\beta^{-\frac{4s-1}{2}} \tag{7.11}$$

Similar computations give

$$\left|E_{D_-\times J_+}(P)\right| \leqslant [v]^2_{K,(x_0-\beta,x_0)\times(x_0+\beta,+\infty)} + \kappa\|v\|_{L^2(\mathbb{R})}\beta^{-\frac{4s-1}{2}} \tag{7.12}$$

Hence, from (7.8), (7.9), (7.11) and (7.12), we conclude that

$$\begin{aligned}&\left|E_{I_-\times I_+}(P)+[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_0-\beta,x_0)\times(x_0,x_0+\beta)}\right|\\ \leqslant\;& \kappa\delta^2\beta^{2\alpha+1-2s_0}+\kappa\|v\|_{L^2(\mathbb{R})}\beta^{-\frac{4s-1}{2}}+[v]^2_{K,(-\infty,x_0-\beta)\times(x_0,+\infty)}+[v]^2_{K,(x_0-\beta,x_0)\times(x_0+\beta,+\infty)}.\end{aligned}$$

This and (7.7) imply that

$$\begin{aligned}&\left|E_{(T_1,T_2)^2}(P)-E_{(T_1,x_0)^2}(P)-E_{(x_0,T_2)^2}(P)+2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_0-\beta,x_0)\times(x_0,x_0+\beta)}\right|\\ &\leqslant \kappa\delta^2\beta^{2\alpha+1-2s}+\kappa\|v\|_{L^2(\mathbb{R})}\beta^{-\frac{4s-1}{2}}+2[v]^2_{K,(-\infty,x_0-\beta)\times(x_0,+\infty)}+2[v]^2_{K,(x_0-\beta,x_0)\times(x_0+\beta,+\infty)}.\end{aligned} \tag{7.13}$$

Now, thanks to (7.13), one can consider a clean point x_0 (according to Definitions 7.1 and 7.2) and glue an optimal trajectory Q_η to a linear interpolation with the integer ζ, close to $Q_\eta(x_0)$. Namely, one can consider

$$P(x):=\begin{cases}Q_\eta(x) & \text{if } x\leqslant x_0,\\ R(x) & \text{if } x>x_0,\end{cases} \tag{7.14}$$

where R is such that $P-Q^\sharp_{\zeta_1,\zeta_2}\in H^1(\mathbb{R})$ and it is defined in $[x_0,x_0+\beta]$ as follows:

$$R(x):=\begin{cases}Q_\eta(x_0)\,(x_0+1-x)+\zeta\,(x-x_0) & \text{if } x\in(x_0,x_0+1),\\ \zeta & \text{if } x\in[x_0+1,x_0+\beta).\end{cases}$$

In this way, and taking $\rho>0$ suitably small, by Definitions 7.1 and 7.2, we know that Q_η is ρ-close to an integer in $[x_0-4\beta,x_0+4\beta]$, with

$$\beta=\beta(\rho)=\frac{|\log\rho|}{8}. \tag{7.15}$$

Moreover, by Lemma 7.4, we have that, for $\alpha\in(0,2s)$,

$$[Q_\eta]_{C^{0,\alpha}(x_0-\beta,x_0+\beta)}\leqslant C\left(\frac{\rho^{1-\frac{\alpha}{2s}}}{|\log\rho|^\alpha}+\rho\right), \tag{7.16}$$

for some $C>0$. Also, we observe that

$$[R]_{C^{0,\alpha}(x_0,x_0+\beta)}\leqslant\kappa\rho.$$

As a consequence of this and (7.16), the function P defined in (7.14) satisfies (7.6) with

$$\delta:=C\left(\frac{\rho^{1-\frac{\alpha}{2s}}}{|\log\rho|^\alpha}+\rho\right) \tag{7.17}$$

and $\alpha\in(0,2s)$. Thus, choosing β as in (7.15) and δ as in (7.17), and recalling (5.10) and (5.27), we infer from estimate (7.13) that

$$\left|E_{(T_1,T_2)^2}(P)-E_{(T_1,x_0)^2}(Q_\eta)-E_{(x_0,T_2)^2}(R)+2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_0-\beta,x_0)\times(x_0,x_0+\beta)}\right|\leqslant\diamondsuit, \tag{7.18}$$

where we use the notation "$\diamondsuit$" to denote quantities that are as small as we wish when ρ is sufficiently small. The smallness of ρ depends on $Q^\sharp_{\zeta_1,\zeta_2}$, η and the structural constants of the kernel and the potential.

We remark that, in virtue of (7.16), we also have that

$$\left|E_{(T_1,T_2)^2}(Q_\eta)-E_{(T_1,x_0)^2}(Q_\eta)-E_{(x_0,T_2)^2}(Q_\eta)+2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_0-\beta,x_0)\times(x_0,x_0+\beta)}\right|\leqslant\diamondsuit. \tag{7.19}$$

8. Stickiness properties of energy minimizers

In this section we show that the minimizers have the tendency to stick at the integers once they arrive sufficiently close to them. For this, we recall that $r \in (0, \min\{\delta_0, r_0\}]$ (with δ_0 and r_0 as in (1.4) and (1.9), respectively) has been fixed at the beginning of Section 5.

Proposition 8.1. *Let $\rho \in (0,1)$. Let Q_η be as in Lemma 5.1. Let $x_1, x_2 \in \mathbb{R}$ be clean points for (ρ, Q_η), according to Definition 7.2, with $x_2 \geqslant x_1 + 4$, and*

$$\max_{i=1,2} |Q_\eta(x_i) - \zeta| \leqslant \rho, \tag{8.1}$$

for some $\zeta \in \mathscr{Z}$. Then

$$\frac{\eta}{2}\int_{x_1}^{x_2} |\dot{Q}_\eta(x)|^2\,dx + \frac{1}{4}[Q_\eta]^2_{K,(x_1,x_2)^2} + \int_{x_1}^{x_2} a(x)\,W(Q_\eta(x))\,dx \leqslant \diamondsuit, \tag{8.2}$$

with $\diamondsuit$ as small as we wish if ρ is suitably small (the smallness of ρ depends on $Q^\sharp_{\zeta_1,\zeta_2}$, and on structural constants, but it is independent of η).

