THE NEUMANN PROBLEM FOR SINGULAR FULLY NONLINEAR OPERATORS

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Abstract. We consider the Neumann problem in C^2 bounded domains for fully nonlinear second order operators which are elliptic, homogenous and perhaps singular or degenerate. Inspired by [9], we define the concept of principal eigenvalue and we characterize it through the maximum principle. Moreover, Lipschitz regularity, uniqueness and existence results for solutions of the Neumann problem are given.

Résumé. On considère le problème de Neumann dans un ouvert borné régulier pour un opérateur elliptique complètement nonlinéaire homogène, qui peut être singulier où dégénéré. Suivant [9], on introduit la notion de valeur propre principal à travers une caractérisation en termes du Principe de Maximum. On donne aussi quelques résultats d'existence, unicité et de régularité Lipschitzienne pour les solutions.

1. Introduction

In this paper we study the maximum principle, principal eigenvalues, regularity and existence for viscosity solutions of the Neumann boundary value problem

$$\begin{cases} F(x, Du, D^2u) + b(x) \cdot Du|Du|^\alpha + (c(x) + \lambda)|u|^\alpha u = g(x) & \text{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of class C^2, $\overrightarrow{n}(x)$ is the exterior normal to the domain Ω at x, $\alpha > -1$, $\lambda \in \mathbb{R}$ and b, c, g are continuous functions on $\overline{\Omega}$. F is a fully nonlinear operator that may be singular at the points where the gradient vanishes. It is defined on $\overline{\Omega} \times \mathbb{R}^N \setminus \{0\} \times S(N)$, where $S(N)$ denotes the space of symmetric matrices on $\mathbb{R}^N$ equipped with the usual ordering, and satisfies the following homogeneity and ellipticity conditions

(F1) For all $t \in \mathbb{R}^*, \mu \geq 0$, $(x, p, X) \in \overline{\Omega} \times \mathbb{R}^N \setminus \{0\} \times S(N)$

$$F(x, tp, \mu X) = |t|^\alpha \mu F(x, p, X).$$

(F2) There exist $a, A > 0$ such that for $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}, M, N \in S(N)$, $N \geq 0$

$$a|p|^\alpha \mathrm{tr}N \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha \mathrm{tr}N.$$

In addition, we will assume on F some Hölder's continuity hypothesis that will be made precise in the next section.

In this class of operators one can consider for example

$$F(Du, D^2u) = |Du|^\alpha \mathcal{M}^{\pm}_{a,A} D^2u,$$

$\alpha > -1$, where $\mathcal{M}^{\pm}_{a,A} D^2u$ are the Pucci's operators (see e.g. [11]), the p-Laplacian

$$\Delta_p u = \mathrm{div}(|Du|^{p-2}Du),$$

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with $\alpha = p-2$, and non-variational generalizations of the p-Laplacian, depending explicitly on x, like the operator

$$F(x, Du, D^2u) = |Du|^{q-2}\text{tr}(B_1(x)D^2u) + c_0|Du|^{q-4}\langle D^2uB_2(x)Du, B_2(x)Du\rangle,$$

with $\alpha = q-2$, where $q > 1$, B_1 and B_2 are θ-Hölderian functions with $\theta > \frac{1}{2}$, which send $\overline{\Omega}$ into *S(N)*, $aI \le B_1 \le AI$, $-\sqrt{a}I \le B_2 \le \sqrt{a}I$ and $c_0 > -1$.

The concept of first eigenvalue has been extended to nonlinear operators which are variational, such as the p-Laplacian with Dirichlet or Neumann boundary conditions, through the method of minimization of the Rayleigh quotient, see e.g. [2] and [20]. That method uses heavily the variational structure and cannot be applied to operators which have not this property. An important step in the study of the eigenvalue problem for nonlinear operators in non-divergence form was made by Lions in [18]. In that paper, using probabilistic and analytic methods, he showed the existence of principal eigenvalues for the uniformly elliptic Bellman operator and obtained results for the related Dirichlet problems. Very recently, many authors, inspired by the famous work of Berestycki, Nirenberg and Varadhan [9], have developed an eigenvalue theory for fully nonlinear operators which are non-variational. Issues similar to those of this paper have been studied for the Dirichlet problem by Birindelli and Demengel in [8]. They assume slightly less general structure conditions on F, but on the other hand, some of their results can be applied to degenerate elliptic equations. The case $\alpha = 0$ has been treated by Quaas [23] and Busca, Esteban and Quaas [10] for the Pucci's operators. Their results have been extended to more general fully nonlinear convex uniformly elliptic operators in [24] by Quaas and Sirakov. See also the work of Ishii and Yoshimura [17] for non-convex operators. All these articles treat Dirichlet boundary conditions.

The techniques of this paper, although partly taken by the previous mentioned articles, use ad hoc test functions depending on the distance function from the boundary of the domain which are suitable for the Neumann boundary conditions.

Comparison principles and the existence results for the Neumann problem have been investigated by Ishii in [14] and Barles in [3] and [4] for degenerate elliptic operators $\mathcal{G}(x, u, Du, D^2u)$ modeled on the Isaacs ones or on the stationary operator associated to the Mean Curvature Equation. In all these papers a fundamental assumption is the monotonicity of $\mathcal{G}(x, r, p, X)$ with respect to r. For the p-Laplace with the zero order term $c(x)|u|^{p-2}u$, $c \le 0$ and $c \not\equiv 0$ and the pure Neumann boundary condition, the comparison principle can be showed through variational techniques, like in the Dirichlet case, see e.g. [19].

We denote

$$G(x, u, Du, D^2u) := F(x, Du, D^2u) + b(x) \cdot Du|Du|^{\alpha} + c(x)|u|^{\alpha}u.$$

It is important to remark that G is homogenous and non-variational. Following the ideas of [9], we define the principal eigenvalue as

$$\begin{aligned}\overline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, v > 0 \text{ bounded viscosity supersolution of}\\ G(x, v, Dv, D^2v) + \lambda v^{\alpha+1} = 0 \text{ in } \Omega, \langle Dv, \overrightarrow{n}\rangle = 0 \text{ on } \partial\Omega\}.\end{aligned} \tag{1.2}$$

$\overline{\lambda}$ is well defined since the above set is not empty; indeed, $-|c|_\infty$ belongs to it, being $v(x) \equiv 1$ a corresponding supersolution. Furthermore it is an interval because if λ belongs to it then so does any $\lambda' < \lambda$.

One of the scope of this work is to prove that $\overline{\lambda}$ is an "eigenvalue" for $-G$ which admits a positive "eigenfunction", in the sense that there exists $\phi > 0$ solution of

$$\begin{cases} G(x, \phi, D\phi, D^2\phi) + \overline{\lambda}\phi^{\alpha+1} = 0 & \text{in} \quad \Omega \\ \langle D\phi, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover, $\overline{\lambda}$ can be characterized as the supremum of those λ for which the operator $G(x, u, Du, D^2u) + \lambda|u|^\alpha u$ with the Neumann boundary condition satisfies the maximum principle. As a consequence $\overline{\lambda}$ is the least "eigenvalue" to which there correspond "eigenfunctions" positive somewhere. These results are applied to obtain existence and uniqueness for the boundary value problem (1.1).

The paper is organized as follows. In the next section we give assumptions and define the concept of solution. In Section 3 we establish a Lipschitz regularity result for viscosity solutions of (1.1). The Section 4 is devoted to the study of the maximum principle for subsolutions of (1.1). In Section 4.1 we show that it holds (even for more general boundary conditions) for $G(x, u, Du, D^2u)$ if $c(x) \leq 0$ and $c \not\equiv 0$, see Theorem 4.5. One of the main result of the paper is that the maximum principle holds for $G(x, u, Du, D^2u) + \lambda|u|^\alpha u$ for any $\lambda < \overline{\lambda}$, as we show in Theorem 4.9 of Section 4.2. In particular it holds for $G(x, u, Du, D^2u)$ if $\overline{\lambda} > 0$. It is natural to wonder if the result of Theorem 4.9 is stronger than that of Theorem 4.5; indeed if $c \equiv 0$, one has $\overline{\lambda} = 0$. A positive answer is given in Section 4.3, where we construct an explicit example of a bounded positive viscosity supersolution of $G(x, v, Dv, D^2v) + \lambda v^{\alpha+1} = 0$ in Ω, $\langle Dv, \overrightarrow{n}\rangle = 0$ on $\partial\Omega$, $\lambda > 0$, with $c(x)$ changing sign. The existence of such v implies, by definition, $\overline{\lambda} > 0$. Finally, in Section 5 we show some existence and comparison theorems.

For fully nonlinear operators it is possible to define another principal eigenvalue

$$\underline{\lambda} := \sup\{\lambda \in \mathbb{R} \mid \exists\, u < 0 \text{ bounded viscosity subsolution of } G(x, u, Du, D^2u) + \lambda|u|^\alpha u = 0 \text{ in } \Omega, \langle Du, \overrightarrow{n}\rangle = 0 \text{ on } \partial\Omega\}.$$

If $F(x, p, X) = -F(x, p, -X)$ then $\overline{\lambda} = \underline{\lambda}$, otherwise $\overline{\lambda}$ may be different from $\underline{\lambda}$.

The classical assumption which guarantees the solvability of the Neumann problem (1.1) with $\lambda = 0$ is $c < 0$ in $\overline{\Omega}$. We show that the right hypothesis for any right-hand side is the positivity of the two principal eigenvalues.

2. Assumptions and definitions

We assume that the operator $F : \overline{\Omega} \times \mathbb{R}^N \setminus \{0\} \times S(N) \to \mathbb{R}$ satisfies the hypothesis (F1) and (F2) given in the introduction and the following Hölder's continuity conditions

(F3) There exist $C_1 > 0$ and $\theta \in (\frac{1}{2}, 1]$ such that for all $x, y \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}$, $X \in S(N)$

$$|F(x, p, X) - F(y, p, X)| \leq C_1|x - y|^\theta |p|^\alpha \|X\|.$$

(F4) There exist $C_2 > 0$ and $\nu \in (\frac{1}{2}, 1]$ such that for all $x \in \overline{\Omega}$, $p \in \mathbb{R}^N \setminus \{0\}$, $p_0 \in \mathbb{R}^N$, $|p_0| \leq \frac{|p|}{2}$, $X \in S(N)$

$$|F(x, p + p_0, X) - F(x, p, X)| \leq C_2|p|^{\alpha-\nu}|p_0|^\nu \|X\|.$$

Here and in what follows we fix the norm $\|X\|$ in $S(N)$ by setting $\|X\| = \sup\{|X\xi| \mid \xi \in \mathbb{R}^N, |\xi| \leq 1\} = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } X\}$.

The domain Ω is supposed to be bounded and of class C^2. In particular, it satisfies the interior sphere condition and the uniform exterior sphere condition, i.e.,

(Ω1) For each $x \in \partial\Omega$ there exist $R > 0$ and $y \in \Omega$ for which $|x - y| = R$ and $B(y, R) \subset \Omega$.

(Ω2) There exists $r > 0$ such that $B(x + r\overrightarrow{n}(x), r) \cap \Omega = \emptyset$ for any $x \in \partial\Omega$.

From the property (Ω2) it follows that

$$\langle y - x, \overrightarrow{n}(x)\rangle \leq \frac{1}{2r}|y - x|^2 \quad \text{for } x \in \partial\Omega \text{ and } y \in \overline{\Omega}. \tag{2.1}$$

Moreover, the C^2-regularity of Ω implies the existence of a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ on which the distance from the boundary

$$d(x) := \inf\{|x-y|, y \in \partial\Omega\}, \quad x \in \overline{\Omega}$$

is of class C^2. We still denote by d a C^2 extension of the distance function to the whole $\overline{\Omega}$. Without loss of generality we can assume that $|Dd(x)| \leq 1$ in $\overline{\Omega}$.

As in [8], here we adopt the notion of viscosity solution for (1.1) adapted to our context. We denote by $USC(\overline{\Omega})$ the set of upper semicontinuous functions on $\overline{\Omega}$ and by $LSC(\overline{\Omega})$ the set of lower semicontinuous functions on $\overline{\Omega}$. Let $g : \overline{\Omega} \to \mathbb{R}$ and $B : \partial\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$.

Definition 2.1. *Any function $u \in USC(\overline{\Omega})$ (resp., $u \in LSC(\overline{\Omega})$) is called* viscosity subsolution *(resp.,* supersolution*) of*

$$\begin{cases} G(x,u,Du,D^2u) = g(x) & in \quad \Omega \\ B(x,u,Du) = 0 & on \quad \partial\Omega, \end{cases}$$

if the following conditions hold

(i) *For every $x_0 \in \Omega$, for all $\varphi \in C^2(\overline{\Omega})$, such that $u - \varphi$ has a local maximum (resp., minimum) on x_0 and $D\varphi(x_0) \neq 0$, one has*

$$G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq \ (resp., \ \leq\ ) \ g(x_0).$$

If $u \equiv k =$const. in a neighborhood of x_0, then

$$c(x_0)|k|^\alpha k \geq \ (resp., \ \leq\ ) \ g(x_0).$$

(ii) *For every $x_0 \in \partial\Omega$, for all $\varphi \in C^2(\overline{\Omega})$, such that $u-\varphi$ has a local maximum (resp., minimum) on x_0 and $D\varphi(x_0) \neq 0$, one has*

$$(-G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) + g(x_0)) \wedge B(x_0, u(x_0), D\varphi(x_0)) \leq 0$$

(resp.,

$$(-G(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) + g(x_0)) \vee B(x_0, u(x_0), D\varphi(x_0)) \geq 0).$$

If $u \equiv k =$const. in a neighborhood of x_0 in $\overline{\Omega}$, then

$$(-c(x_0)|k|^\alpha k + g(x_0)) \wedge B(x_0, k, 0) \leq 0$$

(resp.,

$$(-c(x_0)|k|^\alpha k + g(x_0)) \vee B(x_0, k, 0) \geq 0).$$

A viscosity solution *is a continuous function which is both a subsolution and a supersolution.*

For a detailed presentation of the theory of viscosity solutions and of the boundary conditions in the viscosity sense, we refer the reader to e.g. [12].