Moreover,

$$|Q_\eta(x) - \zeta| \leqslant r/2 \text{ for every } x \in [x_1, x_2]. \tag{8.3}$$

Proof. We define

$$P(x) := \begin{cases} Q_\eta(x) & \text{if } x \in (-\infty, x_1), \\ Q_\eta(x_1)(x_1+1-x) + \zeta(x-x_1) & \text{if } x \in [x_1, x_1+1], \\ \zeta & \text{if } x \in (x_1+1, x_2-1), \\ Q_\eta(x_2)(x-x_2+1) + \zeta(x_2-x) & \text{if } x \in [x_2-1, x_2], \\ Q_\eta(x) & \text{if } x \in (x_2, +\infty). \end{cases}$$

In this way, we have that

$$[P]_{C^{0,1}(x_1,x_2)} \leqslant \rho. \tag{8.4}$$

Moreover, we observe that, if $x \in (x_1, x_2)$, then

$$\begin{aligned} &|P(x) - \zeta| \\ &\leqslant \sup_{y\in(x_1,x_1+1)} |Q_\eta(x_1)(x_1+1-y) + \zeta(y-x_1) - \zeta| + \sup_{y\in(x_2-1,x_2)} |Q_\eta(x_2)(y-x_2-1) + \zeta(x_2-y) - \zeta| \\ &\leqslant |Q_\eta(x_1) - \zeta| + |Q_\eta(x_2) - \zeta| \leqslant 2\rho, \end{aligned} \tag{8.5}$$

thanks to (8.1). Also,

$$\text{if } x, y \in (x_1, x_2), \text{ then } |P(x) - P(y)| \leqslant 2\rho. \tag{8.6}$$

Now, let us estimate $[P]^2_{K,(x_1,x_2)^2}$. We have

$$[P]^2_{K,(x_1,x_2)^2} = [P]^2_{K,(x_1,x_1+1)\times(x_1,x_2)} + [P]^2_{K,(x_1+1,x_2-1)\times(x_1,x_2)} + [P]^2_{K,(x_2-1,x_2)\times(x_1,x_2)}. \tag{8.7}$$

Using (1.4), (8.4) and (8.6), we see that

$$\begin{aligned} &[P]^2_{K,(x_1,x_1+1)\times(x_1,x_2)} \\ &= \int_{x_1}^{x_1+1}\int_{x_1}^{x_1+2} |P(x)-P(y)|^2\,K(x-y)\,dx\,dy + \int_{x_1}^{x_1+1}\int_{x_1+2}^{x_2} |P(x)-P(y)|^2\,K(x-y)\,dx\,dy \\ &\leqslant \Theta_0\rho^2 \int_{x_1}^{x_1+1}\int_{x_1}^{x_1+2} |x-y|^{1-2s}\,dx\,dy + 4\Theta_0\rho^2 \int_{x_1}^{x_1+1}\int_{x_1+2}^{x_2} |x-y|^{-1-2s}\,dx\,dy \\ &\leqslant \kappa\rho^2 \\ &= \diamondsuit. \end{aligned} \tag{8.8}$$

Similarly,

$$[P]^2_{K,(x_2-1,x_2)\times(x_1,x_2)} \leqslant \diamondsuit. \tag{8.9}$$

Finally, making again use of (1.4), (8.4) and (8.6), we compute

(8.10)

$$\begin{aligned}
&[P]^2_{K(x_1+1,x_2-1)\times(x_1,x_2)}\\
&= \int_{x_1+1}^{x_2-1}\int_{x_1}^{x_1+1} |P(x)-P(y)|^2\,K(x-y)\,dx\,dy + \int_{x_1+1}^{x_2-1}\int_{x_2-1}^{x_2} |P(x)-P(y)|^2\,K(x-y)\,dx\,dy\\
&= \int_{x_1+1}^{x_1+2}\int_{x_1}^{x_1+1} |P(x)-P(y)|^2\,K(x-y)\,dx\,dy + \int_{x_1+2}^{x_2-1}\int_{x_1}^{x_1+1} |P(x)-P(y)|^2\,K(x-y)\,dx\,dy\\
&\qquad + \int_{x_1+1}^{x_2-2}\int_{x_2-1}^{x_2} |P(x)-P(y)|^2\,K(x-y)\,dx\,dy + \int_{x_2-2}^{x_2-1}\int_{x_2-1}^{x_2} |P(x)-P(y)|^2\,K_m(x-y)\,dx\,dy\\
&\leqslant \kappa\rho^2\,\Big(\int_{x_1+1}^{x_1+2}\int_{x_1}^{x_1+1} |x-y|^{1-2s}\,dx\,dy + \int_{x_2-2}^{x_2-1}\int_{x_2-1}^{x_2} |x-y|^{1-2s}\,dx\,dy\\
&\qquad\qquad + \int_{x_1+2}^{x_2-1}\int_{x_1}^{x_1+1} |x-y|^{-1-2s}\,dx\,dy + \int_{x_1+1}^{x_2-2}\int_{x_2-1}^{x_2} |x-y|^{-1-2s}\,dx\,dy\Big)\\
&\leqslant \kappa\rho^2\\
&= \diamondsuit.
\end{aligned}$$

Therefore, collecting estimates (8.7), (8.8), (8.9) and (8.10), we get

$$[P]^2_{K,(x_1,x_2)^2} \leqslant \diamondsuit. \tag{8.11}$$

Combining (7.18) (applied here twice, with $x_0 := x_1$ and $x_0 := x_2$) with (8.11) yields, for β as in (7.15),

$$\begin{aligned}
E_{\mathbb{R}^2}(P) &\leqslant E_{(-\infty,x_1)^2}(Q_\eta) + E_{(x_1,x_2)^2}(P) + E_{(x_2,+\infty)^2}(Q_\eta) + \diamondsuit\\
&\qquad - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_1-\beta,x_1)\times(x_1,x_1+\beta)} - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_2-\beta,x_2)\times(x_2,x_2+\beta)}\\
&= E_{(-\infty,x_1)^2}(Q_\eta) + E_{(x_2,+\infty)^2}(Q_\eta) + \diamondsuit - [Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_1,x_2)^2}\\
&\qquad - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_1-\beta,x_1)\times(x_1,x_1+\beta)} - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_2-\beta,x_2)\times(x_2,x_2+\beta)}.
\end{aligned} \tag{8.12}$$