We call strong viscosity subsolutions (resp., supersolutions) the viscosity subsolutions (resp., supersolutions) that satisfy $B(x,u,Du) \leq$ (resp., $\geq$) 0 in the viscosity sense for all $x \in \partial\Omega$. If $\lambda \to B(x,r,p-\lambda\overrightarrow{n})$ is non-increasing in $\lambda \geq 0$, then classical subsolutions (resp., supersolutions) are strong viscosity subsolutions (resp., supersolutions), see [12] Proposition 7.2.

In the above definition the test functions can be substituted by the elements of the semijets $\overline{J}^{2,+}u(x_0)$ when u is a subsolution and $\overline{J}^{2,-}u(x_0)$ when u is a supersolution. For non-singular operators the definitions reduce to the standard ones, see [12].

3. Lipschitz continuity of viscosity solutions

Theorem 3.1. *Let Ω be a bounded domain of class C^2. Suppose that F satisfies (F2)-(F4) and that b, c, g are bounded in Ω. If $u \in C(\overline{\Omega})$ is a viscosity solution of*

$$\begin{cases} F(x, Du, D^2u) + b(x)\cdot Du|Du|^\alpha + c(x)|u|^\alpha u = g(x) & in \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & on \quad \partial\Omega, \end{cases}$$

then

$$|u(x) - u(y)| \le C_0|x - y| \quad \forall x, y \in \overline{\Omega},$$

where C_0 depends on Ω, N, α, a, A, θ, ν, C_1, C_2, $|b|_\infty$, $|c|_\infty$, $|g|_\infty$, and $|u|_\infty$.

The Theorem is an immediate consequence of the next lemma. To prove the lemma we adopt the technique used in Proposition III.1 of [16] for Dirichlet problems, that we modify taking test functions which depend on $d(x)$.

The lemma plays a key role also in the proof of Theorem 4.9 in the next section.

Lemma 3.2. *Assume the hypothesis of Theorem 3.1 and suppose that g and h are bounded functions. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} F(x, Du, D^2u) + b(x)\cdot Du|Du|^\alpha + c(x)|u|^\alpha u = g(x) & in \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & on \quad \partial\Omega, \end{cases}$$

and $v \in LSC(\overline{\Omega})$ a viscosity supersolution of

$$\begin{cases} F(x, Dv, D^2v) + b(x)\cdot Dv|Dv|^\alpha + c(x)|v|^\alpha v = h(x) & in \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & on \quad \partial\Omega, \end{cases}$$

with u and v bounded, or $v \ge 0$ and bounded. If $m = \max_{\overline{\Omega}}(u - v) \ge 0$, then there exists $C_0 > 0$ such that

$$u(x) - v(y) \le m + C_0|x - y| \quad \forall x, y \in \overline{\Omega}, \tag{3.1}$$

where C_0 depends on Ω, N, α, a, A, θ, ν, C_1, C_2, $|b|_\infty$, $|c|_\infty$, $|g|_\infty$, $|h|_\infty$, $|v|_\infty$, m and $|u|_\infty$ or $\sup_{\overline{\Omega}} u$.

Proof. We set

$$\Phi(x) = MK|x| - M(K|x|)^2,$$

and

$$\varphi(x, y) = m + e^{-L(d(x)+d(y))}\Phi(x - y),$$

where L is a fixed number greater than $2/(3r)$ with r the radius in the condition (Ω2) and K and M are two positive constants to be chosen later. If $K|x| \le \frac{1}{4}$, then

$$\Phi(x) \ge \frac{3}{4}MK|x|. \tag{3.2}$$

We define

$$\Delta_K := \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - y| \le \frac{1}{4K}\right\}.$$

We fix M such that

$$\max_{\overline{\Omega}^2}(u(x) - v(y)) \le m + e^{-2Ld_0}\frac{M}{8}, \tag{3.3}$$

where $d_0 = \max_{x\in\overline{\Omega}} d(x)$, and we claim that taking K large enough, one has

$$u(x) - v(y) - \varphi(x, y) \le 0 \quad \text{for } (x, y) \in \Delta_K \cap \overline{\Omega}^2.$$

In this case (3.1) is proven. To show the last inequality we suppose by contradiction that for some $(\overline{x}, \overline{y}) \in \Delta_K \cap \overline{\Omega}^2$

$$u(\overline{x}) - v(\overline{y}) - \varphi(\overline{x}, \overline{y}) = \max_{\Delta_K\cap\overline{\Omega}^2}(u(x) - v(y) - \varphi(x, y)) > 0.$$

Here we have dropped the dependence of $\overline{x}$, $\overline{y}$ on K for simplicity of notations.

Observe that if $v \geq 0$, since from (3.2) $\Phi(x-y)$ is non-negative in Δ_K and $m \geq 0$, one has $u(\overline{x}) > 0$.

Clearly $\overline{x} \neq \overline{y}$. Moreover the point $(\overline{x},\overline{y})$ belongs to $\mathrm{int}(\Delta_K) \cap \overline{\Omega}^2$. Indeed, if $|x-y| = \frac{1}{4K}$, by (3.3) and (3.2) we have

$$u(x) - v(y) \leq m + e^{-2Ld_0}\frac{M}{8} \leq m + e^{-L(d(x)+d(y))}\frac{1}{2}MK|x-y| \leq \varphi(x,y).$$

Since $\overline{x} \neq \overline{y}$ we can compute the derivatives of φ in $(\overline{x},\overline{y})$ obtaining

$$\begin{aligned} D_x\varphi(\overline{x},\overline{y}) &= e^{-L(d(\overline{x})+d(\overline{y}))}MK\Big\{ -L|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)Dd(\overline{x}) \\ &\quad + (1-2K|\overline{x}-\overline{y}|)\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}\Big\}, \\ D_y\varphi(\overline{x},\overline{y}) &= e^{-L(d(\overline{x})+d(\overline{y}))}MK\Big\{ -L|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|)Dd(\overline{y}) \\ &\quad - (1-2K|\overline{x}-\overline{y}|)\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}\Big\}. \end{aligned}$$

Observe that for large K

(3.4)
$$\begin{aligned} e^{-2Ld_0}\frac{MK}{4} &\leq e^{-L(d(\overline{x})+d(\overline{y}))}MK\left(\frac{1}{2} - L|\overline{x}-\overline{y}|\right) \leq |D_x\varphi(\overline{x},\overline{y})|, |D_y\varphi(\overline{x},\overline{y})| \\ &\leq 2MK. \end{aligned}$$

Using (2.1), if $\overline{x} \in \partial\Omega$ we have

$$\begin{aligned} &\langle D_x\varphi(\overline{x},\overline{y}), \overrightarrow{n}(\overline{x})\rangle \\ &= e^{-Ld(\overline{y})}MK\Big\{L|\overline{x}-\overline{y}|(1-K|\overline{x}-\overline{y}|) + (1-2K|\overline{x}-\overline{y}|)\langle\frac{(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|}, \overrightarrow{n}(\overline{x})\rangle\Big\} \\ &\geq e^{-Ld(\overline{y})}MK\Big\{\frac{3}{4}L|\overline{x}-\overline{y}| - (1-2K|\overline{x}-\overline{y}|)\frac{|\overline{x}-\overline{y}|}{2r}\Big\} \\ &\geq \frac{1}{2}e^{-Ld(\overline{y})}MK|\overline{x}-\overline{y}|\left(\frac{3}{2}L - \frac{1}{r}\right) > 0, \end{aligned}$$

since $\overline{x} \neq \overline{y}$ and $L > 2/(3r)$. Similarly, if $\overline{y} \in \partial\Omega$

$$\langle -D_y\varphi(\overline{x},\overline{y}), \overrightarrow{n}(\overline{y})\rangle \leq \frac{1}{2}e^{-Ld(\overline{x})}MK|\overline{x}-\overline{y}|\left(-\frac{3}{2}L + \frac{1}{r}\right) < 0.$$

In view of definition of sub and supersolution, we conclude that

$$\begin{aligned} G(\overline{x}, u(\overline{x}), D_x\varphi(\overline{x},\overline{y}), X) \geq g(\overline{x}) &\quad \text{if } (D_x\varphi(\overline{x},\overline{y}), X) \in \overline{J}^{2,+}u(\overline{x}), \\ G(\overline{y}, v(\overline{y}), -D_y\varphi(\overline{x},\overline{y}), Y) \leq h(\overline{y}) &\quad \text{if } (-D_y\varphi(\overline{x},\overline{y}), Y) \in \overline{J}^{2,-}v(\overline{y}). \end{aligned}$$

Since $(\overline{x},\overline{y}) \in \mathrm{int}\Delta_K \cap \overline{\Omega}^2$, it is a local maximum point of $u(x) - v(y) - \varphi(x,y)$ in $\overline{\Omega}^2$. Then applying Theorem 3.2 in [12], for every $\epsilon > 0$ there exist $X, Y \in S(N)$ such that $(D_x\varphi(\overline{x},\overline{y}), X) \in \overline{J}^{\,2,+}u(\overline{x})$, $(-D_y\varphi(\overline{x},\overline{y}), Y) \in \overline{J}^{\,2,-}v(\overline{y})$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2(\varphi(\overline{x},\overline{y})) + \epsilon(D^2(\varphi(\overline{x},\overline{y})))^2. \tag{3.5}$$

Now we want to estimate the matrix on the right-hand side of the last inequality.

$$\begin{aligned} D^2\varphi(\overline{x},\overline{y}) &= \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}) + D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) \\ &\quad + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}) + e^{-L(d(\overline{x})+d(\overline{y}))}D^2(\Phi(\overline{x}-\overline{y})). \end{aligned}$$

We set

$$A_1 := \Phi(\overline{x}-\overline{y})D^2(e^{-L(d(\overline{x})+d(\overline{y}))}),$$

$$A_2 := D(e^{-L(d(\overline{x})+d(\overline{y}))}) \otimes D(\Phi(\overline{x}-\overline{y})) + D(\Phi(\overline{x}-\overline{y})) \otimes D(e^{-L(d(\overline{x})+d(\overline{y}))}),$$
$$A_3 := e^{-L(d(\overline{x})+d(\overline{y}))} D^2(\Phi(\overline{x}-\overline{y})).$$

Observe that

$$A_1 \le CK|\overline{x}-\overline{y}| \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.6}$$

Here and henceforth C denotes various positive constants independent of K.

For A_2 we have the following estimate

$$A_2 \le CK \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + CK \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.7}$$

Indeed for ξ, $\eta \in \mathbb{R}^N$ we compute

$$\begin{aligned} \langle A_2(\xi,\eta),(\xi,\eta)\rangle &= 2Le^{-L(d(\overline{x})+d(\overline{y}))}\{\langle Dd(\overline{x}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\xi\rangle \\ &+ \langle Dd(\overline{y}) \otimes D\Phi(\overline{x}-\overline{y})(\eta-\xi),\eta\rangle\} \le CK(|\xi|+|\eta|)|\eta-\xi| \\ &\le CK(|\xi|^2+|\eta|^2) + CK|\eta-\xi|^2. \end{aligned}$$

Now we consider A_3. The matrix $D^2(\Phi(\overline{x}-\overline{y}))$ has the form

$$D^2(\Phi(\overline{x}-\overline{y})) = \begin{pmatrix} D^2\Phi(\overline{x}-\overline{y}) & -D^2\Phi(\overline{x}-\overline{y}) \\ -D^2\Phi(\overline{x}-\overline{y}) & D^2\Phi(\overline{x}-\overline{y}) \end{pmatrix},$$

and the Hessian matrix of $\Phi(x)$ is

$$D^2\Phi(x) = \frac{MK}{|x|}\left(I - \frac{x\otimes x}{|x|^2}\right) - 2MK^2 I. \tag{3.8}$$

If we choose

$$\epsilon = \frac{|\overline{x}-\overline{y}|}{2MKe^{-L(d(\overline{x})+d(\overline{y}))}},$$

then we have the following estimates

$$\begin{aligned} &\epsilon A_1^2 \le CK|\overline{x}-\overline{y}|^3 I_{2N}, \quad \epsilon A_2^2 \le CK|\overline{x}-\overline{y}| I_{2N}, \\ &\epsilon(A_1A_2 + A_2A_1) \le CK|\overline{x}-\overline{y}|^2 I_{2N}, \\ &\epsilon(A_1A_3 + A_3A_1) \le CK|\overline{x}-\overline{y}| I_{2N}, \quad \epsilon(A_2A_3 + A_3A_2) \le CKI_{2N}, \end{aligned} \tag{3.9}$$

where $I_{2N} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then using (3.6), (3.7), (3.9) and observing that

$$(D^2(\Phi(\overline{x}-\overline{y})))^2 = \begin{pmatrix} 2(D^2\Phi(\overline{x}-\overline{y}))^2 & -2(D^2\Phi(\overline{x}-\overline{y}))^2 \\ -2(D^2\Phi(\overline{x}-\overline{y}))^2 & 2(D^2\Phi(\overline{x}-\overline{y}))^2 \end{pmatrix},$$

from (3.5) we conclude that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le O(K) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

where

$$B = CKI + e^{-L(d(\overline{x})+d(\overline{y}))}\left[D^2\Phi(\overline{x}-\overline{y}) + \frac{|\overline{x}-\overline{y}|}{MK}(D^2\Phi(\overline{x}-\overline{y}))^2\right].$$

The last inequality can be rewritten as follows

$$\begin{pmatrix} \widetilde{X} & 0 \\ 0 & -\widetilde{Y} \end{pmatrix} \le \begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

with $\widetilde{X} = X - O(K)I$ and $\widetilde{Y} = Y + O(K)I$.