On the other hand, by (7.19) (again applied here twice, with $x_0 := x_1$ and $x_0 := x_2$), we have that

$$\begin{aligned}
E_{\mathbb{R}^2}(Q_\eta) &\geqslant E_{(-\infty,x_1)^2}(Q_\eta) + E_{(x_1,x_2)^2}(Q_\eta) + E_{(x_2,+\infty)^2}(Q_\eta) + \diamondsuit\\
&\qquad - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_1-\beta,x_1)\times(x_1,x_1+\beta)} - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_2-\beta,x_2)\times(x_2,x_2+\beta)}.
\end{aligned} \tag{8.13}$$

Subtracting (8.13) to (8.12), we get

$$E_{\mathbb{R}^2}(P) - E_{\mathbb{R}^2}(Q_\eta) \leqslant -[Q_\eta]^2_{K(x_1,x_2)^2} + \diamondsuit. \tag{8.14}$$

In addition, by (1.9) and (8.5), we see that if $x \in (x_1,x_2)$ then $W(P(x)) \leqslant 4C_0\rho^2$. Using this and the fact that $W(P(x)) = W(\zeta) = 0$ if $x \in (x_1+1, x_2-1)$, we conclude that

$$\int_{x_1}^{x_2} W(P(x))\,dx = \int_{x_1}^{x_1+1} W(P(x))\,dx + \int_{x_2-1}^{x_2} W(P(x))\,dx \leqslant 8C_0\,\rho^2.$$

Thus, by the minimality of Q_η for I_η (defined in (5.7)) and (8.14),

$$\begin{aligned}
0 &\leqslant I_\eta(P) - I_\eta(Q_\eta)\\
&\leqslant \eta\rho - \frac{\eta}{2}\int_{x_1}^{x_2} |\dot Q_\eta(x)|^2\,dx - \frac14 [Q_\eta]^2_{K,(x_1,x_2)^2} - \int_{x_1}^{x_2} a(x)\,W(Q_\eta(x))\,dx + \diamondsuit,
\end{aligned}$$

which proves (8.2).

Now we prove (8.3). For this, we assume by contradiction that there exists $\tilde{x} \in [x_1, x_2]$ such that $|Q_\eta(\tilde{x}) - \zeta| > r/2$.

By Corollary 6.2, we have that Q_η is Hölder continuous (with uniform bound). Hence, since $|Q_\eta(x_1) - \zeta| \leqslant \rho < r/2$, we obtain that there exists $\hat{x} \in [x_1, x_2]$ such that

$$|Q(\hat{x}) - \zeta| = \frac{r}{2}. \tag{8.15}$$

In particular, there exists ℓ independent of η such that, for any $x \in [\hat{x} - \ell, \hat{x} + \ell]$ and $\alpha \in (0, 2s)$,

$$|Q_\eta(x) - Q_\eta(\hat{x})| \leqslant \kappa\, |x - \hat{x}|^\alpha \leqslant \frac{r}{4}.$$

This and (8.15) imply that, if $x \in [\hat{x} - \ell, \hat{x} + \ell]$,

$$Q_\eta(x) \in \overline{B_{3r/4}(\zeta) \setminus B_{r/4}(\zeta)}$$

and thus

$$\operatorname{dist}\big(Q_\eta(x), \mathscr{Z}\big) \geqslant \frac{r}{4},$$

for all $x \in [\hat{x} - \ell, \hat{x} + \ell]$. This, (1.9) and (1.10) give that

$$\int_{\hat{x}-\ell}^{\hat{x}+\ell} a(x) W(Q_\eta(x))\, dx \geqslant \underline{a} \int_{\hat{x}-\ell}^{\hat{x}+\ell} W(Q_\eta(x))\, dx \geqslant 2\ell\, \underline{a} \inf_{\operatorname{dist}(\tau,\, \mathscr{Z}) \geqslant r/4} W(\tau) =: c.$$

Hence, noticing that $(\hat{x} - \ell, \hat{x} + \ell) \subseteq (x_1, x_2)$, we obtain that

$$\int_{x_1}^{x_2} a(x) W(Q_\eta(x))\, dx \geqslant c,$$

and this is in contradiction with (8.2) for small ρ. Then, the proof of (8.3) is now complete. □

9. Unconstrained minimization for a perturbed problem

Here, recalling the setting of Section 5, we show that if b_1 and b_1 are sufficiently separated, then the constrained minimizer, whose existence has been established in Lemma 5.1, is in fact an unconstrained minimizer. The idea for this is that the "excursion" of the minimizer will occur at the points "favored by the wells of a" (recall the non-degeneracy condition in (1.12)), which can be placed suitably far from the constraints.

Fixed $\zeta_1 \neq \zeta_2 \in \mathscr{Z}$, we consider the minimizer $Q_\eta = Q_\eta^{\zeta_1,\zeta_2}$ for I_η as given in Lemma 5.1. Let also

$$I_{\zeta_1} := \inf_{\zeta_2 \in \mathscr{Z} \setminus \{\zeta_1\}} I_\eta(Q_\eta^{\zeta_1,\zeta_2}). \tag{9.1}$$

We remark that, by (5.26), only a finite number of integer points ζ_2 takes part to the minimization procedure in (9.1). Accordingly, we can write

$$I_{\zeta_1} = \min_{\zeta_2 \in \mathscr{Z} \setminus \{\zeta_1\}} I_\eta(Q_\eta^{\zeta_1,\zeta_2}) \tag{9.2}$$

and define $\mathscr{A}(\zeta_1)$ the family of all $\zeta_2 \in \mathscr{Z}$ attaining such minimum.

In what follows we make explicit the dependence of the set $\Gamma(b_1, b_2)$, defined in (5.6), on ζ_1 and ζ_2 and we denote it by $\Gamma(b_1, b_2, \zeta_1, \zeta_2)$.

Lemma 9.1. *There exists $\rho_* > 0$, possibly depending on $Q^\sharp_{\zeta_1,\zeta_2}$ and on structural constants, such that if $\rho \in (0, \rho_*]$ the following statement holds.*

Let $\zeta_1 \in \mathscr{Z}$ and $\zeta_2 \in \mathscr{A}(\zeta_1)$. Let $Q_\eta^{\zeta_1,\zeta_2}$ be as in Lemma 5.1. Assume that there exist $\zeta \in \mathscr{Z}$ and a clean point $x_* \in (b_1 + 1, b_2 - 1)$ such that $Q_\eta^{\zeta_1,\zeta_2}(x_*) \in \overline{B_\rho(\zeta)}$.

Then $\zeta \in \{\zeta_1, \zeta_2\}$.

Proof. Suppose by contradiction that $\zeta \notin \{\zeta_1, \zeta_2\}$. We define

$$P(x) := \begin{cases} Q_\eta^{\zeta_1,\zeta_2}(x) & \text{if } x \leqslant x_*, \\ Q_\eta^{\zeta_1,\zeta_2}(x_*)(x_*+1-x) + \zeta(x-x_*) & \text{if } x \in (x_*, x_*+1), \\ \zeta & \text{if } x > x_*+1. \end{cases}$$

By construction, P belongs to the set $\Gamma(b_1, b_2, \zeta_1, \zeta)$ and $\zeta \neq \zeta_1$. Therefore, using the minimality of $Q_\eta^{\zeta_1,\zeta_2}$,

$$I_\eta(Q_\eta^{\zeta_1,\zeta_2}) \leqslant I_\eta(P). \tag{9.3}$$

On the other hand, using (7.18), we see that for β defined as in (7.15)

$$\begin{aligned} E_{(\mathbb{R}^2)}(P) &\leqslant E_{(-\infty,x_*)^2}(P) + E_{(x_*,+\infty)^2}(P) - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_*-\beta,x_*)\times(x_*,x_*+\beta)} + \diamondsuit \\ &\leqslant E_{(-\infty,x_*)^2}(Q_\eta^{\zeta_1,\zeta_2}) - [Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_*,+\infty)^2} - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_*-\beta,x_*)\times(x_*,x_*+\beta)} + \diamondsuit. \end{aligned} \tag{9.4}$$

Moreover, by (7.19),

$$E_{\mathbb{R}^2}(Q_\eta^{\zeta_1,\zeta_2}) \geqslant E_{(-\infty,x_*)^2}(Q_\eta^{\zeta_1,\zeta_2}) + E_{(x_*,+\infty)^2}(Q_\eta^{\zeta_1,\zeta_2}) - 2[Q^\sharp_{\zeta_1,\zeta_2}]^2_{K,(x_*-\beta,x_*)\times(x_*,x_*+\beta)} + \diamondsuit. \tag{9.5}$$

Estimates (9.3), (9.4) and (9.5) imply that

$$0 \leqslant I_\eta(P) - I_\eta(Q_\eta^{\zeta_1,\zeta_2}) \leqslant \int_{x_*}^{+\infty} a(x)\left[W(P(x)) - W(Q_\eta^{\zeta_1,\zeta_2}(x))\right] dx + \diamondsuit. \tag{9.6}$$

Now we use that $\zeta \neq \zeta_2$ and that $|Q_\eta^{\zeta_1,\zeta_2}(b_2) - \zeta_2| \leqslant \frac{5}{4}r$ (recall (5.6)) to find $y_* \in [x_*, b_2]$ for which $Q_\eta^{\zeta_1,\zeta_2}(y_*) = \zeta_2 + \frac{1}{2}$ or $Q_\eta^{\zeta_1,\zeta_2}(y_*) = \zeta_2 - \frac{1}{2}$. Assume, without loss of generality, that $Q_\eta^{\zeta_1,\zeta_2}(y_*) = \zeta_2 + \frac{1}{2}$. Then, by Corollary 6.2, there exists $\ell > 0$ independent of η such that $Q_\eta^{\zeta_1,\zeta_2}(x)$ stays at distance at least 1/4 from $\mathscr{Z}$ for all $x \in [y_*, y_* + \ell]$. Accordingly,

$$\int_{x_*}^{+\infty} a(x)\, W(Q_\eta^{\zeta_1,\zeta_2}(x))\, dx \geqslant \underline{a} \int_{y_*}^{y_*+\ell} W(Q_\eta^{\zeta_1,\zeta_2}(x))\, dx \geqslant \underline{a}\,\ell \inf_{\operatorname{dist}(\tau,\mathscr{Z})\geqslant 1/4} W(\tau) =: \tilde{c}.$$

Plugging this into (9.6) and using the definition of P, we obtain

$$0 \leqslant I_\eta(P) - I_\eta(Q_\eta^{\zeta_1,\zeta_2}) \leqslant \diamondsuit - \tilde{c},$$

which is a contradiction for ρ small enough. This completes the proof of Lemma 9.1. □

Proposition 9.2. *There exist b_1, $b_2 \in \mathbb{R}$ and $Q^\star_\eta \in \Gamma(b_1, b_2)$ such that*

$$I_\eta(Q^\star_\eta) \leqslant I_\eta(Q) \text{ for all } Q \text{ s.t. } Q - Q^\sharp_{\zeta_1,\zeta_2} \in H^1(\mathbb{R}). \tag{9.7}$$

Also, letting $v^\star_\eta := Q^\star_\eta - Q^\sharp_{\zeta_1,\zeta_2}$, it holds that

$$[v^\star_\eta]_{H^1(\mathbb{R})} \leqslant \frac{\kappa}{\eta}, \tag{9.8}$$

$$[v^\star_\eta]_{K,\mathbb{R}\times\mathbb{R}} \leqslant \kappa, \tag{9.9}$$

$$\|v^\star_\eta\|_{L^\infty(\mathbb{R})} \leqslant \kappa, \tag{9.10}$$

$$\|v^\star_\eta\|_{L^2(\mathbb{R})} \leqslant \kappa \tag{9.11}$$

$$\text{and} \qquad \|v^\star_\eta\|_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa \text{ for all } \alpha \in (0, 2s), \tag{9.12}$$

for some $\kappa > 0$, which possibly depends on $Q^\sharp_{\zeta_1,\zeta_2}$ and on structural constants.

Proof. We stress that the main difference between (5.8) and (9.7) is that the competitors in (9.7) do not need to be in $\Gamma(b_1, b_2)$ and so $Q^\star_\eta$ is a free minimizer. The proof of Proposition 9.2 is a slight modification of the proof of Theorem 9.4 in [DPV17], and we refer to it for more details.