Now we want to get a good estimate for $\mathrm{tr}(\widetilde{X} - \widetilde{Y})$, as in [16]. For that aim let

$$0 \le P := \frac{(\overline{x}-\overline{y})\otimes(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|^2} \le I.$$

Since $\widetilde{X}-\widetilde{Y}\leq 0$ and $\widetilde{X}-\widetilde{Y}\leq 4B$, we have

$$\mathrm{tr}(\widetilde{X}-\widetilde{Y})\leq \mathrm{tr}(P(\widetilde{X}-\widetilde{Y}))\leq 4\mathrm{tr}(PB).$$

We have to compute $\mathrm{tr}(PB)$. From (3.8), observing that the matrix $(1/|x|^2)x\otimes x$ is idempotent, i.e., $[(1/|x|^2)x\otimes x]^2=(1/|x|^2)x\otimes x$, we compute

$$(D^2\Phi(x))^2=\frac{M^2K^2}{|x|^2}(1-4K|x|)\left(I-\frac{x\otimes x}{|x|^2}\right)+4M^2K^4I.$$

Then, since $\mathrm{tr}P=1$ and $4K|\overline{x}-\overline{y}|\leq 1$, we have

$$\begin{aligned}\mathrm{tr}(PB)&=CK+e^{-L(d(\overline{x})+d(\overline{y}))}(-2MK^2+4MK^3|\overline{x}-\overline{y}|)\\&\leq CK-e^{-L(d(\overline{x})+d(\overline{y}))}MK^2<0,\end{aligned}$$

for large K. This gives

$$|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|=-\mathrm{tr}(\widetilde{X}-\widetilde{Y})\geq 4e^{-L(d(\overline{x})+d(\overline{y}))}MK^2-4CK\geq CK^2,$$

for large K. Since $\|B\|\leq \frac{CK}{|\overline{x}-\overline{y}|}$, we have

$$\|B\|^{\frac12}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac12}\leq\left(\frac{CK}{|\overline{x}-\overline{y}|}\right)^{\frac12}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac12}\leq\frac{C}{K^{\frac12}|\overline{x}-\overline{y}|^{\frac12}}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|.$$

The Lemma III.I in [16] ensures the existence of a universal constant C depending only on N such that

$$\|\widetilde{X}\|,\|\widetilde{Y}\|\leq C\{|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|+\|B\|^{\frac12}|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|^{\frac12}\}.$$

Thanks to the above estimates we can conclude that

$$\|\widetilde{X}\|,\ \|\widetilde{Y}\|\leq C|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|\left(1+\frac{1}{K^{\frac12}|\overline{x}-\overline{y}|^{\frac12}}\right). \tag{3.10}$$

Now, using the assumptions (F2), (F3) and (F4) concerning F, the definition of $\widetilde{X}$ and $\widetilde{Y}$ and the fact that u and v are respectively sub and supersolution we compute

$$\begin{aligned}g(\overline{x})-c(\overline{x})|u(\overline{x})|^\alpha u(\overline{x})&\leq F(\overline{x},D_x\varphi,X)+b(\overline{x})\cdot D_x\varphi|D_x\varphi|^\alpha\\&\leq F(\overline{x},D_x\varphi,\widetilde{X})+|D_x\varphi|^\alpha O(K)+b(\overline{x})\cdot D_x\varphi|D_x\varphi|^\alpha\\&\leq F(\overline{y},-D_y\varphi,\widetilde{Y})+C_1|\overline{x}-\overline{y}|^\theta|D_x\varphi|^\alpha\|\widetilde{X}\|\\&\quad+CK^\nu|\overline{x}-\overline{y}|^\nu|D_x\varphi|^{\alpha-\nu}\|\widetilde{X}\|+a|D_y\varphi|^\alpha\mathrm{tr}(\widetilde{X}-\widetilde{Y})\\&\quad+|D_x\varphi|^\alpha O(K)+b(\overline{x})\cdot D_x\varphi|D_x\varphi|^\alpha\\&\leq b(\overline{y})\cdot D_y\varphi|D_y\varphi|^\alpha-c(\overline{y})|v(\overline{y})|^\alpha v(\overline{y})+h(\overline{y})\\&\quad+C_1|\overline{x}-\overline{y}|^\theta|D_x\varphi|^\alpha\|\widetilde{X}\|+CK^\nu|\overline{x}-\overline{y}|^\nu|D_x\varphi|^{\alpha-\nu}\|\widetilde{X}\|\\&\quad+a|D_y\varphi|^\alpha\mathrm{tr}(\widetilde{X}-\widetilde{Y})+|D_y\varphi|^\alpha\vee|D_x\varphi|^\alpha O(K)\\&\quad+b(\overline{x})\cdot D_x\varphi|D_x\varphi|^\alpha.\end{aligned}$$

From this inequalities, using (3.4), (3.10) and the fact that $\theta,\nu>\frac12$ we get

$$\begin{aligned}&g(\overline{x})-h(\overline{y})-c(\overline{x})|u(\overline{x})|^\alpha u(\overline{x})+c(\overline{y})|v(\overline{y})|^\alpha v(\overline{y})\leq|D_y\varphi|^\alpha\vee|D_x\varphi|^\alpha[a\mathrm{tr}(\widetilde{X}-\widetilde{Y})\\&+C_1|\overline{x}-\overline{y}|^\theta\|\widetilde{X}\|+C|\overline{x}-\overline{y}|^\nu\|\widetilde{X}\|+O(K)]\leq CK^\alpha[a\mathrm{tr}(\widetilde{X}-\widetilde{Y})+o(|\mathrm{tr}(\widetilde{X}-\widetilde{Y})|)].\end{aligned}$$

If both u and v are bounded, then the first member in the last inequalities is bounded from below by $-|g|_\infty-|h|_\infty-|c|_\infty(|u|_\infty^{\alpha+1}+|v|_\infty^{\alpha+1})$. Otherwise, if v is non-negative and bounded, then $u(\overline{x})\geq 0$ and that quantity is greater than $-|g|_\infty-|h|_\infty-|c|_\infty(\sup u)^{\alpha+1}-|c|_\infty|v|_\infty^{\alpha+1}$. On the other hand, the last member goes to $-\infty$ as $K\to+\infty$, hence taking K large enough we obtain a contradiction and this concludes the proof. □

Remark 3.3. If F satisfies (F2) and (F3), u is a subsolution of $G(x,u,Du,D^2u)=g$, v is a supersolution of $G(x,v,Dv,D^2v)=h$ in Ω, $u\le v$ on $\partial\Omega$ and $m>0$ then the estimate (3.1) still holds for any $x,y\in\Omega$. To prove this define $\varphi = m+MK|x|-M(K|x|)^2$ and follow the proof of Lemma 3.2.

Since the Lipschitz estimate depends only on the bounds of the solution, of g and on the structural constants, an immediate consequence of Theorem 3.1 is the following compactness criterion that will be useful in the last section.

Corollary 3.4. *Assume the hypothesis of Theorem 3.1 on Ω, F and b. Suppose that $(g_n)_n$ is a sequence of continuous and uniformly bounded functions and $(u_n)_n$ is a sequence of uniformly bounded viscosity solutions of*

$$\begin{cases} F(x,Du_n,D^2u_n)+b(x)\cdot Du_n|Du_n|^\alpha = g_n(x) & \text{in} \quad \Omega \\ \langle Du_n, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then the sequence $(u_n)_n$ is relatively compact in $C(\overline{\Omega})$.

4. The Maximum Principle and the principal eigenvalues

We say that the operator $G(x,u,Du,D^2u)$ with the Neumann boundary condition satisfies the maximum principle if whenever $u\in USC(\overline{\Omega})$ is a viscosity subsolution of

$$\begin{cases} G(x,u,Du,D^2u)=0 & \text{in} \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then $u\le 0$ in $\overline{\Omega}$.

We first prove that the maximum principle holds under the classical assumption $c\le 0$, also for domain which are not of class C^2 and with more general boundary conditions. Then we show that the operator $G(x,u,Du,D^2u)+\lambda|u|^\alpha u$ with the Neumann boundary condition satisfies the maximum principle for any $\lambda<\overline{\lambda}$. This is the best result that one can expect, indeed, as we will see in the last section, $\overline{\lambda}$ admits a positive eigenfunction which provides a counterexample to the maximum principle for $\lambda\ge\overline{\lambda}$.

Finally, we give an example of $c(x)$ which changes sign in Ω and such that the associated principal eigenvalue $\overline{\lambda}$ is positive.

4.1. The case $c(x)\le 0$.

In this subsection we assume that Ω is of class C^1 and satisfies the interior sphere condition (Ω1). We need the comparison principle between sub and supersolutions of the Dirichlet problem when $c<0$ in Ω. This result is proven in [8] under different assumptions on F and b; thanks to the estimate (3.1), see Remark 3.3, we can show it using the same strategy of [8], if F satisfies the conditions (F2) and (F3) and b is continuous and bounded on Ω.

Theorem 4.1. *Let Ω be bounded. Assume that (F2) and (F3) hold, that b, c and g are continuous and bounded on Ω and $c<0$ in Ω. If $u\in USC(\overline{\Omega})$ and $v\in LSC(\overline{\Omega})$ are respectively sub and supersolution of*

$$F(x,Du,D^2u)+b(x)\cdot Du|Du|^\alpha + c(x)|u|^\alpha u = g(x) \quad \text{in } \Omega,$$

and $u\le v$ on $\partial\Omega$ then $u\le v$ in Ω.

For convenience of the reader we postpone the proof of the theorem to the next subsection.

The previous comparison result allows us to establish the strong minimum and maximum principles, for sub and supersolutions of the Neumann problem even with the following more general boundary condition

$$f(x,u)+\langle Du,\overrightarrow{n}(x)\rangle = 0 \quad x\in\partial\Omega,$$

for some $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$. We do not assume any regularity on f.

Proposition 4.2. *Let Ω be a C^1 domain satisfying ($\Omega 1$). Assume that (F1)-(F3) hold, that b and c are bounded and continuous on Ω and that $f(x, 0) \le 0$ for all $x \in \partial\Omega$. If $v \in LSC(\overline{\Omega})$ is a non-negative viscosity supersolution of*

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha + c(x)|v|^\alpha v = 0 & \text{in} \quad \Omega \\ f(x, v) + \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

then either $v \equiv 0$ or $v > 0$ in $\overline{\Omega}$.

Proof. The assumption $(F2)$ and the fact that $F(x, p, 0) = 0$ imply that

$$F(x, p, M) \ge |p|^\alpha \mathcal{M}^-_{a,A} M = |p|^\alpha (atr(M^+) - Atr(M^-)) =: H(p, M),$$

where $M = M^+ - M^-$ is the minimal decomposition of M into positive and negative symmetric matrices. It follows, since v is non-negative, that it suffices to prove the proposition when v is a supersolution of the Neumann problem for the equation

$$H(Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha - |c|_\infty v^{1+\alpha} = 0 \quad \text{in } \Omega. \tag{4.2}$$

Moreover we can assume $|c|_\infty > 0$. Following the proof of Theorem 2 in [8] it can be showed that $v > 0$ in Ω. We prove that v cannot vanish on the boundary of Ω. We suppose by contradiction that x_0 is some point in $\partial\Omega$ on which $v(x_0) = 0$. For the interior sphere condition (Ω1) there exist $R > 0$ and $y \in \Omega$ such that the ball centered in y and of radius R, $B(y, R)$, is contained in Ω and $x_0 \in \partial B(y, R)$. Fixed $0 < \rho < R$, let us construct a subsolution of (4.2) in the annulus $\rho < |x-y| = r < R$. Let us consider the function $\phi(x) = e^{-kr} - e^{-kR}$, where k is a positive constant to be determined. If we compute the derivatives of ϕ we get

$$D\phi(x) = -ke^{-kr}\frac{(x-y)}{r},\ D^2\phi(x) = \left(k^2e^{-kr} + \frac{k}{r}e^{-kr}\right)\frac{(x-y)\otimes(x-y)}{r^2} - \frac{k}{r}e^{-kr}I.$$

The eigenvalues of $D^2\phi(x)$ are k^2e^{-kr} of multiplicity 1 and $-ke^{-kr}/r$ of multiplicity $N - 1$. Then

$$\begin{aligned} &H(D\phi, D^2\phi) + b(x) \cdot D\phi|D\phi|^\alpha - |c|_\infty \phi^{1+\alpha} \\ &\ge e^{-(\alpha+1)kr}\left(ak^{\alpha+2} - \left(A\frac{N-1}{\rho} + |b|_\infty\right)k^{\alpha+1} - |c|_\infty\right). \end{aligned}$$

Take k such that

$$ak^{\alpha+2} - \left(A\frac{N-1}{\rho} + |b|_\infty\right)k^{\alpha+1} - |c|_\infty > \epsilon,$$

for some $\epsilon > 0$, then ϕ is a strict subsolution of the equation (4.2). Now choose $m > 0$ such that

$$m(e^{-k\rho} - e^{-kR}) = v_1 := \inf_{|x-y|=\rho} v(x) > 0,$$

and define $w(x) = m(e^{-kr} - e^{-kR})$. By homogeneity w is still a subsolution of (4.2) in the annulus $\rho < |x - y| < R$, moreover $w = v_1 \le v$ if $|x - y| = \rho$ and $w = 0 \le v$ if $|x - y| = R$. Then by the comparison principle, Theorem 4.1, $w \le v$ in the entire annulus.

Now let δ be a positive number smaller than $R - \rho$. In $B(x_0, \delta) \cap \overline{\Omega}$ it is again $w \le v$, in fact where $|x - y| > R$ it is $w < 0 \le v$; moreover $w(x_0) = v(x_0) = 0$. Then w is a test function for v at x_0. But

$$H(Dw(x_0), D^2w(x_0)) + b(x_0) \cdot Dw(x_0)|Dw(x_0)|^\alpha - |c|_\infty w^{1+\alpha}(x_0) > 0,$$

and

$$f(x_0, w(x_0)) + \langle Dw(x_0), \overrightarrow{n}(x_0)\rangle = f(x_0, 0) + \frac{\partial w}{\partial \overrightarrow{n}}(x_0) \le -kme^{-kR} < 0.$$

This contradicts the definition of v. Finally v cannot be zero in $\overline{\Omega}$. □

Remark 4.3. By Proposition 4.2 the supersolutions in the definition (1.2) are positive in the whole $\overline{\Omega}$.