Let $\zeta_1 \in \mathscr{Z}$ and $\zeta_2 \in \mathscr{A}(\zeta_1)$. Let $Q^\star_\eta := Q_\eta^{\zeta_1,\zeta_2}$ be as in Lemma 5.1 and let $v^\star_\eta := Q^\star_\eta - Q^\sharp_{\zeta_1,\zeta_2}$. Then by Lemma 5.1, Corollary 5.2 and Corollary 6.2 we have that $v^\star_\eta$ satisfies (9.8)-(9.12).

To prove (9.7), we fix $\rho \in (0,r)$, to be taken sufficiently small, and we set

$$b_1 = m_1 \quad \text{and} \quad b_2 = m_2,$$

with m_1, m_2 given by (1.12). To prove Proposition 9.2, we want to show that $Q^\star_\eta$ does not touch the constraints of $\Gamma(b_1,b_2,\zeta_1,\zeta_2)$. Assume by contradiction that

$$\text{there exists } x_1 \leqslant b_1 = m_1 \text{ such that either } Q^\star_\eta(x_1) = \Phi(x_1) \text{ or } Q^\star_\eta(x_1) = \Psi(x_1), \tag{9.13}$$

the other case being similar. In particular, by (5.4) and (5.5), we have that $|Q^\star_\eta(x_1) - \zeta_1| \geqslant \frac{3}{4}r$. Also, by (9.12), we know that $[Q^\star_\eta]_{C^{0,\alpha}(\mathbb{R})} \leqslant \kappa$, for $\alpha \in (0,2s)$. Thus, by Lemma 7.3, if

$$\omega \geqslant \frac{\kappa_1 \kappa^{\frac{1}{\alpha}}}{\rho^{2+\frac{1}{\alpha}}} |\log \rho| + 1,$$

we conclude that

$$\text{there exist a clean point } x_* \in (m_1+1, m_1+\omega) \text{ and } \zeta \in \mathscr{Z} \text{ such that } Q^\star_\eta(x_*) \in \overline{B_\rho(\zeta)}. \tag{9.14}$$

Furthermore, by Lemma 9.1, we have that $\zeta \in \{\zeta_1, \zeta_2\}$. Now, arguing as in [DPV17] and using (9.13), we see that we must actually have that

$$\zeta = \zeta_2 \tag{9.15}$$

and that $Q^\star_\eta(x) \in \overline{B_{\frac{r}{2}}}(\zeta_2)$ for any $x \geqslant x_*$. In particular, since by (1.11), $x_* \leqslant m_1 + \omega \leqslant m_2 - \theta$, we have that

$$Q^\star_\eta(x) \in \overline{B_{\frac{r}{2}}}(\zeta_2) \text{ for any } x \geqslant m_2 - \theta. \tag{9.16}$$

Now we define $P(x) := Q^\star_\eta(x-\theta)$. Due to (9.16), we have that $P \in \Gamma(b_1,b_2,\zeta_1,\zeta_2)$ and therefore, by the minimality of $Q^\star_\eta$,

$$\begin{aligned} 0 \leqslant I(P) - I(Q^\star_\eta) &= \int_{\mathbb{R}} a(x)\, W(P(x))\, dx - \int_{\mathbb{R}} a(x)\, W(Q^\star_\eta(x))\, dx \\ &= \int_{\mathbb{R}} a(x)\, W(Q^\star_\eta(x-\theta))\, dx - \int_{\mathbb{R}} a(x)\, W(Q^\star_\eta(x))\, dx \\ &= \int_{\mathbb{R}} \big[a(x+\theta) - a(x)\big]\, W(Q^\star_\eta(x))\, dx. \end{aligned} \tag{9.17}$$

Now, we observe that $Q^\star_\eta(m_1) \in \overline{B_{\frac{5}{4}r}(\zeta_1)}$ and $Q^\star_\eta(x_*) \in \overline{B_\rho(\zeta_2)}$, due to (9.14) and (9.15). Therefore, since Q^*_η is continuous, there exists $y_* \in (m_1, m_1+\omega)$ such that either $Q^*_\eta(y_*) = \zeta_1 + \frac{1}{2}$ or $Q^*_\eta(y_*) = \zeta_1 - \frac{1}{2}$. Assume without loss of generality that $Q^*_\eta(y_*) = \zeta_1 + \frac{1}{2}$. Then by the Hölder continuity of $Q^\star_\eta$, there exists an interval $J_* \subset (m_1, m_1+\omega)$ of uniform length and centered at y_* such that $Q^\star_\eta(x)$ stays at distance $1/4$ from $\mathscr{Z}$ for any $x \in J_*$. Therefore, using (1.12), we get

$$\begin{aligned} \int_{m_1-\omega}^{m_1+\omega} \big[a(x+\theta) - a(x)\big]\, W(Q^\star_\eta(x))\, dx &\leqslant \int_{J_*} \big[a(x+\theta) - a(x)\big]\, W(Q^\star_\eta(x))\, dx \\ &\leqslant -\gamma \int_{J_*} W(Q^\star_\eta(x))\, dx \leqslant -\tilde{\gamma} \inf_{\operatorname{dist}(\tau,\mathscr{Z}) \geqslant 1/4} =: -\hat{\gamma}. \end{aligned} \tag{9.18}$$

Now, by (5.27) and the continuity of $Q^\star_\eta$, we know that there exists a sequence of points $y_k \geqslant b_2 = m_2$ with $y_k \to +\infty$ as $k \to +\infty$, such that y_k is a clean point for $Q^\star_\eta$ and $Q^\star_\eta(y_k) \in \overline{B_\rho(\zeta_2)}$. Then, recalling (9.14) and (9.15), by (8.2) and (1.10), we have that

$$\int_{x_*}^{y_k} \big[a(x+\theta) - a(x)\big]\, W(Q^\star_\eta(x))\, dx \leqslant \diamondsuit.$$

On that account, sending $k \to +\infty$, we obtain that

$$\int_{m_1+\omega}^{+\infty} \big[a(x+\theta) - a(x)\big]\, W(Q^\star_\eta(x))\, dx \leqslant \diamondsuit. \tag{9.19}$$

On the other hand, by arguing as in [DPV17], we have that

$$\int_{-\infty}^{m_1-\omega} \big[a(x+\theta) - a(x)\big]\, W(Q^\star_\eta(x))\, dx \leqslant \diamondsuit. \tag{9.20}$$

By plugging (9.18), (9.19) and (9.20) into (9.17), we conclude that

$$0 \leqslant -\hat\gamma + \diamondsuit.$$

The latter inequality is negative for ρ sufficiently small, and so we have obtained the desired contradiction. This proves (9.7).

□

10. Vanishing viscosity method and proof of Theorem 1.1

Now we consider the free minimizer constructed in Proposition 9.2 and we send $\eta \to 0$. The uniform estimates in (9.9), (9.10), (9.11) and (9.12) will allow us to pass to the limit and obtain a free minimizer, hence a solution, of the original nonlocal problem, thus completing the proof of Theorem 1.1.

This perturbative technique may be thought as a nonlocal counterpart of the so-called vanishing viscosity method for Hamilton-Jacobi equations, in which a small viscosity term is added as a perturbation to obtain solutions of the original equation.

To this aim, we consider I_0 to be the energy functional corresponding to the choice $\eta := 0$ in (5.7), namely the one in (1.14).

Then, for any $\eta > 0$, we take $Q^\star_\eta$ to be the free minimizer given by Proposition 9.2. We consider an infinitesimal sequence $\eta_j \to 0$ and let $Q^\star_j := Q^\star_{\eta_j}$ and $v^j_\star := Q^\star_j - Q^\sharp_{\zeta_1,\zeta_2}$.

Since the estimates in (9.9), (9.10), (9.11) and (9.12) are uniform in η_j, up to a subsequence we can assume that $v^\star_j$ converges to some $v^\star$ locally uniformly in $\mathbb{R}$ and weakly in the Hilbert space induced by $[\cdot]_{K,\mathbb{R}\times\mathbb{R}}$. Then, we set $Q^\star := v^\star + Q^\sharp_{\zeta_1,\zeta_2}$.

By passing to the limit in (9.7), we obtain (1.17). Also, from (9.9) and (9.11) we obtain (1.18) and from (9.10) and (9.12) we obtain (1.19).

Since $Q^\star$ is a minimizer of I_0, by differentiating the energy functional we obtain (1.13) (in the distributional sense, and thus also in the viscosity sense, due to [SV14]).

Since from (1.18) and (1.19) $v^\star$ is uniformly continuous and also in $L^2(\mathbb{R})$, it follows that

$$\lim_{x\to\pm\infty} v^\star(x) = 0.$$

This implies (1.16). The proof of Theorem 1.1 is thus completed.

Appendix A. A general Sobolev Inequality

For completeness, in this appendix, we provide a Sobolev Inequality in the fractional setting, used here on page 15. Most of the settings considered in the literature deal with the case of homogeneous kernels, corresponding to Sobolev spaces of fractional order. The result we present here is general enough to comprise also truncated kernels (as the ones on the left hand side of (1.4)) and so can be applied in our context.

Proposition A.1. *Let $N \in \mathbb{N}$, $N \geqslant 1$, $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Let $r_0 > 0$. Then there exists a positive constant C, possibly depending on N, p, s and r_0, such that, for any measurable and compactly supported function $f : \mathbb{R}^N \to \mathbb{R}$, we have that*

$$\|f\|_{L^{p^*_s}(\mathbb{R}^N)} \leqslant C \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\, \chi_{[0,r_0]}(|x-y|)\, dx\, dy \right)^{\frac1p},$$

where

$$p_s^* := \frac{Np}{N - sp}.$$

Proof. The proof combines the classical Sobolev Inequality in the fractional setting, an extension method and a covering argument. The details go as follows. We fix $\rho_0 > 0$ such that the diameter of the N-dimensional cube of side $2\rho_0$ is less than or equal to r_0. Then, we cover $\mathbb{R}^N$ with a grid of adjacent cubes $\mathcal{Q}_k$ of side $2\rho_0$, $k \in \mathbb{N}$. Notice that, by construction,

$$\text{if } x,\, y \in \mathcal{Q}_k, \text{ then } |x-y| \leqslant r_0. \tag{A.1}$$

Also, each $\mathcal{Q}_k$ is a Lipschitz domain and so it is an extension domain for the fractional Sobolev norm: namely (see e.g. Theorem 5.4 in [DNPV12]) there exists an extension function $\tilde{f}_k$ such that $\tilde{f}_k = f$ in $\mathcal{Q}_k$ and

$$\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{f}_k(x)-\tilde{f}_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{1}{p}} \leqslant C\left(\iint_{\mathcal{Q}_k\times\mathcal{Q}_k} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{1}{p}}. \tag{A.2}$$

Here and below, $C > 0$ may vary from line to line and depends only on N, p, s and r_0.

Moreover, the classical Sobolev Inequality in fractional Sobolev spaces (see e.g. Theorem 6.5 in [DNPV12]) gives that

$$\|f\|_{L^{p_s^*}(\mathcal{Q}_k)} = \|\tilde{f}_k\|_{L^{p_s^*}(\mathcal{Q}_k)} \leqslant \|\tilde{f}_k\|_{L^{p_s^*}(\mathbb{R}^N)} \leqslant C\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\tilde{f}_k(x)-\tilde{f}_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{1}{p}}.$$

From this and (A.2), we find that

$$\int_{\mathcal{Q}_k} |f(x)|^{p_s^*}\,dx = \|f_k\|^{p_s^*}_{L^{p_s^*}(\mathcal{Q}_k)} \leqslant C\left(\iint_{\mathcal{Q}_k\times\mathcal{Q}_k} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{p_s^*}{p}}. \tag{A.3}$$

Now we observe that, for any a, $b \geqslant 0$, and any $m \in [1,+\infty)$, it holds that

$$a^m + b^m \leqslant (a+b)^m. \tag{A.4}$$

To check this, we consider the function

$$[0,+\infty) \ni t \mapsto g(t) := \frac{t^m+1}{(t+1)^m}.$$

We have that

$$g(0) = 1 = \lim_{t\to+\infty} g(t),$$

hence there exists a maximum point $t_\star \in [0,+\infty)$ for g. We show that $t_\star = 0$. Indeed, if not, it would be an interior critical point, and so $g'(t_\star) = 0$. This identity would give that

$$m t_\star^{m-1}(t_\star+1)^m = m(t_\star^m+1)(t_\star+1)^{m-1},$$

and so $t_\star^{m-1}(t_\star+1) = t_\star^m + 1$, which implies $t_\star^m + t_\star^{m-1} = t_\star^m + 1$ and thus $t_\star = 1$. Since $g(1) = \frac{2}{2^m} < 1 = g(0)$, we reach a contradiction with the maximality of $t_\star$.