Proposition 4.4. *Let Ω be a C^1 domain satisfying (Ω1). Assume that (F1)-(F3) hold, that b and c are bounded and continuous on Ω and that $f(x,0) \geq 0$ for all $x \in \partial\Omega$. If $u \in USC(\overline{\Omega})$ is a non-positive viscosity subsolution of* (4.1) *then either $u \equiv 0$ or $u < 0$ in $\overline{\Omega}$.*

Proof. The proof is similar to the proof of Proposition 4.2, observing that (F1) and the fact that $F(x,p,0) = 0$ imply that

$$F(x,p,M) \leq |p|^\alpha (A\mathrm{tr}(M^+) - a\mathrm{tr}(M^-)).$$

□

For $x \in \partial\Omega$, let us introduce $S(x)$, the symmetric operator corresponding to the second fundamental form of $\partial\Omega$ in x oriented with the exterior normal to Ω.

Theorem 4.5 (Maximum Principle for $c \leq 0$)**.** *Assume the hypothesis of Proposition 4.4. In addition suppose that Ω is bounded, $c \leq 0$, $c \not\equiv 0$ and $r \to f(x,r)$ is non-decreasing on $\mathbb{R}$. If $u \in USC(\overline{\Omega})$ is a viscosity subsolution of* (4.1) *then $u \leq 0$ in $\overline{\Omega}$. The same conclusion holds also if $c \equiv 0$ in the following two cases*

(i) *Ω is a C^2 domain and there exists $\overline{x} \in \partial\Omega$ such that $S(\overline{x}) \leq 0$, $\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle > 0$ and $f(\overline{x}, r) > 0$ for any $r > 0$;*

(ii) *There exists $\overline{x} \in \partial\Omega$ such that $f(\overline{x}, r) > 0$ for any $r > 0$ and u is a strong subsolution.*

Proof. Let u be a subsolution of (4.1) and $c \not\equiv 0$. First let us suppose $u \equiv k$ =const. By definition

$$c(x)|k|^\alpha k \geq 0 \qquad \text{in } \Omega,$$

which implies $k \leq 0$.

Now we assume that u is not a constant. We argue by contradiction; suppose that $\max_{\overline{\Omega}} u = u(x_0) > 0$, for some $x_0 \in \overline{\Omega}$. Define $\tilde{u}(x) := u(x) - u(x_0)$. Since $c \leq 0$ and f is non-decreasing, $\tilde{u}$ is a non-positive subsolution of (4.1). Then, from Proposition 4.4, either $u \equiv u(x_0)$ or $u < u(x_0)$ in $\overline{\Omega}$. In both cases we get a contradiction.

Let us turn to the case $c \equiv 0$. Suppose that Ω is a C^2 domain, $S(\overline{x}) \leq 0$, $\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle > 0$ and $f(\overline{x}, r) > 0$ for any $r > 0$ and some point $\overline{x} \in \partial\Omega$. We have to prove that u cannot be a positive constant. Suppose by contradiction that $u \equiv k$. In general, if ϕ is a C^2 function, $\overline{x} \in \partial\Omega$ and $S(\overline{x}) \leq 0$, then $(D\phi(\overline{x}) - \lambda\overrightarrow{n}(\overline{x}), D^2\phi(\overline{x})) \in J^{2,+}\phi(\overline{x})$, for $\lambda \geq 0$ (see [12] Remark 2.7). Hence $(-\lambda\overrightarrow{n}(\overline{x}), 0) \in J^{2,+}u(\overline{x})$. But

$$f(\overline{x}, k) - \lambda\langle \overrightarrow{n}(\overline{x}), \overrightarrow{n}(\overline{x})\rangle = f(\overline{x}, k) - \lambda > 0,$$

for $\lambda > 0$ small enough, and

$$G(\overline{x}, k, -\lambda\overrightarrow{n}(\overline{x}), 0) = -\lambda^{\alpha+1}\langle b(\overline{x}), \overrightarrow{n}(\overline{x})\rangle < 0.$$

This contradicts the definition of u.

Finally, if u is a strong subsolution, $f(\overline{x}, r) > 0$ for $r > 0$ and some $\overline{x} \in \partial\Omega$, $u \equiv k > 0$, then the boundary condition is not satisfied at $\overline{x}$ for $p = 0$. □

Remark 4.6. Under the same assumptions of Theorem 4.5, but now with f satisfying $f(x,0) \leq 0$ for all $x \in \partial\Omega$ and with $f(\overline{x}, r) < 0$ for any $r < 0$ and some $\overline{x} \in \partial\Omega$ in (i) and (ii), using Proposition 4.2 we can prove the minimum principle, i.e., if $u \in LSC(\overline{\Omega})$ is a viscosity supersolution of (4.1) then $u \geq 0$ in $\overline{\Omega}$.

Remark 4.7. C^2 convex sets satisfy the condition $S \leq 0$ in every point of the boundary.

Remark 4.8. If $c \equiv 0$ and $f \equiv 0$ a counterexample to the maximum principle is given by the positive constants.

4.2. **The threshold for the Maximum Principle.** In this subsection and in the rest of the paper we always assume that Ω is bounded and of class C^2, that F satisfies (F1)-(F4), that b and c are continuous on $\overline{\Omega}$.

Theorem 4.9 (Maximum Principle for $\lambda < \overline{\lambda}$)**.** *Let $\lambda < \overline{\lambda}$ and let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of*

$$\begin{cases} G(x, u, Du, D^2u) + \lambda|u|^\alpha u = 0 & in \quad \Omega \\ \langle Du, \overrightarrow{n}(x)\rangle = 0 & on \quad \partial\Omega, \end{cases} \tag{4.3}$$

then $u \leq 0$ in $\overline{\Omega}$.

Remark 4.10. Similarly it is possible to prove that if $\lambda < \underline{\lambda}$ and v is a supersolution of (4.3) then $v \geq 0$ in $\overline{\Omega}$.

Corollary 4.11. *The quantities $\overline{\lambda}$ and $\underline{\lambda}$ are finite.*

Proof. It suffices to observe that $\overline{\lambda}$, $\underline{\lambda} \leq |c|_\infty$, since when the zero order coefficient is $c(x)+|c|_\infty$ the maximum and the minimum principles do not hold. The theorems fail respectively for the positive and negative constants. □

In the proof of Theorem 4.9 the Lemma 3.2 is one of the main ingredient. Furthermore, we need the following two results. The first one is an adaptation of Lemma 1 of [8] for supersolutions of the Neumann boundary value problem; the second one is a lemma due to Barles and Ramaswamy, [6].

Lemma 4.12. *Let $v \in LSC(\overline{\Omega})$ be a viscosity supersolution of*

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha - \beta(v(x)) = g(x) & in \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & on \quad \partial\Omega, \end{cases}$$

for some functions $g, \beta \in USC(\overline{\Omega})$. Suppose that $\overline{x} \in \overline{\Omega}$ is a strict local minimum of $v(x) + C|x - \overline{x}|^q e^{-kd(x)}$, $k > \frac{q}{2r}$, where r is the radius in the condition (Ω2) and $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$. Moreover suppose that v is not locally constant around $\overline{x}$. Then

$$-\beta(v(\overline{x})) \leq g(\overline{x}).$$

Remark 4.13. Similarly, if β, $g \in LSC(\overline{\Omega})$, $u \in USC(\overline{\Omega})$ is a supersolution, $\overline{x}$ is a strict local maximum of $u(x) - C|x - \overline{x}|^q e^{-kd(x)}$, $k > \frac{q}{2r}$, $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$ and u is not locally constant around $\overline{x}$, it can be proved that

$$-\beta(u(\overline{x})) \geq g(\overline{x}).$$

Lemma 4.14. *If $X, Y \in S(N)$ satisfy*

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

then we have

$$X - Y \leq -\frac{1}{2\zeta}(tX + (1-t)Y)^2 \quad \text{for all } t \in [0, 1].$$

Proof of Theorem 4.9. Let $\tau \in]\lambda, \overline{\lambda}[$, then by definition there exists $v > 0$ in $\overline{\Omega}$ bounded viscosity supersolution of

$$\begin{cases} G(x, v, Dv, D^2v) + \tau v^{\alpha+1} = 0 & \text{in} \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.4}$$

We argue by contradiction that u has a positive maximum in $\overline{\Omega}$. As in [8], we define $\gamma' := \sup_{\overline{\Omega}}(u/v) > 0$ and $w = \gamma v$, with $\gamma \in (0, \gamma')$ to be determined. By homogeneity, w is still a supersolution of (4.4). Let $\overline{y} \in \overline{\Omega}$ be such that $u(\overline{y})/v(\overline{y}) = \gamma'$. Since $u(\overline{y}) - w(\overline{y}) = (\gamma' - \gamma)v(\overline{y}) > 0$, the supremum of $u - w$ is strictly positive, then by upper semicontinuity there exists $\overline{x} \in \overline{\Omega}$ such that

$$u(\overline{x}) - w(\overline{x}) = \max_{\overline{\Omega}}(u - w) = m > 0.$$

Clearly $u(\overline{x}) > w(\overline{x}) > 0$, moreover $u(\overline{x}) \le \gamma' v(\overline{x}) = \frac{\gamma'}{\gamma} w(\overline{x})$, from which

$$w(\overline{x}) \ge \frac{\gamma}{\gamma'} u(\overline{x}). \tag{4.5}$$

Fix $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$ and $k > q/(2r)$, where r is the radius in the condition (Ω2), and define for $j \in \mathbb{N}$ the functions $\phi \in C^2(\overline{\Omega} \times \overline{\Omega})$ and $\psi \in USC(\overline{\Omega} \times \overline{\Omega})$ by

$$\phi(x,y) = \frac{j}{q}|x-y|^q e^{-k(d(x)+d(y))}, \quad \psi(x,y) = u(x) - w(y) - \phi(x,y).$$

Let $(x_j, y_j) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of ψ, then $m = \psi(\overline{x}, \overline{x}) \le u(x_j) - w(y_j) - \phi(x_j, y_j)$, from which

$$\frac{j}{q}|x_j - y_j|^q \le (u(x_j) - w(y_j) - m)e^{k(d(x_j)+d(y_j))} \le C, \tag{4.6}$$

where C is independent of j. The last relation implies that, up to subsequence, x_j and y_j converge to some $\overline{z} \in \overline{\Omega}$ as $j \to +\infty$. Classical arguments show that

$$\lim_{j\to+\infty} \frac{j}{q}|x_j - y_j|^q = 0, \quad \lim_{j\to+\infty} u(x_j) = u(\overline{z}), \quad \lim_{j\to+\infty} w(y_j) = w(\overline{z}),$$

and

$$u(\overline{z}) - w(\overline{z}) = m.$$

Claim 1 *For j large enough, there exist x_j and y_j such that (x_j, y_j) is a maximum point of ψ and $x_j \ne y_j$.*

Indeed if $x_j = y_j$ we have

$$\psi(x_j, x) = u(x_j) - w(x) - \frac{j}{q}|x - x_j|^q e^{-k(d(x_j)+d(x))} \le \psi(x_j, x_j) = u(x_j) - w(x_j),$$

and

$$\psi(x, x_j) = u(x) - w(x_j) - \frac{j}{q}|x - x_j|^q e^{-k(d(x)+d(x_j))} \le \psi(x_j, x_j) = u(x_j) - w(x_j).$$

Then x_j is a minimum point for

$$W(x) := w(x) + \frac{j}{q} e^{-kd(x_j)} |x - x_j|^q e^{-kd(x)},$$

and a maximum point for

$$U(x) := u(x) - \frac{j}{q} e^{-kd(x_j)} |x - x_j|^q e^{-kd(x)}.$$

We first exclude that x_j is both a strict local minimum and a strict local maximum. Indeed in that case, if u and w are not locally constant around x_j, by Lemma 4.12

$$(c(x_j) + \tau) w(x_j)^{\alpha+1} \le (c(x_j) + \lambda) u(x_j)^{\alpha+1}.$$

The same result holds if u or w are locally constant by definition of sub and supersolution. The last inequality leads to a contradiction, as we will see at the end of the proof. Hence x_j cannot be both a strict local minimum and a strict local maximum. In the first case there exist $\delta > 0$ and $R > \delta$ such that

$$w(x_j) = \min_{\substack{\delta \le |x-x_j| \le R \\ x \in \overline{\Omega}}} \left(w(x) + \frac{j}{q}|x - x_j|^q e^{-k(d(x_j)+d(x))} \right)$$
$$= w(y_j) + \frac{j}{q}|y_j - x_j|^q e^{-k(d(x_j)+d(y_j))},$$

for some $y_j \neq x_j$, so that (x_j, y_j) is still a maximum point for ψ. In the other case, similarly, one can replace x_j by a point $y_j \neq x_j$ such that (y_j, x_j) is a maximum for ψ. This concludes the Claim 1.

Now computing the derivatives of ϕ we get

$$D_x\phi(x,y) = j|x-y|^{q-2}e^{-k(d(x)+d(y))}(x-y) - k\frac{j}{q}|x-y|^q e^{-k(d(x)+d(y))}Dd(x),$$

and

$$D_y\phi(x,y) = -j|x-y|^{q-2}e^{-k(d(x)+d(y))}(x-y) - k\frac{j}{q}|x-y|^q e^{-k(d(x)+d(y))}Dd(y).$$

Denote $p_j := D_x\phi(x_j,y_j)$ and $r_j := -D_y\phi(x_j,y_j)$. Since $x_j \neq y_j$, p_j and r_j are different from 0 for j large enough. Indeed

$$|p_j|, |r_j| \ge j|x_j - y_j|^{q-1}e^{-k(d(x_j)+d(y_j))}\left(1 - \frac{k}{q}|x_j - y_j|\right) \ge \frac{j}{2}|x_j - y_j|^{q-1}e^{-2kd_0},$$

where $d_0 = \max_{\overline{\Omega}} d(x)$. Using (2.1), if $x_j \in \partial\Omega$ then

$$\langle p_j, \overrightarrow{n}(x_j)\rangle \ge j|x_j - y_j|^q e^{-kd(y_j)}\left(-\frac{1}{2r} + \frac{k}{q}\right) > 0,$$

and if $y_j \in \partial\Omega$ then

$$\langle r_j, \overrightarrow{n}(y_j)\rangle \le j|x_j - y_j|^q e^{-kd(x_j)}\left(\frac{1}{2r} - \frac{k}{q}\right) < 0,$$

since $k > q/(2r)$ and $x_j \neq y_j$. In view of definition of sub and supersolution we conclude that

$$G(x_j, u(x_j), p_j, X) + \lambda u(x_j)^{\alpha+1} \ge 0 \quad \text{if } (p_j, X) \in \overline{J}^{2,+}u(x_j),$$
$$G(y_j, w(y_j), r_j, Y) + \tau w(y_j)^{\alpha+1} \le 0 \quad \text{if } (r_j, Y) \in \overline{J}^{2,-}w(y_j).$$

Then the previous relations hold for $(x_j, y_j) \in \overline{\Omega}^2$, provided j is large.

Now, applying Theorem 3.2 of [12] for any $\epsilon > 0$ there exist $X_j, Y_j \in S(N)$ such that $(p_j, X_j) \in \overline{J}^{2,+}u(x_j)$, $(r_j, Y_j) \in \overline{J}^{2,-}w(y_j)$ and

(4.7)
$$-\left(\frac{1}{\epsilon} + \|D^2\phi(x_j,y_j)\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_j & 0 \\ 0 & -Y_j \end{pmatrix} \le D^2\phi(x_j,y_j) + \epsilon(D^2\phi(x_j,y_j))^2.$$

Claim 2 *X_j and Y_j satisfy*

$$-\zeta_j\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_j - \widetilde{X_j} & 0 \\ 0 & -Y_j + \widetilde{Y_j} \end{pmatrix} \le \zeta_j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{4.8}$$

where $\zeta_j = Cj|x_j - y_j|^{q-2}$, for some positive constant C independent of j and some matrices $\widetilde{X_j}$, $\widetilde{Y_j} = O(j|x_j - y_j|^q)$.

To prove the claim we need to estimate $D^2\phi(x_j,y_j)$.

$$D^2\phi(x_j,y_j)=\frac{j}{q}|x_j-y_j|^qD^2(e^{-k(d(x_j)+d(y_j))})+D(e^{-k(d(x_j)+d(y_j))})\otimes\frac{j}{q}D(|x_j-y_j|^q)$$
$$+\frac{j}{q}D(|x_j-y_j|^q)\otimes D(e^{-k(d(x_j)+d(y_j))})+e^{-k(d(x_j)+d(y_j))}\frac{j}{q}D^2(|x_j-y_j|^q).$$

We denote

$$A_1:=\frac{j}{q}|x_j-y_j|^qD^2(e^{-k(d(x_j)+d(y_j))}),$$
$$A_2:=De^{-k(d(x_j)+d(y_j))}\otimes\frac{j}{q}D(|x_j-y_j|^q)+\frac{j}{q}D(|x_j-y_j|^q)\otimes D(e^{-k(d(x_j)+d(y_j))}),$$
$$A_3:=e^{-k(d(x_j)+d(y_j))}\frac{j}{q}D^2(|x_j-y_j|^q).$$

For A_1 and A_3 we have

$$A_1\le Cj|x_j-y_j|^q\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix},$$
$$A_3\le (q-1)j|x_j-y_j|^{q-2}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}.$$

Here and henceforth, as usual, the letter C denotes various constants independent of j. Now we consider the quantity $\langle A_2(\xi,\eta),(\xi,\eta)\rangle$ for ξ, $\eta\in\mathbb{R}^N$. We have

$$\begin{aligned}\langle A_2(\xi,\eta),(\xi,\eta)\rangle &= 2kj|x_j-y_j|^{q-2}e^{-k(d(x_j)+d(y_j))}[\langle Dd(x_j)\otimes(x_j-y_j)(\eta-\xi),\xi\rangle\\ &\quad+\langle Dd(y_j)\otimes(x_j-y_j)(\eta-\xi),\eta\rangle]\\ &\le Cj|x_j-y_j|^{q-1}|\xi-\eta|(|\xi|+|\eta|)\\ &\le Cj|x_j-y_j|^{q-1}\left(\frac{|\xi-\eta|^2}{|x_j-y_j|}+\frac{(|\xi|+|\eta|)^2}{4}|x_j-y_j|\right)\\ &\le C\left[j|x_j-y_j|^{q-2}|\xi-\eta|^2+j|x_j-y_j|^q(|\xi|^2+|\eta|^2)\right].\end{aligned}$$

The last inequality can be rewritten equivalently in this way

$$A_2\le Cj|x_j-y_j|^{q-2}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}+Cj|x_j-y_j|^q\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}.$$

Finally if we choose

$$\epsilon=\frac{1}{j|x_j-y_j|^{q-2}},$$

we get the same estimates for the matrix $\epsilon(D^2\phi(x_j,y_j))^2$. In conclusion we have

$$D^2\phi(x_j,y_j)+\epsilon(D^2\phi(x_j,y_j))^2\le Cj|x_j-y_j|^{q-2}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}$$
$$+Cj|x_j-y_j|^q\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}.$$

Hence, since $\|D^2\phi(x_j,y_j)\|\le Cj|x_j-y_j|^{q-2}$, (4.7) implies (4.8) and the Claim 2 is proved.

Claim 3 *$F(x_j,p_j,X_j-\widetilde{X_j})-F(y_j,r_j,Y_j-\widetilde{Y_j})\le o_j$, where $o_j\to 0$ as $j\to+\infty$.*

First we need to know that the quantity $j|x_j-y_j|^{q-1}$ is bounded uniformly in j. This is a simple consequence of Lemma 3.2. Indeed, since $m>0$ and w is positive and bounded, the estimate (3.1) holds for u and w; then using it in (4.6) and dividing by $|x_j-y_j|\neq 0$ we obtain

$$\frac{j}{q}|x_j-y_j|^{q-1}\le C_0e^{k(d(x_j)+d(y_j))}\le C.$$

Consequently, there exists $R>0$ such that for large j

$$C\zeta_j|x_j-y_j| \le \frac{j}{2}|x_j-y_j|^{q-1}e^{-k(d(x_j)+d(y_j))} \le |p_j|,\ |r_j| \le 2j|x_j-y_j|^{q-1} \le R. \tag{4.9}$$

Denote for simplicity $Z_j := X_j - \widetilde{X_j}$ and $W_j := Y_j - \widetilde{Y_j}$. By (4.8) and Lemma 4.14 with $t=0$, we have

$$Z_j - W_j \le -\frac{1}{2\zeta_j}W_j^2.$$

As in the appendix of [5] we use the previous relation, the Cauchy-Schwarz's inequality and the properties of F to get the estimate of the claim

$$\begin{aligned}F(x_j,p_j,Z_j)-F(y_j,r_j,W_j) &= F(x_j,p_j,Z_j)-F(x_j,p_j,W_j)+F(x_j,p_j,W_j)\\ &\quad -F(y_j,p_j,W_j)+F(y_j,p_j,W_j)-F(y_j,r_j,W_j)\\ &\le -\frac{a}{2\zeta_j}|p_j|^\alpha \mathrm{tr}W_j^2 + C_1|x_j-y_j|^\theta|p_j|^\alpha\|W_j\|\\ &\quad + C_2|p_j|^{\alpha-\nu}|p_j-r_j|^\nu\|W_j\| \le -\frac{a}{2\zeta_j}|p_j|^\alpha\mathrm{tr}W_j^2\\ &\quad + \frac{a}{4\zeta_j}|p_j|^\alpha\mathrm{tr}W_j^2 + \frac{C_1^2|x_j-y_j|^{2\theta}|p_j|^{2\alpha}\zeta_j}{a|p_j|^\alpha}\\ &\quad + \frac{a}{4\zeta_j}|p_j|^\alpha\mathrm{tr}W_j^2 + \frac{C_2^2|p_j|^{2(\alpha-\nu)}|p_j-r_j|^{2\nu}\zeta_j}{a|p_j|^\alpha}\\ &= C\zeta_j|x_j-y_j|^{2\theta}|p_j|^\alpha + C\zeta_j|p_j|^{\alpha-2\nu}|p_j-r_j|^{2\nu}.\end{aligned}$$

Now consider the first term of the last quantity. Using (4.9) we have

$$C\zeta_j|x_j-y_j|^{2\theta}|p_j|^\alpha \le \frac{C\zeta_j|x_j-y_j|^{2\theta}|p_j|^{\alpha+1}}{\zeta_j|x_j-y_j|} \le CR^{\alpha+1}|x_j-y_j|^{2\theta-1},$$

and the last term goes to 0 as $j\to+\infty$ since $\theta>\frac{1}{2}$. It remains to estimate $C\zeta_j|p_j|^{\alpha-2\nu}|p_j-r_j|^{2\nu}$. Observe that

$$|p_j-r_j| \le 2k\frac{j}{q}|x_j-y_j|^q = C\zeta_j|x_j-y_j|^2,$$

then we have

$$\begin{aligned}C\zeta_j|p_j|^{\alpha-2\nu}|p_j-r_j|^{2\nu} &= C|p_j|^{\alpha+1}\frac{\zeta_j}{|p_j|}\frac{|p_j-r_j|^{2\nu}}{|p_j|^{2\nu}} \le \frac{CR^{\alpha+1}}{|x_j-y_j|}|x_j-y_j|^{2\nu}\\ &= CR^{\alpha+1}|x_j-y_j|^{2\nu-1}.\end{aligned}$$

Also the last quantity goes to 0 as $j\to+\infty$ since $\nu>\frac{1}{2}$ and this concludes the Claim 3.

Now using the properties of F and the fact that u and w are respectively sub and supersolution we compute

$$\begin{aligned}-(\lambda+c(x_j))u(x_j)^{\alpha+1} &\le F(x_j,p_j,X_j)+b(x_j)\cdot p_j|p_j|^\alpha\\ &\le F(x_j,p_j,X_j-\widetilde{X_j})+b(x_j)\cdot p_j|p_j|^\alpha + |p_j|^\alpha O\left(j|x_j-y_j|^q\right)\\ &\le F(y_j,r_j,Y_j-\widetilde{Y_j})+b(x_j)\cdot p_j|p_j|^\alpha + |p_j|^\alpha O\left(j|x_j-y_j|^q\right)+o_j\\ &\le -(\tau+c(y_j))w(y_j)^{\alpha+1}+b(x_j)\cdot p_j|p_j|^\alpha - b(y_j)\cdot r_j|r_j|^\alpha\\ &\quad + (|p_j|^\alpha\vee|r_j|^\alpha)O\left(j|x_j-y_j|^q\right)+o_j.\end{aligned}$$

Sending $j\to+\infty$ we obtain

$$-(\lambda+c(\overline{z}))u(\overline{z})^{\alpha+1} \le -(\tau+c(\overline{z}))w(\overline{z})^{\alpha+1}. \tag{4.10}$$

Indeed $o_j \to 0$ as $j \to +\infty$ and

$(|p_j|^\alpha \vee |r_j|^\alpha) O\left(j|x_j - y_j|^q\right) \le C(j|x_j - y_j|^{q-1})^{\alpha+1}|x_j - y_j| \le CR^{\alpha+1}|x_j - y_j| \to 0$ as $j \to +\infty$. Moreover, up to subsequence $p_j,\, r_j \to p_0 \in \mathbb{R}^N$. If $p_0 \neq 0$ then

$$b(x_j)\cdot p_j|p_j|^\alpha,\; b(y_j)\cdot r_j|r_j|^\alpha \to b(\overline{z})\cdot p_0|p_0|^\alpha$$

and so the difference goes to 0, otherwise

$$|b(x_j)\cdot p_j||p_j|^\alpha \le |b(x_j)||p_j|^{\alpha+1} \to 0 \quad \text{as } j \to +\infty.$$

The same result holds for $b(y_j)\cdot r_j|r_j|^\alpha$.

If $\tau + c(\overline{z}) > 0$, from (4.5) and (4.10) we have

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le -(\tau + c(\overline{z}))\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1} u(\overline{z})^{\alpha+1},$$

and taking γ sufficiently close to γ' in order that $\frac{\tau\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1}-\lambda}{1-\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1}} > |c|_\infty$, we get a contradiction. Finally if $\tau + c(\overline{z}) \le 0$ we obtain

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le -(\tau + c(\overline{z}))w(\overline{z})^{\alpha+1} \le -(\tau + c(\overline{z}))u(\overline{z})^{\alpha+1},$$

once more a contradiction since $\lambda < \tau$. □

Proof of Lemma 4.12. Without loss of generality we can assume that $\overline{x} = 0$.