Having shown that the maximum point for g is reached at $t_\star = 0$, we have that $g(t) \leqslant 1$ for all $t \geqslant 0$ and therefore, for any a, $b \geqslant 0$ (with, say $b \neq 0$) we see that

$$\frac{a^m+b^m}{(a+b)^m} = \frac{(a/b)^m+1}{\big((a/b)+1\big)^m} = g(a/b) \leqslant 1,$$

which establishes (A.4).

Now, if $\beta_k \geqslant 0$, with $k \in \mathbb{N}$, fixed $k_0 \in \mathbb{N}$, using (A.4) we find that

$$\begin{aligned}\sum_{k=0}^{k_0} \beta_k^m &= \beta_0^m + \beta_1^m + \sum_{k=2}^{k_0} \beta_k^m \leqslant= (\beta_0+\beta_1)^m + \sum_{k=2}^{k_0} \beta_k^m \leqslant= (\beta_0+\beta_1)^m + \beta_2^m + \sum_{k=3}^{k_0} \beta_k^m \\ &\leqslant (\beta_0+\beta_1+\beta_2)^m + \sum_{k=3}^{k_0} \beta_k^m \leqslant \ldots \leqslant \left(\sum_{k=0}^{k_0} \beta_k\right)^m \leqslant \left(\sum_{k\in\mathbb{N}} \beta_k\right)^m .\end{aligned}$$

Thus, sending $k_0 \to +\infty$,

$$\sum_{k\in\mathbb{N}} \beta_k^m \leqslant \left(\sum_{k\in\mathbb{N}} \beta_k\right)^m .$$

Hence, we use this inequality with $\beta_k := \iint_{\mathbb{Q}_k\times\mathbb{Q}_k} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy$ and $m := \frac{p_s^*}{p} = \frac{N}{N-sp} > 1$. In this way, recalling (A.1), we obtain that

$$\begin{aligned}\sum_{k\in\mathbb{N}} \left(\iint_{\mathbb{Q}_k\times\mathbb{Q}_k} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right)^{\frac{p_s^*}{p}} &\leqslant \left(\sum_{k\in\mathbb{N}} \iint_{\mathbb{Q}_k\times\mathbb{Q}_k} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right)^{\frac{p_s^*}{p}} \\ &= \left(\sum_{k\in\mathbb{N}} \iint_{\mathbb{Q}_k\times\mathbb{Q}_k} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,\chi_{[0,r_0]}(|x-y|)\,dx\,dy \right)^{\frac{p_s^*}{p}} \\ &\leqslant \left(\sum_{k\in\mathbb{N}} \iint_{\mathbb{Q}_k\times\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,\chi_{[0,r_0]}(|x-y|)\,dx\,dy \right)^{\frac{p_s^*}{p}} \\ &= \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,\chi_{[0,r_0]}(|x-y|)\,dx\,dy \right)^{\frac{p_s^*}{p}} .\end{aligned}$$

Exploiting this inequality and (A.3), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |f(x)|^{p_s^*}\,dx &= \sum_{k\in\mathbb{N}} \int_{\mathbb{Q}_k} |f(x)|^{p_s^*}\,dx \leqslant C \sum_{k\in\mathbb{N}} \left(\iint_{\mathbb{Q}_k\times\mathbb{Q}_k} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right)^{\frac{p_s^*}{p}} \\ &\leqslant C \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\,\chi_{[0,r_0]}(|x-y|)\,dx\,dy \right)^{\frac{p_s^*}{p}} ,\end{aligned}$$

as desired. □

Appendix B. Discontinuity and oscillatory behavior at infinity for functions in Sobolev spaces with low fractional exponents

We recall here that functions belonging to the fractional Sobolev space $H^s(\mathbb{R})$ with $s \in \left(0, \frac{1}{2}\right)$ are not necessarily continuous, and they do not need to converge to zero at infinity.

To construct a simple example, let $\varphi \in C_0^\infty\big(\mathbb{R}, [0,1]\big)$ with $\varphi(0) = 1$. Given a sequence b_k, let

$$\varphi_{b_k}(x) := \varphi\left(e^k(x-b_k)\right). \tag{B.1}$$

Then

$$\|\varphi_{b_k}\|_{L^2(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} \left|\varphi\left(e^k(x-b_k)\right)\right|^2 \, dx} = e^{-\frac{k}{2}} \sqrt{\int_{\mathbb{R}} \left|\varphi\left(X\right)\right|^2 \, dX} = \text{const}\, e^{-\frac{k}{2}}$$

and

$$[\varphi_{b_k}]_{H^s(\mathbb{R})} = \sqrt{\iint_{\mathbb{R}\times\mathbb{R}} \frac{\left|\varphi\left(e^k(x-b_k)\right)-\varphi\left(e^k(y-b_k)\right)\right|^2}{|x-y|^{1+2s}}\, dx\, dy}$$
$$= e^{-\frac{(1-2s)k}{2}} \sqrt{\iint_{\mathbb{R}\times\mathbb{R}} \frac{\left|\varphi\left(X\right)-\varphi\left(Y\right)\right|^2}{|X-Y|^{1+2s}}\, dX\, dY} = \text{const}\, e^{-\frac{(1-2s)k}{2}}.$$