Since the minimum is strict there exists a small $\delta > 0$ such that

$$v(0) < v(x) + C|x|^q e^{-kd(x)} \quad \text{for any } x \in \overline{\Omega},\ 0 < |x| \le \delta.$$

Since v is not locally constant and $q > 1$, for any $n > \delta^{-1}$ there exists $(t_n, z_n) \in B(0, \frac{1}{n})^2 \cap \overline{\Omega}^2$ such that

$$v(t_n) > v(z_n) + C|z_n - t_n|^q e^{-kd(z_n)}.$$

Consequently, for $n > \delta^{-1}$ the minimum of the function $v(x) + C|x - t_n|^q e^{-kd(x)}$ in $\overline{B}(0,\delta) \cap \overline{\Omega}$ is not achieved on t_n. Indeed

$$\min_{|x|\le\delta,\, x\in\overline{\Omega}} (v(x) + C|x - t_n|^q e^{-kd(x)}) \le v(z_n) + C|z_n - t_n|^q e^{-kd(z_n)} < v(t_n).$$

Let $y_n \neq t_n$ be some point in $\overline{B}(0,\delta)\cap\overline{\Omega}$ on which the minimum is achieved. Passing to the limit as n goes to infinity, t_n goes to 0 and, up to subsequence, y_n converges to some $y \in \overline{B}(0,\delta) \cap \overline{\Omega}$. By the lower semicontinuity of v and the fact that 0 is a local minimum of $v(x) + C|x|^q e^{-kd(x)}$ we have

$$v(0) \le v(y) + C|y|^q e^{-kd(y)} \le \liminf_{n\to+\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}),$$

and using that $v(0) + C|t_n|^q e^{-kd(0)} \ge v(y_n) + C|y_n - t_n|^q e^{-kd(y_n)}$, one has

$$v(0) \ge \limsup_{n\to+\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}).$$

Then

$$v(0) = v(y) + C|y|^q e^{-kd(y)} = \lim_{n\to+\infty} (v(y_n) + C|y_n|^q e^{-kd(y_n)}).$$

Since 0 is a strict local minimum of $v(x) + C|x|^q e^{-kd(x)}$, the last equalities imply that $y = 0$ and $v(y_n)$ goes to $v(0)$ as $n \to +\infty$. Then for large n, y_n is an interior point of $B(0,\delta)$ so that the function

$$\varphi(x) = v(y_n) + C|y_n - t_n|^q e^{-kd(y_n)} - C|x - t_n|^q e^{-kd(x)}$$

is a test function for v at y_n. Moreover, the gradient of φ

$$D\varphi(x) = -Cq|x - t_n|^{q-2} e^{-kd(x)}(x - t_n) + kC|x - t_n|^q e^{-kd(x)} Dd(x)$$

is different from 0 at $x = y_n$ for small δ, indeed

$$|D\varphi(y_n)| \geq C|y_n - t_n|^{q-1} e^{-kd(y_n)}(q - k|y_n - t_n|) \geq C|y_n - t_n|^{q-1} e^{-kd(y_n)}(q - 2k\delta) > 0.$$

Using (2.1), if $y_n \in \partial\Omega$ we have

$$\langle D\varphi(y_n), \overrightarrow{n}(y_n)\rangle \leq C|y_n - t_n|^q \left(\frac{q}{2r} - k\right) < 0,$$

since $k > q/(2r)$. Then we conclude that

$$F(y_n, D\varphi(y_n), D^2\varphi(y_n)) + b(y_n) \cdot D\varphi(y_n)|D\varphi(y_n)|^\alpha - \beta(v(y_n)) \leq g(y_n).$$

This inequality together with the condition (F2) implies that

$$-|D\varphi(y_n)|^\alpha A\mathrm{tr}(D^2\varphi(y_n))^- + b(y_n) \cdot D\varphi(y_n)|D\varphi(y_n)|^\alpha - \beta(v(y_n)) \leq g(y_n). \tag{4.11}$$

Observe that $D^2\varphi(y_n) = |y_n - t_n|^{q-2}M$, where M is a matrix such that $\mathrm{tr}M^+$ and $\mathrm{tr}M^-$ are bounded by a constant independent of δ and n. Hence, from (4.11) we get

$$C_0|y_n - t_n|^{\alpha(q-1)+q-2} - \beta(v(y_n)) \leq g(y_n),$$

for some constant C_0, where the exponent $\alpha(q-1)+q-2 = q(\alpha+1)-(\alpha+2) > 0$. Passing to the limit, since β and g are upper semicontinuous we get

$$-\beta(v(0)) \leq g(0),$$

which is the desired conclusion. □

We conclude sketching the proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose by contradiction that $\max_{\overline{\Omega}}(u - v) = m > 0$. Since $u \leq v$ on the boundary, the supremum is achieved inside Ω. Let us define for $j \in \mathbb{N}$ and some $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$

$$\psi(x, y) = u(x) - v(y) - \frac{j}{q}|x - y|^q.$$

Suppose that (x_j, y_j) is a maximum point for ψ in $\overline{\Omega}^2$. Then $|x_j - y_j| \to 0$ as $j \to +\infty$ and up to subsequence $x_j, y_j \to \overline{x}$, $u(x_j) \to u(\overline{x})$, $v(y_j) \to v(\overline{x})$ and $j|x_j - y_j|^q \to 0$ as $j \to +\infty$. Moreover, $\overline{x}$ is such that $u(\overline{x}) - v(\overline{x}) = m$ and we can choose $x_j \neq y_j$. Recalling by Remark 3.3 that the estimate (3.1) holds in Ω, we can proceed as in the proof of Theorem 4.9 to get

$$-c(\overline{x})|u(\overline{x})|^\alpha u(\overline{x}) \leq -c(\overline{x})|v(\overline{x})|^\alpha v(\overline{x}).$$

This is a contradiction since $c(\overline{x}) < 0$. □

4.3. **The Maximum Principle for $c(x)$ changing sign: an example.** In the previous subsections we have proved that $G(x, u, Du, D^2u)$ with the Neumann boundary condition satisfies the maximum principle if $c(x) \leq 0$ or without condition on the sign of $c(x)$ provided $\overline{\lambda} > 0$. In this subsection we want to prove that these two cases do not coincide, i.e., that there exists some $c(x)$ which changes sign in Ω such that the associated principal eigenvalue $\overline{\lambda}$ is positive. To prove this, by definition of $\overline{\lambda}$, it suffices to find a function $c(x)$ changing sign for which there exists a bounded positive supersolution of

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha + c(x)|v|^\alpha v = -m & \text{in } \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

where $m > 0$.

In the rest of this subsection we will construct an explicit example of such function. For simplicity, let us suppose that $b \equiv 0$ and Ω is the ball of center 0 and radius R. We will look for c such that:

$$\begin{cases} c(x) < 0 & \text{if } R-\epsilon < |x| \le R \\ c(x) \le -\beta_1 & \text{if } \rho < |x| \le R-\epsilon \\ c(x) \le \beta_2 & \text{if } |x| \le \rho, \end{cases} \tag{4.13}$$

where $0 < \rho < R - \epsilon$ and ϵ, β_1, β_2 are positive constants which satisfy a suitable inequality. Remark that in the ball of radius ρ, $c(x)$ may assume positive values.

In order to construct a supersolution, we define the function

$$v(x) := \begin{cases} D & \text{if } R-\epsilon < |x| \le R \\ E|x|^2 - E(R+\rho-\epsilon)|x| + D + E\rho(R-\epsilon) & \text{if } \rho < |x| \le R-\epsilon \\ D + 1 - e^{k(|x|-\rho)} & \text{if } |x| \le \rho, \end{cases} \tag{4.14}$$

where D, E, k are positive constants to be chosen later.

Lemma 4.15. *The function v defined in* (4.14) *has the following properties*

- (i) *v is continuous on $\overline{B}(0,R)$ and of class C^2 in the sets $B(0,\rho)\setminus\{0\}$, $B(0,R-\epsilon)\setminus\overline{B}(0,\rho)$, $\overline{B}(0,R)\setminus\overline{B}(0,R-\epsilon)$;*
- (ii) *v is positive provided $D > \frac{E}{4}(R-\rho-\epsilon)^2$;*
- (iii) *$J^{2,-}v(x) = \emptyset$ if $x = 0$, $|x| = R-\epsilon$ and if $|x| = \rho$ provided $E(R-\rho-\epsilon) > k$.*

Proof. The proof of (i) is a very simple calculation.

For (ii) we observe that v is positive if $R-\epsilon \le |x| \le R$ and $|x| \le \rho$ since $D, k > 0$. In the region $\{\rho \le |x| \le R-\epsilon\}$ v is positive on the boundary where takes the value D, while in the interior $Dv(x) = 2Ex - E(R+\rho-\epsilon)\frac{x}{|x|} = 0$ if $|x| = \frac{R+\rho-\epsilon}{2}$. In such points $v(x) = -\frac{E}{4}(R-\rho-\epsilon)^2 + D$, then they are global minimums where v takes positive value if $D > \frac{E}{4}(R-\rho-\epsilon)^2$.

Now we turn to (iii). Let $\widehat{x} \in \Omega$ be such that $|\widehat{x}| = \rho$ and let $(p, X) \in J^{2,-}v(\widehat{x})$, then by definition of semi-jet

$$v(x) \ge v(\widehat{x}) + \langle p, x-\widehat{x}\rangle + \frac{1}{2}\langle X(x-\widehat{x}), x-\widehat{x}\rangle + o(|x-\widehat{x}|^2), \tag{4.15}$$

as $x \to \widehat{x}$. If we take $x = \widehat{x} + t\overrightarrow{n}(\widehat{x})$, for $t > 0$, where $\overrightarrow{n}(\widehat{x}) = \frac{\widehat{x}}{|\widehat{x}|}$ is the exterior normal to the sphere of radius ρ at $\widehat{x}$, then $|x| > \rho$ and dividing (4.15) by t we have

$$\frac{v(\widehat{x} + t\overrightarrow{n}(\widehat{x})) - v(\widehat{x})}{t} \ge p_n + O(t),$$

where $p_n = p \cdot \overrightarrow{n}(\widehat{x})$. Letting $t \to 0^+$ we get

$$p_n \le \langle 2E\widehat{x} - E(R+\rho-\epsilon)\frac{\widehat{x}}{|\widehat{x}|}, \frac{\widehat{x}}{|\widehat{x}|}\rangle = -E(R-\rho-\epsilon).$$

On the other hand, if we take $x = \widehat{x} - t\overrightarrow{n}(\widehat{x})$, $t > 0$, in (4.15) and divide by $-t$, letting $t \to 0^+$ we get

$$p_n \ge \langle -ke^{k(|\widehat{x}|-\rho)}\frac{\widehat{x}}{|\widehat{x}|}, \frac{\widehat{x}}{|\widehat{x}|}\rangle = -k.$$

In conclusion

$$E(R-\rho-\epsilon) \le -p_n \le k.$$

Assuming the hypothesis in (iii) the previous condition cannot never be satisfied, then $J^{2,-}v(\widehat{x}) = \emptyset$.

In the same way it can be proved that if $\widehat{x} \in \Omega$ is such that $|\widehat{x}| = R - \rho$ and $(p, X) \in J^{2,-}v(\widehat{x})$ then

$$E(R-\rho-\epsilon) \le p_n \le 0,$$

and clearly also this condition cannot be satisfied, consequently $J^{2,-}v(\widehat{x}) = \emptyset$.

Finally it is easy to see that $J^{2,-}v(0) = \emptyset$. □

Proposition 4.16. *There exist ϵ, β_1, $\beta_2 > 0$ such that for any $c(x)$ satisfying* (4.13) *the function v defined in* (4.14) *is a positive continuous viscosity solution of* (4.12).

Proof. Clearly v satisfies the boundary condition. Since the semi-jet $J^{2,-}v(x)$ is empty if $|x| = \rho$, $|x| = R - \epsilon$ and $x = 0$, in such points we have nothing to test. In $B(0,\rho) \setminus \{0\}$, $B(0,R-\epsilon) \setminus \overline{B}(0,\rho)$, $B(0,R) \setminus \overline{B}(0,R-\epsilon)$ v is of class C^2, then it suffices to prove that v is a classical supersolution of (4.12) in these open sets.
Case I: $R - \epsilon < |x| < R$.

Since $c < 0$ and continuous on $\{R - \epsilon \le |x| \le R\}$, we have

$$c(x)v^{\alpha+1} = c(x)D^{\alpha+1} \le -m_1 < 0. \tag{4.16}$$

Hence, by definition v is supersolution.
Case II: $\rho < |x| < R - \epsilon$.

In this set

$$Dv(x) = E[2|x| - (R+\rho-\epsilon)]\frac{x}{|x|}, \quad D^2v(x) = 2EI - E(R+\rho-\epsilon)\frac{1}{|x|}\left(I - \frac{x \otimes x}{|x|^2}\right).$$

Since $-(R - \rho - \epsilon) \le 2|x| - (R + \rho - \epsilon) \le R - \rho - \epsilon$, using (F2) we compute

$$F(x, Dv, D^2v) \le E^{\alpha+1}(R-\rho-\epsilon)^\alpha \left[\frac{2AN(R-\epsilon) - a(R+\rho-\epsilon)N + a(R+\rho-\epsilon)}{R-\epsilon}\right]$$
$$= E^{\alpha+1}\frac{(R-\rho-\epsilon)^\alpha}{R-\epsilon}\{N[(A-a)(R-\epsilon) + A(R-\epsilon) - a\rho] + a(R+\rho-\epsilon)\}.$$

Observe that all the factors in the last member are positive. Using the last computation, the fact that in the minimum points v takes the value $D - \frac{E}{4}(R - \rho - \epsilon)^2$ (see the proof of Lemma 4.15) and that $c \le -\beta_1$, we have

$$\begin{aligned} F(x, Dv, D^2v) + c(x)v^{\alpha+1} \le E^{\alpha+1}\frac{(R-\rho-\epsilon)^\alpha}{R-\epsilon}&\{N[(A-a)(R-\epsilon) + A(R-\epsilon) - a\rho] \\ + a(R+\rho-\epsilon)\} - \beta_1 &\left[D - \frac{E}{4}(R-\rho-\epsilon)^2\right]^{\alpha+1} =: -m_2. \end{aligned} \tag{4.17}$$

The above quantity is negative if

$$D > \frac{E}{4}(R-\rho-\epsilon)^2 + EC, \tag{4.18}$$

where

$$C := \frac{(R-\rho-\epsilon)^{\frac{\alpha}{\alpha+1}}}{\beta_1^{\frac{1}{\alpha+1}}(R-\epsilon)^{\frac{1}{\alpha+1}}}\{N[(A-a)(R-\epsilon) + A(R-\epsilon) - a\rho] + a(R+\rho-\epsilon)\}^{\frac{1}{\alpha+1}} > 0.$$

Case III: $0 < |x| < \rho$.

Here we have

$$Dv(x) = -ke^{k(|x|-\rho)}\frac{x}{|x|}, \quad D^2v(x) = -k^2e^{k(|x|-\rho)}\frac{x \otimes x}{|x|^2} - ke^{k(|x|-\rho)}\frac{1}{|x|}\left(I - \frac{x \otimes x}{|x|^2}\right).$$

Then

$$\begin{aligned} F(x, Dv, D^2v) + c(x)v^{\alpha+1} &\le -k^{\alpha+1}e^{(\alpha+1)k(|x|-\rho)}a\left(k + \frac{N-1}{|x|}\right) + \beta_2(D+1 \\ - e^{k(|x|-\rho)})^{\alpha+1} &\le -k^{\alpha+1}e^{-(\alpha+1)k\rho}a\left(k + \frac{N-1}{\rho}\right) + \beta_2(D + 1 - e^{-k\rho})^{\alpha+1} =: -m3. \end{aligned} \tag{4.19}$$

The last quantity is negative if

$$\beta_2 < \frac{k^{\alpha+1}e^{-(\alpha+1)k\rho}a\left(k+\frac{N-1}{\rho}\right)}{(D+1-e^{-k\rho})^{\alpha+1}}. \tag{4.20}$$

Since E must satisfy the condition in (iii) of Lemma 4.15, we choose

$$E := \frac{k}{R-\rho-\epsilon'}, \tag{4.21}$$

for $\epsilon < \epsilon' < R-\rho$. Furthermore we take

$$D := \frac{E}{4}(R-\rho-\epsilon)^2 + EC + \epsilon = \frac{k(R-\rho-\epsilon)^2}{4(R-\rho-\epsilon')} + \frac{kC}{R-\rho-\epsilon'} + \epsilon. \tag{4.22}$$

With this choice of D, (4.18) is satisfied and v is positive by (ii) of Lemma 4.15. Observe that

$$D \to k\left\{\frac{R-\rho}{4} + \left[\frac{2NAR-(N-1)a(R+\rho)}{\beta_1 R(R-\rho)}\right]^{\frac{1}{\alpha+1}}\right\}$$

as $\epsilon, \epsilon' \to 0^+$.

Finally we can write the relation between β_1 and β_2:

$$\beta_2 < \frac{k^{\alpha+1}e^{-(\alpha+1)k\rho}a\left(k+\frac{N-1}{\rho}\right)}{\left(k\left\{\frac{R-\rho}{4} + \left[\frac{2NAR-(N-1)a(R+\rho)}{\beta_1 R(R-\rho)}\right]^{\frac{1}{\alpha+1}}\right\} + 1 - e^{-k\rho}\right)^{\alpha+1}}. \tag{4.23}$$

Suppose that (4.23) holds for some $k>0$, then we can choose $\epsilon' > \epsilon > 0$ so small that

$$\beta_2 < \frac{k^{\alpha+1}e^{-(\alpha+1)k\rho}a\left(k+\frac{N-1}{\rho}\right)}{(D+1-e^{-k\rho})^{\alpha+1}},$$

where D is defined by (4.22). Define E as in (4.21), then v is a positive supersolution of (4.12) with m the minimum between the quantity m_1, m_2 and m_3 defined respectively in (4.16), (4.17) and (4.19). Observe that the size of ϵ is given by (4.23). □

Remark 4.17. If we call $UB(\beta_2)$ the upper bound of β_2 in (4.23), we can see that if we choose $k=\frac{1}{\rho}$ then $UB(\beta_2)$ goes to $+\infty$ as $\rho \to 0^+$, that is, if the set where c is positive becomes smaller then the values of c in this set can be very large. On the contrary, for any value of k, if $\rho \to R^-$ then $UB(\beta_2)$ goes to 0. Finally, for any k, if $\beta_1 \to 0^+$ then again $UB(\beta_2)$ goes to 0. So there is a sort of balance between β_1 and β_2. This behavior can be explained by the following example: consider the equation $\Delta v + c(x)v = 0$ which is a subcase of our equation and suppose that $v>0$ in $\overline{\Omega}$ is a classical solution of $\Delta v + c(x)v \le 0$ in Ω, $\frac{\partial v}{\partial \overrightarrow{n}} \ge 0$ on $\partial\Omega$. Then dividing by v and integrating by part we get

$$\int_\Omega c(x)dx \le -\int_\Omega \frac{|Dv|^2}{v^2}dx - \int_{\partial\Omega} \frac{1}{v}\frac{\partial v}{\partial \overrightarrow{n}}dS \le 0, \tag{4.24}$$

the first inequality being strict if $\Delta v + c(x)v \not\equiv 0$. If the supersolution is C^2 piecewise with $J^{2,-}v = \emptyset$ in the non-regular points, as the one constructed before, then we can repeat this computation in any set where v is C^2 getting again

$$\int_\Omega c(x)dx < 0.$$

Remark 4.18. The construction above can be repeated for any C^2 domain. The assumptions on c and the supersolution v can be rewritten respectively as follows

$$\begin{cases} c(x) < 0 & \text{if } d(x) < \epsilon \\ c(x) \le -\beta_1 & \text{if } \epsilon \le d(x) < \delta \\ c(x) \le \beta_2 & \text{if } d(x) \ge \delta, \end{cases}$$

$$v(x) := \begin{cases} D & \text{if } d(x) < \epsilon \\ E(\delta+\epsilon-d(x))^2 + E(\delta+\epsilon)(d(x)-\epsilon-\delta) + D + E\epsilon\delta & \text{if } \epsilon \le d(x) < \delta \\ D + 1 - e^{k(\delta - d(x))} & \text{if } d(x) \ge \delta, \end{cases}$$

where $0 < \epsilon < \delta$ and $d(x)$ is precisely the distance function, not one of its C^2 extensions. We recall some properties of the distance function:

- There exists $\mu > 0$ such that d is of class C^2 in $\Omega_\mu := \{x \in \overline{\Omega} | \, d(x) < \mu\}$ and the eigenvalues of the hessian matrix of d at x are 0 and $\frac{k_i}{1+d(x)k_i}$, $i = 1, ..., N-1$, where k_i are the principal curvatures of $\partial\Omega$ corresponding to the directions orthogonal to $\overrightarrow{n}$ at the point $y = x - d(x)Dd(x)$;
- d is semi-concave in Ω, i.e., there exists $s_0 > 0$ such that $d(x) - \frac{s_0}{2}|x|^2$ is concave;
- If $J^{2,-}d(x) \neq \emptyset$, d is differentiable at x and $|Dd(x)| = 1$.

We choose δ so small that in $\Omega_{\delta+\delta'}$ d is of class C^2 for some small $\delta' > 0$. Then, as in previous example, where δ was $R-\rho$, it can be proved that v is continuous on $\overline{\Omega}$, positive if $D > \frac{E}{4}(\delta-\epsilon)^2$ and of class C^2 on Ω_ϵ, $\Omega_\delta \setminus \overline{\Omega}_\epsilon$. Furthermore, $J^{2,-}v(x) = \emptyset$ if $d(x) = \epsilon$ and if $d(x) = \delta$ provided $E(\delta - \epsilon) > k$.

Let K be such that $|k_i(x)| \le K$ for all i and all $x \in \partial\Omega$. Then, if $\epsilon < d(x) < \delta$ we have the following estimate

$$\begin{aligned} F(x, Dv, D^2v) + c(x)v^{\alpha+1} \le E^{\alpha+1}(\delta-\epsilon)^\alpha \Big\{ 2A + [A\delta + (A-2a)\epsilon](N-1)K \\ + [(2A-a)\delta - a\epsilon]\frac{(N-1)K}{1-\delta K} \Big\} - \beta_1 \left[D - \frac{E}{4}(\delta-\epsilon)^2 \right]^{\alpha+1}. \end{aligned}$$

Now suppose $d(x) > \delta$, then $v(x) = D + 1 - e^{k(\delta - d(x))}$. Let $\overline{x} \in \Omega$ be such that $d(\overline{x}) > \delta$ and let ψ be a C^2 function such that $(v-\psi)(x) \ge (v-\psi)(\overline{x}) = 0$ for all x in a small neighborhood of $\overline{x}$. Then the function ϕ defined as

$$\phi(x) := -\frac{1}{k}\log(D + 1 - \psi(x)) + \delta$$

is a C^2 function in a neighborhood of $\overline{x}$, such that $(d-\phi)(x) \ge (d-\phi)(\overline{x}) = 0$. This implies that $J^{2,-}d(\overline{x}) \neq \emptyset$. According to some of the properties of d recalled before, on such point d is differentiable, $D\phi(\overline{x}) = Dd(\overline{x})$ and $D^2\phi(\overline{x}) \le s_0 I$. Then it easy to check that for $k > \frac{s_0 AN}{a}$

$$\begin{aligned} F(\overline{x}, D\psi(\overline{x}), D^2\psi(\overline{x})) + c(\overline{x})v^{\alpha+1} \le k^{\alpha+1}e^{-(\alpha+1)k(R-\delta)}(s_0AN - ka) \\ + \beta_2(D + 1 - e^{-k(R-\delta)})^{\alpha+1}, \end{aligned}$$

where $R := \max_{\overline{\Omega}} d(x)$.

We can repeat the argument used before to conclude that v is a positive supersolution of (4.12) if ϵ is small enough and β_1 and β_2 satisfy the following inequality for some $k > \frac{s_0 AN}{a}$

$$\beta_2 < \frac{k^{\alpha+1}e^{-(\alpha+1)k(R-\delta)}(ka - s_0AN)}{\left\{ k\left[\frac{\delta}{4} + \left[\frac{2A+(N-1)\delta K[A+(2A-a)(1-\delta K)^{-1}]}{\beta_1\delta} \right]^{\frac{1}{\alpha+1}} \right] + 1 - e^{-k(R-\delta)} \right\}^{\alpha+1}}.$$

Of course the relation between β_1 and β_2 can be bettered if we have more informations about the domain Ω.

5. Some existence results

This section is devoted to the problem of the existence of a solution of

$$(5.1)\qquad \begin{cases} F(x,Du,D^2u)+b(x)\cdot Du|Du|^\alpha+(c(x)+\lambda)|u|^\alpha u=g(x) & \text{in}\quad \Omega\\ \langle Du,\overrightarrow{n}(x)\rangle=0 & \text{on}\quad \partial\Omega. \end{cases}$$

The first existence result for (5.1) is obtained when $\lambda=0$ and $c<0$, via Perron's method. Using it, we will prove the existence of a positive solution of (5.1) when g is non-positive and $\lambda<\overline{\lambda}$, without condition on the sign of c. Then we will show the existence of a positive principal eigenfunction corresponding to $\overline{\lambda}$, that is a solution of (5.1) when $g\equiv 0$ and $\lambda=\overline{\lambda}$. For the last two results we will follow the proof given in [8] for the analogous theorems with the Dirichlet boundary condition.

Symmetrical results can be obtained for the eigenvalue $\underline{\lambda}$.

Finally, we will prove that the Neumann problem (5.1) is solvable for any right-hand side if $\lambda<\min\{\overline{\lambda},\underline{\lambda}\}$.

Comparison results guarantee for (5.1) the uniqueness of the solution when $c<0$, of the positive solution when $\lambda<\overline{\lambda}$ and $g<0$ and of the negative solution when $\lambda<\underline{\lambda}$ and $g>0$.

Theorem 5.1. *Suppose that $c<0$ and g is continuous on $\overline{\Omega}$. If $u\in USC(\overline{\Omega})$ and $v\in LSC(\overline{\Omega})$ are respectively viscosity sub and supersolution of*

$$(5.2)\qquad \begin{cases} F(x,Du,D^2u)+b(x)\cdot Du|Du|^\alpha+c(x)|u|^\alpha u=g(x) & \textit{in}\quad \Omega\\ \langle Du,\overrightarrow{n}(x)\rangle=0 & \textit{on}\quad \partial\Omega, \end{cases}$$

with u and v bounded or $v\geq 0$ and bounded, then $u\leq v$ in $\overline{\Omega}$. Moreover (5.2) *has a unique viscosity solution.*

Proof. We suppose by contradiction that $\max_{\overline{\Omega}}(u-v)=m>0$. Repeating the proof of Theorem 4.9 taking v as w, we arrive to the following inequality

$$-c(\overline{z})|u(\overline{z})|^\alpha u(\overline{z})\leq -c(\overline{z})|v(\overline{z})|^\alpha v(\overline{z}),$$

where $\overline{z}\in\overline{\Omega}$ is such that $u(\overline{z})-v(\overline{z})=m>0$. This is a contradiction since $c(\overline{z})<0$.

The existence of a solution follows from Perron's method of Ishii [15] and the comparison result just proved, provided there is a bounded subsolution and a bounded supersolution of (5.2). Since c is negative and continuous on $\overline{\Omega}$, there exists $c_0>0$ such that $c(x)\leq -c_0$ for every $x\in\overline{\Omega}$. Then

$$u_1:=-\left(\frac{|g|_\infty}{c_0}\right)^{\frac{1}{\alpha+1}},\quad u_2:=\left(\frac{|g|_\infty}{c_0}\right)^{\frac{1}{\alpha+1}}$$

are respectively a bounded sub and supersolution of (5.2).