We now consider the superposition of the functions φ_{b_k} with the choices $b_k := k$ and $b_k := 1/k$. Namely, if we set

$$\Phi(x) := \sum_{k=1}^{+\infty} \varphi_{1/k}(x) + \sum_{k=1}^{+\infty} \varphi_k(x),$$

when $s \in \left(0, \frac{1}{2}\right)$ we have that

$$\|\Phi\|_{H^s(\mathbb{R})} \leqslant \sum_{k=1}^{+\infty} \|\varphi_{1/k}\|_{H^s(\mathbb{R})} + \sum_{k=1}^{+\infty} \|\varphi_k\|_{H^s(\mathbb{R})} \leqslant \text{const} \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{2}} + e^{-\frac{(1-2s)k}{2}}\right) \leqslant \text{const}\,.$$

Nevertheless Φ is not continuous at the origin, and

$$\limsup_{x\to+\infty} \Phi(x) > 0 = \liminf_{x\to+\infty} \Phi(x).$$

The case of $H^{1/2}(\mathbb{R})$ is slightly more delicate, since simple examples based on scaling, such as the one provided in (B.1), do not work in this case (and, in fact, functions in $H^{1/2}(\mathbb{R})$ have nicer properties in terms of topology than those in $H^s(\mathbb{R})$ with $s \in \left(0, \frac{1}{2}\right)$, see e.g. [BN95]). Nevertheless, also functions in $H^{1/2}(\mathbb{R})$ are not necessarily continuous and they do not necessarily converge to zero at infinity. To construct an example of these behaviors, as depicted in Figure 1, we consider the function

$$\mathbb{R}^2 \ni X \mapsto \psi(X) := \begin{cases} \log(1-\log|X|) & \text{if } X \in B_1 \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that

$$\psi \in H^1(\mathbb{R}^2). \tag{B.2}$$

To check this, we notice that

$$\psi \text{ is supported in } B_1, \tag{B.3}$$

where it holds that

$$|\nabla\psi(X)| = \frac{1}{|X|\left(1-\log|X|\right)}.$$

Therefore, using polar coordinates and the change of variable $t := -\log\rho$, we find that

$$[\psi]^2_{H^1(\mathbb{R})} = \int_{B_1} \frac{1}{|X|^2\left(1-\log|X|\right)^2}\, dX = 2\pi \int_0^1 \frac{1}{\rho\left(1-\log\rho\right)^2}\, d\rho = 2\pi \int_0^{+\infty} \frac{1}{(1+t)^2}\, dt < +\infty.$$

This, together with (B.3) and the Poincaré Inequality, proves (B.2).

Then, from (B.2) and the Trace Theorem (see e.g. formula (3.19) in [DNPV12]), we obtain that

$$\text{the function } \mathbb{R} \ni x \mapsto \bar\psi(x) := \psi(x,0) \text{ belongs to } H^{1/2}(\mathbb{R}). \tag{B.4}$$

Now we define the sequence of functions, for $k \in \mathbb{Z}$ and $X = (x,y) \in \mathbb{R}\times\mathbb{R}$,

$$\psi_k(X) = \psi_k(x,y) := e^{-|k|}\bar\psi\big(e^{|k|}(x-e^k)\big).$$

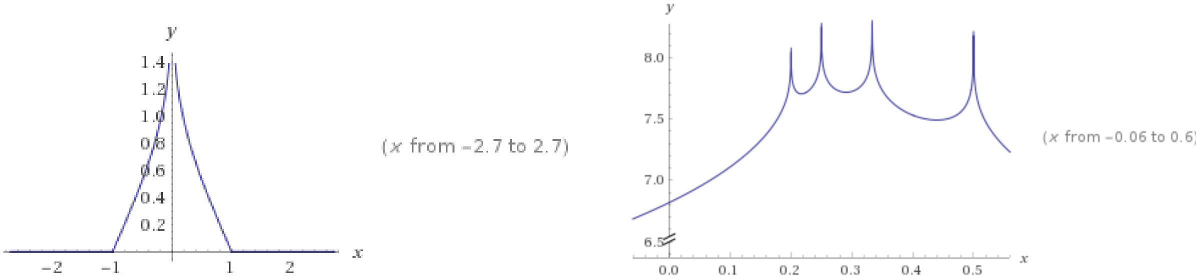

FIGURE 1. *The function $\bar\psi$ and sketch of the construction of the function Ψ.*

Then, in view of (B.4) we have that

$$\begin{aligned}\|\psi_k\|_{L^2(\mathbb{R})} &= e^{-|k|}\sqrt{\int_{\mathbb{R}} \left|\bar\psi\big(e^{|k|}(x-e^k)\big)\right|^2\,dx} = e^{-\frac{3|k|}{2}}\sqrt{\int_{\mathbb{R}}\left|\bar\psi(\eta)\right|^2\,d\eta}\\ &= e^{-\frac{3|k|}{2}}\,\|\bar\psi\|_{L^2(\mathbb{R})} = \text{const}\,e^{-\frac{3|k|}{2}}\end{aligned}$$

and
$$\begin{aligned}[\psi_k]_{H^{1/2}(\mathbb{R})} &= e^{-|k|}\sqrt{\iint_{\mathbb{R}\times\mathbb{R}} \frac{\left|\bar\psi\big(e^{|k|}(x-e^k)\big)-\bar\psi\big(e^{|k|}(y-e^k)\big)\right|^2}{|\bar x-\bar y|^2}\,dx\,dy}\\ &= e^{-|k|}\sqrt{\iint_{\mathbb{R}\times\mathbb{R}}\frac{|\bar\psi(\eta)-\bar\psi(\xi)|^2}{|\eta-\xi|^2}\,d\eta\,d\xi} = e^{-|k|}\,[\bar\psi]_{H^{1/2}(\mathbb{R})} = \text{const}\,e^{-|k|}.\end{aligned}$$

Consequently, if we set

$$\mathbb{R}\mapsto \Psi(x) := \sum_{k\in\mathbb{Z}} \psi_k(x),$$

it holds that Ψ is not continuous (and not even locally bounded) and it does not go to zero at infinity, but it belongs to $H^{1/2}(\mathbb{R})$ since

$$\|\Psi\|_{H^{1/2}(\mathbb{R})} \leqslant \sum_{k\in\mathbb{Z}} \|\psi_k\|_{H^{1/2}(\mathbb{R})} \leqslant \text{const}\sum_{k\in\mathbb{Z}}\left(\text{const}\,e^{-\frac{3|k|}{2}} + \text{const}\,e^{-|k|}\right) \leqslant \text{const}\,.$$

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