Put

$$u(x):=\sup\{\varphi(x)|\, u_1\leq\varphi\leq u_2 \text{ and } \varphi \text{ is a subsolution of (5.2) }\},$$

then u is a solution of (5.2). We first show that the upper semicontinuous envelope of u defined as

$$u^*(x):=\lim_{\rho\downarrow 0}\sup\{u(y): y\in\overline{\Omega} \text{ and } |y-x|\leq\rho\}$$

is a subsolution of (5.2). Indeed if $(p,X)\in J^{2,+}u(x_0)$ and $p\neq 0$ then by the standard arguments of the Perron's method it can be proved that $G(x_0,u(x_0),p,X)\geq g(x_0)$ if $x_0\in\Omega$ and $(-G(x_0,u(x_0),p,X)+g(x_0))\wedge\langle p,\overrightarrow{n}(x_0)\rangle\leq 0$ if $x_0\in\partial\Omega$.

Now suppose $u^* \equiv k$ in a neighborhood of $x_0 \in \overline{\Omega}$. If $x_0 \in \partial\Omega$ clearly u^* is subsolution in x_0. Assume that x_0 is an interior point of Ω. We may choose a sequence of subsolutions $(\varphi_n)_n$ and a sequence of points $(x_n)_n$ in Ω such that $x_n \to x_0$ and $\varphi_n(x_n) \to k$. Suppose that $|x_n - x_0| < a_n$ with a_n decreasing to 0 as $n \to +\infty$. If, up to subsequence, φ_n is constant in $B(x_0, a_n)$ for any n, then passing to the limit in the relation $c(x_n)|\varphi_n(x_n)|^\alpha \varphi_n(x_n) \geq g(x_n)$ we get $c(x_0)|k|^\alpha k \geq g(x_0)$ as desired. Otherwise, suppose that for any n φ_n is not constant in $B(x_0, a_n)$. Repeating the argument of Lemma 4.12 we find a sequence $\{(t_n, y_n)\}_{n\in\mathbb{N}} \subset \Omega^2$ and a small $\delta > 0$ such that $|t_n - x_0| < a_n$, $|y_n - x_0| \leq \delta$, $t_n \neq y_n$, $\varphi_n(x) - |x - t_n|^q \leq \varphi_n(y_n) - |y_n - t_n|^q$ for any $x \in B(x_0, \delta)$, with $q > \max\{2, \frac{\alpha+2}{\alpha+1}\}$ and $u^* \equiv k$ in $\overline{B}(x_0, \delta)$. Up to subsequence $y_n \to y \in \overline{B}(x_0, \delta)$ as $n \to +\infty$. We have

$$\begin{aligned} k = \lim_{n\to+\infty} (\varphi_n(x_n) - |x_n - t_n|^q) &\leq \liminf_{n\to+\infty} (\varphi_n(y_n) - |y_n - t_n|^q) \\ &\leq \limsup_{n\to+\infty} (\varphi_n(y_n) - |y_n - t_n|^q) \leq k - |y - x_0|^q. \end{aligned}$$

The last inequalities imply that $y = x_0$ and $\varphi_n(y_n) \to k$. Then for large n, y_n is an interior point of $B(x_0, \delta)$ and $\phi_n(x) := \varphi_n(y_n) - |y_n - t_n|^q + |x - t_n|^q$ is a test function for φ_n at y_n. Passing to the limit as $n \to +\infty$ in the relation $G(y_n, \varphi_n(y_n), D\phi_n(y_n), D^2\phi_n(y_n)) \geq g(y_n)$, we get again $c(x_0)|k|^\alpha k \geq g(x_0)$. In conclusion u^* is a subsolution of (5.2). Since $u_1 \leq u^* \leq u_2$, it follows from the definition of u that $u = u^*$.

Finally the lower semicontinuous envelope of u defined as

$$u_*(x) := \lim_{\rho\downarrow 0} \inf\{u(y) : y \in \overline{\Omega} \text{ and } |y - x| \leq \rho\}$$

is a supersolution. Indeed, if it is not, the Perron's method provides a viscosity subsolution of (5.2) greater than u, contradicting the definition of u. If $u_* \equiv k$ in a neighborhood of $x_0 \in \Omega$ and $c(x_0)|k|^\alpha k > g(x_0)$ then for small δ and ρ, the subsolution is

$$u_{\delta,\rho}(x) := \begin{cases} \max\{u(x), k + \frac{\delta\rho^2}{8} - \delta|x - x_0|^2\} & \text{if } |x - x_0| < \rho, \\ u(x) & \text{otherwise.} \end{cases}$$

Hence u_* is a supersolution of (5.2) and then, by comparison, $u^* = u \leq u_*$, showing that u is continuous and is a solution.

The uniqueness of the solution is an immediate consequence of the comparison principle just proved. □

Theorem 5.2. *Suppose $g \in LSC(\overline{\Omega})$, $h \in USC(\overline{\Omega})$, $h \leq 0$, $h \leq g$ and $g(x) > 0$ if $h(x) = 0$. Let $u \in USC(\overline{\Omega})$ be a viscosity subsolution of* (5.1) *and $v \in LSC(\overline{\Omega})$ be a bounded positive viscosity supersolution of* (5.1) *with g replaced by h. Then $u \leq v \quad in\ \overline{\Omega}$.*

Remark 5.3. The existence of a such v implies $\lambda \leq \overline{\lambda}$.

Proof. It suffices to prove the theorem for $h < g$. Indeed, for $l > 1$ the function defined by $v_l := lv$ is a supersolution of (5.1) with right-hand side $l^{\alpha+1}h(x)$. By the assumptions on h and g, $l^{\alpha+1}h < g$. If $u \leq lv$ for any $l > 1$, passing to the limit as $l \to 1^+$, one obtains $u \leq v$ as desired.

Hence we can assume $h < g$. By upper semicontinuity $\max_{\overline{\Omega}}(h - g) = -M < 0$. Suppose by contradiction that $u > v$ somewhere in Ω. Then there exists $\overline{y} \in \overline{\Omega}$ such that

$$\gamma' := \frac{u(\overline{y})}{v(\overline{y})} = \max_{x\in\overline{\Omega}} \frac{u(x)}{v(x)} > 1.$$

Define $w = \gamma v$ for some $1 \le \gamma < \gamma'$. Since $h \le 0$ and $\gamma \ge 1$, $\gamma^{\alpha+1} h \le h$ and then w is still a supersolution of (5.1) with right-hand side h. The supremum of $u - w$ is strictly positive then, by upper semicontinuity, there exists $\overline{x} \in \overline{\Omega}$ such that $u(\overline{x}) - w(\overline{x}) = \max_{\overline{\Omega}}(u - w) > 0$. We have $u(\overline{x}) > w(\overline{x})$ and $w(\overline{x}) \ge \frac{\gamma}{\gamma'} u(\overline{x})$. Repeating the proof of Theorem 4.9, we get

$$g(\overline{z}) - (\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le h(\overline{z}) - (\lambda + c(\overline{z}))w(\overline{z})^{\alpha+1},$$

where $\overline{z}$ is some point in $\overline{\Omega}$ where the maximum of $u - w$ is attained. If $\lambda + c(\overline{z}) \le 0$, then

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le h(\overline{z}) - g(\overline{z}) - (\lambda + c(\overline{z}))w(\overline{z})^{\alpha+1} < -(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1},$$

which is a contradiction. If $\lambda + c(\overline{z}) > 0$, then

$$-(\lambda + c(\overline{z}))u(\overline{z})^{\alpha+1} \le h(\overline{z}) - g(\overline{z}) - (\lambda + c(\overline{z})) \left(\frac{\gamma}{\gamma'}\right)^{\alpha+1} u(\overline{z})^{\alpha+1}.$$

If we choose γ sufficiently close to γ' in order that

$$|\lambda + c|_\infty \left[\left(\frac{\gamma}{\gamma'}\right)^{\alpha+1} - 1\right] (\max_{\overline{\Omega}} u)^{\alpha+1} \ge -\frac{M}{2},$$

we get once more a contradiction. □

Theorem 5.4. *Suppose that $\lambda < \overline{\lambda}$, $g \le 0$, $g \not\equiv 0$ and g is continuous on $\overline{\Omega}$, then there exists a positive viscosity solution of* (5.1)*. If $g < 0$, the positive solution is unique.*

Proof. If $\lambda < -|c|_\infty$ then the existence of the solution is guaranteed by Theorem 5.1. Let us suppose $\lambda \ge -|c|_\infty$ and define by induction the sequence $(u_n)_n$ by $u_1 = 0$ and u_{n+1} as the solution of

$$\begin{cases} F(x, Du_{n+1}, D^2u_{n+1}) + b(x) \cdot Du_{n+1}|Du_{n+1}|^\alpha \\ \quad +(c(x) - |c|_\infty - 1)|u_{n+1}|^\alpha u_{n+1} = g - (\lambda + |c|_\infty + 1)|u_n|^\alpha u_n & \text{in} \quad \Omega \\ \langle Du_{n+1}, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

which exists by Theorem 5.1. By the comparison principle, since $g \le 0$ and $g \not\equiv 0$ the sequence is positive and increasing. We use the argument of Theorem 7 of [8] to prove that $(u_n)_n$ is also bounded. Suppose that it is not, then dividing by $|u_{n+1}|_\infty^{\alpha+1}$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_{n+1} is a solution of

$$\begin{cases} F(x, Dv_{n+1}, D^2v_{n+1}) + b(x) \cdot Dv_{n+1}|Dv_{n+1}|^\alpha \\ \quad +(c(x) - |c|_\infty - 1)v_{n+1}^{\alpha+1} = \frac{g}{|u_{n+1}|_\infty^{\alpha+1}} - (\lambda + |c|_\infty + 1)\frac{u_n^{\alpha+1}}{|u_{n+1}|_\infty^{\alpha+1}} & \text{in} \quad \Omega \\ \langle Dv_{n+1}, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By Corollary 3.4, $(v_n)_n$ converges to a positive function v with $|v|_\infty = 1$ which satisfies

$$\begin{cases} F(x, Dv, D^2v) + b(x) \cdot Dv|Dv|^\alpha + (c(x) + \lambda)v^{\alpha+1} \\ \quad = (\lambda + |c|_\infty + 1)(1 - k)v^{\alpha+1} \ge 0 & \text{in} \quad \Omega \\ \langle Dv, \overrightarrow{n}(x)\rangle = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $k := \lim_{n\to+\infty} \frac{|u_n|_\infty^{\alpha+1}}{|u_{n+1}|_\infty^{\alpha+1}} \le 1$. This contradicts the maximum principle, Theorem 4.9.

Then $(u_n)_n$ is bounded and letting n go to infinity, by the compactness result, the sequence converges to a function u which is a solution. Moreover, the solution is positive in $\overline{\Omega}$ by the strong minimum principle, Proposition 4.2.

If $g < 0$, the uniqueness of the positive solution follows from Theorem 5.2. $\square$

Theorem 5.5 (Existence of principal eigenfunctions)**.** *There exists $\phi > 0$ in $\overline{\Omega}$ viscosity solution of*

$$\begin{cases} F(x, D\phi, D^2\phi) + b(x) \cdot D\phi|D\phi|^\alpha + (c(x) + \overline{\lambda})\phi^{\alpha+1} = 0 & in \quad \Omega \\ \langle D\phi, \overrightarrow{n}(x)\rangle = 0 & on \quad \partial\Omega. \end{cases}$$

Moreover ϕ is Lipschitz continuous on $\overline{\Omega}$.

Proof. Let λ_n be an increasing sequence which converges to $\overline{\lambda}$. Let u_n be the positive solution of (5.1) with $\lambda = \lambda_n$ and $g \equiv -1$. By Theorem 5.4 the sequence $(u_n)_n$ is well defined. Following the argument of the proof of Theorem 8 of [8], we can prove that it is unbounded, otherwise one would contradict the definition of $\overline{\lambda}$. Then, up to subsequence $|u_n|_\infty \to +\infty$ as $n \to +\infty$ and defining $v_n := \frac{u_n}{|u_n|_\infty}$ one gets that v_n satisfies (5.1) with $\lambda = \lambda_n$ and $g \equiv -\frac{1}{|u_n|_\infty^{\alpha+1}}$. By Corollary 3.4, we can extract a subsequence converging to a positive function ϕ with $|\phi|_\infty = 1$ which is the desired solution. By Theorem 3.1 the solution is also Lipschitz continuous on $\overline{\Omega}$. $\square$

Remark 5.6. With the same arguments used in the proofs of Theorems 5.2, 5.4 and 5.5 one can prove: the comparison result between $u \in USC(\overline{\Omega})$ bounded and negative viscosity subsolution of (5.1) and $v \in LSC(\overline{\Omega})$ supersolution of (5.1) with g replaced by h, provided $g \geq 0$, $h \leq g$ and $h(x) < 0$ if $g(x) = 0$; the existence of a negative viscosity solution of (5.1), for $\lambda < \underline{\lambda}$ and $g \geq 0$, $g \not\equiv 0$; the existence of a negative Lipschitz first eigenfunction corresponding to $\underline{\lambda}$, i.e., a solution of (5.1) with $\lambda = \underline{\lambda}$ and $g \equiv 0$.

Theorem 5.7. *Suppose that $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$ and g is continuous on $\overline{\Omega}$, then there exists a viscosity solution of* (5.1)*.*

Proof. If $g \equiv 0$, by the maximum and minimum principles the only solution is $u \equiv 0$. Let us suppose $g \not\equiv 0$. Since $\lambda < \min\{\overline{\lambda}, \underline{\lambda}\}$ by Theorem 5.4 and Remark 5.6 there exist v_0 positive viscosity solution of (5.1) with right-hand side $-|g|_\infty$ and u_0 negative viscosity solution of (5.1) with right-hand side $|g|_\infty$.

Let us suppose $\lambda + |c|_\infty \geq 0$. Let $(u_n)_n$ be the sequence defined in the proof of Theorem 5.4 with $u_1 = u_0$, then by comparison Theorem 5.1 we have $u_0 = u_1 \leq u_2 \leq ... \leq v_0$. Hence, by the compactness Corollary 3.4 the sequence converges to a continuous function which is the desired solution. $\square$

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